Well-balanced bicharacteristic-based scheme for two-layer shallow water flows including wet/dry fronts

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Outline

1. Introduction
   - Aim
   - FVEG framework
   - Two-layer shallow water equations
2. Numerical scheme
   - Predictor step
   - Corrector step
     - Flux discretization
     - Source discretization
     - Preserve positivity
     - Adaption at shoreline
3. Numerical results

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Aim

aim: construct a scheme for the two-layer 2D shallow water equations

- in the FVEG framework
- positivity preserving
- handle wet/dry fronts
- well-balanced

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FVEG framework

We consider hyperbolic equations with source terms, i.e. balance laws

$$w_t + \operatorname{div} f(w) = S(w, \nabla w, \nabla \sigma)$$

$$w_t + \sum_{k=1}^{d} A_k w_{x_k} = s(w, \nabla \sigma)$$

- σ a smooth scalar-valued function
- $f := \left(f_1, \ldots, f_d\right)$ a smooth matrix-valued function
- $A_i := \frac{df_k}{dw} - \widetilde{s}_k(w)$
- $\Omega \subseteq \mathbb{R}^d$ spatial domain

FVEG framework

The FVEG framework consists of two step

- corrector step
- predictor step

FVEG framework

corrector step (FV update)

FVEG framework

corrector step (FV update)

$$W_{i,j}^{n+1} = W_{i,j}^{n} - \frac{1}{|\Omega_{i,j}|} \int_{t_n}^{t_{n+1}} \int_{\partial\Omega_{i,j}} f(W^*)n + \frac{\Delta t}{|\Omega_{i,j}|} \int_{\Omega_{i,j}} S(W^*, \nabla W^*, \nabla\sigma)\, dt$$

- $W_{i,j}^{n} := \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} w(x,t)\, d^{(d)}\Omega_{ij}$

FVEG framework

corrector step (FV update)

$$W_{i,j}^{n+1} = W_{i,j}^{n} - \frac{\Delta t}{|\Omega_{i,j}|} \int_{\partial\Omega_{i,j}} f(W^*)n + \frac{\Delta t}{|\Omega_{i,j}|} \int_{\Omega_{i,j}} S(W^*, \nabla W^*, \nabla\sigma)$$

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FVEG framework

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- $W_{i,j}^n := \frac{1}{|\Omega_{ij}|} \int\limits_{\Omega_{ij}} w(x,t)\, d^{(d)}\Omega_{ij}$

predictor step (Evolution operator)

$$W^* = E_{\Delta t/2} R W^n$$

- $E_{\Delta t/2}$ evolution operator
- R recovery operator (e.g. bilinear reconstruction)

FVEG framework

predictor step (Evolution operator ($E_{\Delta t/2}$))

FVEG framework

predictor step (Evolution operator ($E_{\Delta t/2}$))

$$W(P) = \frac{1}{|S^{d-1}|} \int\limits_{S^{d-1}} \sum_{j=1}^{n} r_\eta^j \, l_\eta^j \Big\{ W(\widetilde{Q}_j(\eta))$$

$$+ \int\limits_{t_s}^{t_e} s(W(Q_j(t,\eta)), \nabla\sigma(q_j(t,\eta))) + \sum_{k=1}^{d} \left(\partial_{\eta_k} \lambda_\eta^j I - \widetilde{A}_k \right) W_{x_k}(Q_j(t,\eta))\, dt \Big\}\, d\eta$$

- $q_j(t,\eta) := x_e - \nabla_\eta \lambda_\eta^j (t_e - t)$, $Q_j(t,\eta) := \left(q_j(t,\eta), t\right)^T$
- $\widetilde{q}_j(\eta) := q_j(t_s,\eta)$, $\widetilde{Q}_j(\eta) := Q_j(t_s,\eta)$
- $\widetilde{A}_k := A_k(\widetilde{W})$ linearized matrices
- $r_\eta^j, l_\eta^j, \lambda_\eta^j$ eigenstructure of $A(\eta) := \sum_{k=1}^{d} \eta_k A_k$, $\eta \in S^{d-1}$

FVEG framework

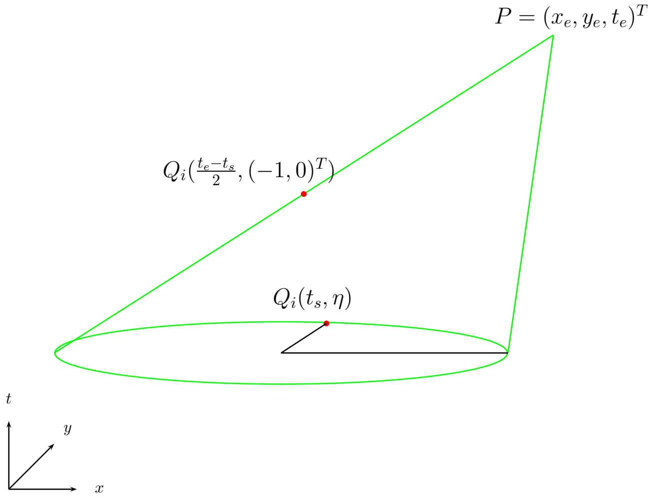

Abbildung: **Bicharacteristic cone**

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Two-layer Shallow water equations

$$
\begin{cases}
\partial_t h_1 + \operatorname{div}(h_1 \vec{u}_1) = 0, \\
\partial_t (h_1 \vec{u}_1) + \operatorname{div}\left(h_1 \vec{u}_1 \vec{u}_1^T\right) + g h_1 \nabla (h_1 + h_2 + b) + f h_1 \vec{u}_1^{\perp} = 0, \\
\partial_t h_2 + \operatorname{div}(h_2 \vec{u}_2) = 0, \\
\partial_t (h_2 \vec{u}_2) + \operatorname{div}\left(h_2 \vec{u}_2 \vec{u}_2^T\right) + g h_2 \nabla (r h_1 + h_2 + b) + f h_2 \vec{u}_2^{\perp} = 0,
\end{cases} \tag{1}
$$

- h_i height of i-th layer
- $\vec{u}_i := \begin{pmatrix} u_i \\ v_i \end{pmatrix}$ velocity of i-th layer
- $0 < r := \frac{\rho_1}{\rho_2} < 1$ density fraction
- $(.)^{\perp}$ ccw rotation by $\frac{\pi}{2}$

Two-layer Shallow water equations

$$\begin{cases} \partial_t h_1 + \operatorname{div}(h_1 \vec{u}_1) = 0, \\ \partial_t (h_1 \vec{u}_1) + \operatorname{div}\left(h_1 \vec{u}_1 \vec{u}_1^T\right) + \frac{g}{2}\nabla(h_1^2) = -gh_1 \nabla(h_2 + b) - fh_1 \vec{u}_1^{\perp}, \\ \partial_t h_2 + \operatorname{div}(h_2 \vec{u}_2) = 0, \\ \partial_t (h_2 \vec{u}_2) + \operatorname{div}\left(h_2 \vec{u}_2 \vec{u}_2^T\right) + \frac{g}{2}\nabla(h_2^2) = -gh_2 \nabla(rh_1 + b) - fh_2 \vec{u}_2^{\perp}, \end{cases}$$

- h_i height of i-th layer
- $\vec{u}_i := \begin{pmatrix} u_i \\ v_i \end{pmatrix}$ velocity of i-th layer
- $0 < r := \frac{\rho_1}{\rho_2} < 1$ density fraction
- $(.)^{\perp}$ ccw rotation by $\frac{\pi}{2}$

Eigenstructure

characteristic polynomial

$$
\begin{aligned}
p(t,\eta) = &\quad p_1(t,\eta)\cdot p_2(t,\eta)\\
= &\quad \underbrace{(\eta^T\vec{u}_1 - t)(\eta^T\vec{u}_2 - t)}_{=:p_1(t,\eta)}\\
&\quad \underbrace{\left[\left((\eta^T\vec{u}_1 - t)^2 - \|\eta\|_2^2 g h_1\right)\left((\eta^T\vec{u}_2 - t)^2 - \|\eta\|_2^2 g h_2\right) - \|\eta\|_2^4 r g^2 h_1 h_2\right]}_{=:p_2(t,\eta)}
\end{aligned}
$$

Eigenstructure

characteristic polynomial

$$p(t,\eta) = p_1(t,\eta)\cdot p_2(t,\eta)$$

$$= \underbrace{(\eta^T\vec{u}_1 - t)(\eta^T\vec{u}_2 - t)}_{=:p_1(t,\eta)}$$

$$\underbrace{\left[\left((\eta^T\vec{u}_1 - t)^2 - \|\eta\|_2^2 g h_1\right)\left((\eta^T\vec{u}_2 - t)^2 - \|\eta\|_2^2 g h_2\right) - \|\eta\|_2^4 r g^2 h_1 h_2\right]}_{=:p_2(t,\eta)}$$

- always four real roots $\lambda_{2/5} := \eta^T\vec{u}_{1/2}$ and $\lambda_{1/3}$, surface waves

Eigenstructure

characteristic polynomial

$$p(t,\eta) = p_1(t,\eta)\cdot p_2(t,\eta)$$

$$= \underbrace{(\eta^T\vec{u}_1 - t)(\eta^T\vec{u}_2 - t)}_{=:p_1(t,\eta)}$$

$$\underbrace{\left[\left((\eta^T\vec{u}_1 - t)^2 - \|\eta\|_2^2 gh_1\right)\left((\eta^T\vec{u}_2 - t)^2 - \|\eta\|_2^2 gh_2\right) - \|\eta\|_2^4 rg^2 h_1 h_2\right]}_{=:p_2(t,\eta)}$$

- always four real roots $\lambda_{2/5} := \eta^T\vec{u}_{1/2}$ and $\lambda_{1/3}$, surface waves
- two possibly complex roots $\lambda_{4/6}$, interface waves

Hyperbolic region

Lemma

(1) *is hyperbolic* $\iff$ $\forall \eta \ : \ p_2(t_{max}(\eta), \eta) > 0.$

Eigenstructure

$$
\begin{aligned}
T &:= \frac{1}{3}\left(\left(\eta^T\left(\vec{u}_1 - \vec{u}_2\right)\right)^2 - g\left(h_1 + h_2\right)\right),\\
C_1 &:= T + g\left(h_1 + h_2\right) \geq 0,\\
C_2 &:= T^2 + \frac{4}{3} g^2 h_1 h_2 (1 - r) \geq 0,\\
C_3 &:= T^3 - 2g^2 h_1 h_2 \left[(2 + r)T + rg\left(h_1 + h_2\right)\right],\\
D &:= \sqrt{C_1 + \left(\sqrt{C_3^2 - C_2^3} + C_3\right)^{1/3} + \frac{C_2}{\left(\sqrt{C_3^2 - C_2^3} + C_3\right)^{1/3}}},\\
K_{1/2} &:= \sqrt{3C_1 - D^2 \mp \frac{2g\left(h_1 - h_2\right)\left(\eta^T(\vec{u}_1 - \vec{u}_2)\right)}{D}},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2}\eta^T\left(\vec{u}_1 + \vec{u}_2\right) - \frac{1}{2}\left(D + K_1\right),\\
\lambda_3 &= \frac{1}{2}\eta^T\left(\vec{u}_1 + \vec{u}_2\right) + \frac{1}{2}\left(D + K_2\right),\\
\lambda_4 &= \frac{1}{2}\eta^T\left(\vec{u}_1 + \vec{u}_2\right) - \frac{1}{2}\left(D - K_1\right),\\
\lambda_6 &= \frac{1}{2}\eta^T\left(\vec{u}_1 + \vec{u}_2\right) + \frac{1}{2}\left(D - K_2\right).
\end{aligned}
$$

Hyperbolic region

Lemma

$$(1) \textit{ is hyperbolic} \iff \forall \eta : \Im(D) = 0, D \neq 0.$$

Steady states

we consider solutions constant along streamlines, i.e.

$$\left(\partial_t + \nabla^T \vec{u}_i\right) h_i = 0, \quad \left(\partial_t + \nabla^T \vec{u}_i\right) \vec{u}_i = 0, \quad i = 1, 2$$

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then (1) rewrites to ($h_i \neq 0$)

$$\begin{cases} \operatorname{div}(\vec{u}_1) = \operatorname{div}(\vec{u}_2) = 0, \\ g\nabla\varepsilon = -f\vec{u}_1^{\perp}, \\ g\nabla\widetilde{\varepsilon} = -f\vec{u}_2^{\perp}, \end{cases} \implies \text{geostrophic equilibrium}$$

$\varepsilon := h_1 + h_2 + b$ and $\widetilde{\varepsilon} := rh_1 + h_2 + b$ are locally one-dimensional

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$\varepsilon := h_1 + h_2 + b$ and $\widetilde{\varepsilon} := rh_1 + h_2 + b$ are locally one-dimensional

$$\vec{u}_1 = \vec{u}_2 = 0 \Rightarrow \nabla\varepsilon = \nabla\widetilde{\varepsilon} = 0 \quad \Longrightarrow \quad \text{lake at rest}$$

Outline

Predictor step

for prediction rewrite system in primitive variables

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for prediction rewrite system in primitive variables

$$V_t + A_1 V_x + A_2 V_y = s(\nabla b, V),$$

$$V = \begin{pmatrix} \varepsilon \\ \vec{u}_1 \\ \omega \\ \vec{u}_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} u_1 & \varepsilon - \omega & 0 & u_2 - u_1 & \omega - b & 0 \\ g & u_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2 & \omega - b & 0 \\ rg & 0 & 0 & (1-r)g & u_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_2 \end{pmatrix},$$

$$s = \begin{pmatrix} \vec{u}_2^T \nabla b \\ -f\vec{u}_1^\perp \\ \vec{u}_2^T \nabla b \\ -f\vec{u}_2^\perp \end{pmatrix}, \quad A_2 = \begin{pmatrix} v_1 & 0 & \varepsilon - \omega & v_2 - v_1 & 0 & \omega - b \\ 0 & v_1 & 0 & 0 & 0 & 0 \\ g & 0 & v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 & 0 & \omega - b \\ 0 & 0 & 0 & 0 & v_2 & 0 \\ rg & 0 & 0 & (1-r)g & 0 & v_2 \end{pmatrix}.$$

where $\omega := h_2 + b$, $\varepsilon := h_1 + \omega$

Predictor step

recall ($d = 2, n = 6$ now)

$$V(P) = \frac{1}{|S^{d-1}|} \int\limits_{S^{d-1}} \sum_{j=1}^{n} r_\eta^j \, l_\eta^j \left\{ V(\widetilde{Q}_j(\eta)) + \int\limits_{t_s}^{t_e} s(V(Q_j(t,\eta)), \nabla\sigma(q_j(t,\eta))) + \sum_{k=1}^{d} \left(\partial_{\eta_k} \lambda_\eta^j I - \widetilde{A}_k \right) V_{x_k}(Q_j(t,\eta))\, dt \right\} d\eta$$

Predictor step

recall ($d = 2, n = 6$ now)

$$V(P) = \frac{1}{|S^{d-1}|} \int\limits_{S^{d-1}} \sum_{j=1}^{n} r_\eta^j \, l_\eta^j \left\{ V(\widetilde{Q_j}(\eta)) + (t_e - t_s) \left(s(V(\widetilde{Q_j}(\eta)), \nabla\sigma(\widetilde{q_j}(\eta))) + \sum_{k=1}^{d} \left(\partial_{\eta_k} \lambda_\eta^j I - \widetilde{A_k} \right) V_{x_k}(\widetilde{Q_j}(\eta)) \right) \right\} d\eta$$

- time integral: rectangle rule at $t = t_s$

Predictor step

recall ($d = 2, n = 6$ now)

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- time integral: rectangle rule at $t = t_s$
- η integral:
  - parametrize $\eta(\theta) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \implies \int\limits_{S^1} (.) d\eta = \int\limits_0^{2\pi} (.) d\theta$

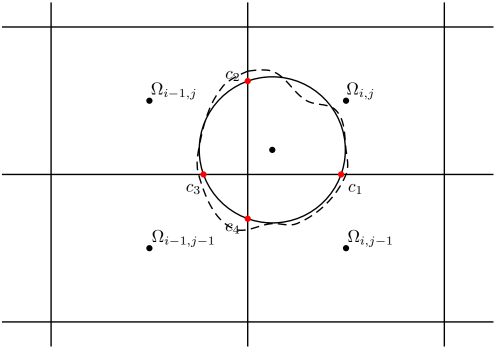
Predictor step

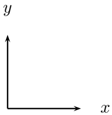
recall ($d = 2, n = 6$ now)

$$V(P) = \frac{1}{|S^{d-1}|} \int\limits_{S^{d-1}} \sum_{j=1}^{n} r_\eta^j l_\eta^j \left\{ V(\widetilde{Q}_j(\eta)) + (t_e - t_s)\left(s(V(\widetilde{Q}_j(\eta)), \nabla\sigma(\widetilde{q}_j(\eta))) + \sum_{k=1}^{d} \left(\partial_{\eta_k} \lambda_\eta^j I - \widetilde{A}_k \right) V_{x_k}(\widetilde{Q}_j(\eta)) \right) \right\} d\eta$$

- time integral: rectangle rule at $t = t_s$
- η integral:
  - parametrize $\eta(\theta) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \Longrightarrow \int\limits_{S^1} (.)d\eta = \int\limits_0^{2\pi} (.)d\theta$
  - how to approximate $\int\limits_0^{2\pi} (.)d\theta$?

Predictor step





Predictor step

Lemma

Evolution operator is exactly well-balanced for lake at rest.

Outline

Flux discretization

$$\int_{\partial\Omega_{i,j}} f(W^*)n = \sum_{k=1}^{4} \int_{\partial\Omega_{i,j}^k} f(W^*)n_{i,j}^k \approx \sum_{k=1}^{4} |\partial\Omega_{i,j}^k| \sum_{l=1}^{K} \alpha_l f(W(x_{i,k}^l, t_n + \frac{\Delta t}{2}))\, n_{i,j}^k$$

- K point quadrature rule (midpoint, Simpson,...)
- $x_{i,k}^l$ quadrature points
- α_l weights

Source discretization

$$\int_{\Omega_{i,j}} S(W^*, \nabla W^*, \nabla \sigma) \approx \; ?$$

Source discretization

$$\int_{\Omega_{i,j}} S(W^*, \nabla W^*, \nabla \sigma) \approx \ ?$$

$$\partial_t(rh_1u_1 + h_2u_2) + \partial_x\left(rh_1u_1^2 + h_2u_2^2 + \frac{g}{2}\left(rh_1^2 + 2rh_1h_2 + h_2^2\right)\right)$$
$$+ \partial_y\left(rh_1u_1v_1 + h_2u_2v_2\right)$$
$$= -g\left(rh_1 + h_2\right)\partial_x b + f\left(rh_1v_1 + h_2v_2\right),$$

Source discretization

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$$+ \partial_y\left(rh_1u_1v_1 + h_2u_2v_2\right)$$
$$= -g\left(rh_1 + h_2\right)\partial_x b + f\left(rh_1v_1 + h_2v_2\right),$$

$$\Rightarrow$$

$$rgh_1\partial_x(b + h_2) + gh_2\partial_x(b + rh_1) = rg\partial_x(h_1h_2) + g(rh_1 + h_2)\partial_x b.$$

Source discretization

$$\partial_t(rh_1u_1+h_2u_2)+\partial_x\left(rh_1u_1^2+h_2u_2^2+\frac{g}{2}\left(rh_1^2+2rh_1h_2+h_2^2\right)\right)$$
$$+\partial_y\left(rh_1u_1v_1+h_2u_2v_2\right)$$
$$=-g\left(rh_1+h_2\right)\partial_x b+f\left(rh_1v_1+h_2v_2\right),$$

$$\Rightarrow$$

$$rgh_1\partial_x(b+h_2)+gh_2\partial_x(b+rh_1)=rg\partial_x(h_1h_2)+g(rh_1+h_2)\partial_x b.$$

$$\frac{h_1^R+h_1^L}{2}\left(h_2^R-h_2^L\right)+\frac{h_2^R+h_2^L}{2}\left(h_1^R-h_1^L\right)=h_1^Rh_2^R-h_1^Lh_2^L$$

Source discretization

$$\int_{\Omega_{i,j}} S(W^*, \nabla W^*, \nabla\sigma) \approx \sum_{k=1}^{4} \int_{\partial\Omega_{i,j}^k} \frac{1}{r}\overline{C_1} \cdot \mathrm{diag}(n_{i,j}^k) \cdot \begin{pmatrix} K_2 + rh_1 + b \\ L_2 + rh_1 + b \end{pmatrix} + \overline{C_4} \cdot \mathrm{diag}(n_{i,j}^k) \cdot \begin{pmatrix} K_1 + h_2 + b \\ L_1 + h_2 + b \end{pmatrix} d^{(1)}\partial\Omega_{i,j}^k$$

$$\overline{C_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -rg\mu_x h_2 & 0 \\ 0 & -rg\mu_y h_2 \end{bmatrix}, \overline{C_4} = \begin{bmatrix} 0 & 0 \\ -g\mu_x h_1 & 0 \\ 0 & -g\mu_y h_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\partial_x K_i = -\frac{f}{g} v_i$, $\partial_x L_i = \frac{f}{g} u_i$, $i = 1, 2$

Source discretization

$$\int_{\partial\Omega_{i,j}} f(W^*)n \approx \sum_{k=1}^{4} |\partial\Omega_{i,j}^k| \sum_{l=1}^{K} \alpha_l F(W(x_{i,k}^l, t_n + \frac{\Delta t}{2}))\, n_{i,j}^k$$

$$\int_{\Omega_{i,j}} S(W^*, \nabla W^*, \nabla\sigma) \approx \sum_{k=1}^{4} \int_{\partial\Omega_{i,j}^k} \frac{1}{r}\overline{C_1} \cdot \mathrm{diag}(n_{i,j}^k) \cdot \begin{pmatrix} K_2 + rh_1 + b \\ L_2 + rh_1 + b \end{pmatrix}$$

$$+ \overline{C_4} \cdot \mathrm{diag}(n_{i,j}^k) \cdot \begin{pmatrix} K_1 + h_2 + b \\ L_1 + h_2 + b \end{pmatrix} d^{(1)}\partial\Omega_{i,j}^k$$

Source discretization

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- apply same quadrature as for fluxes

Source discretization

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$$\int_{\Omega_{i,j}} S(W^*, \nabla W^*, \nabla\sigma) \approx \sum_{k=1}^{4} |\partial\Omega_{i,j}^k| \sum_{l=1}^{K} \alpha_l \Big(\frac{1}{r}\overline{C_1} \cdot \mathrm{diag}(n_{i,j}^k) \cdot \begin{pmatrix} K_2 + rh_1 + b \\ L_2 + rh_1 + b \end{pmatrix} + \overline{C_4} \cdot \mathrm{diag}(n_{i,j}^k) \cdot \begin{pmatrix} K_1 + h_2 + b \\ L_1 + h_2 + b \end{pmatrix} \Big)$$

Corrector step

Lemma

Corrector step is exactly well-balanced for lake at rest and third order well-balanced for the geostrophic equilibrium.

Preserve positivity

$$(h_1)_{i,j}^{n+1} = (h_1)_i^n - \frac{(\Delta t)_{1,i}}{|\Omega_{i,j}|}\sum_{k=1}^{4}|\partial\Omega_{i,j}^k|\,\left(F_i^k\right)^{out} - \frac{1}{|\Omega_{i,j}|}\sum_{k=1}^{4}(\Delta t)_{1,k}^{in}\,|\partial\Omega_{i,j}^k|\,\left(F_i^k\right)^{in} \geq 0.$$

$$\left(F_i^k\right)^{in} := \min(0, F_i^k), \qquad \left(F_i^k\right)^{out} := \max(0, F_i^k).$$

Preserve positivity

$$(h_1)_{i,j}^{n+1} = (h_1)_i^n - \frac{(\Delta t)_{1,i}}{|\Omega_{i,j}|} \sum_{k=1}^{4} |\partial\Omega_{i,j}^k| \left(F_i^k\right)^{out} - \frac{1}{|\Omega_{i,j}|} \sum_{k=1}^{4} (\Delta t)_{1,k}^{in} |\partial\Omega_{i,j}^k| \left(F_i^k\right)^{in} \geq 0.$$

Preserve positivity

$$(h_1)_{i,j}^{n+1} = (h_1)_i^n - \frac{(\Delta t)_{1,i}}{|\Omega_{i,j}|} \sum_{k=1}^{4} |\partial\Omega_{i,j}^k| \left(F_i^k\right)^{out} - \frac{1}{|\Omega_{i,j}|} \sum_{k=1}^{4} (\Delta t)_{1,k}^{in} |\partial\Omega_{i,j}^k| \left(F_i^k\right)^{in} \geq 0.$$

information travels at finite speed $\Rightarrow$ if condition is not fulfilled, reduce time step

$$(\Delta t)_{1,i} := \frac{|\Omega_{i,j}|\,(h_1)_i^n}{\sum_{k=1}^{4} |\partial\Omega_{i,j}^k| \left(F_i^k\right)^{out}} \geq 0.$$

Preserve positivity

consider

$$\underbrace{(h_1 u_1)_t + (h_1 u_1^2 + \frac{1}{2} g h_1^2)_x + (h_1 u_1 v_1)_y}_{\text{flux}} = \underbrace{-g h_1 (h_2 + b)_x + f h_1 v_1}_{source}$$

lake at rest means $\partial_x (h_1 + h_2 + b) = 0$, but previous adaption introduces different timings for flux and source

Preserve positivity

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$$\underbrace{(h_1 u_1)_t + (h_1 u_1^2 + \frac{1}{2} g h_1^2)_x + (h_1 u_1 v_1)_y}_{\text{flux}} = \underbrace{-g h_1 (h_2 + b)_x + f h_1 v_1}_{\textit{source}}$$

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Preserve positivity

consider

$$(h_1 u_1)_t + \underbrace{(h_1 u_1^2)_x + (h_1 u_1 v_1)_y}_{\text{flux}} = \underbrace{-gh_1(h_1 + h_2 + b)_x + fh_1 v_1}_{source}$$

lake at rest means $\partial_x(h_1 + h_2 + b) = 0$, but previous adaption introduces different timings for flux and source

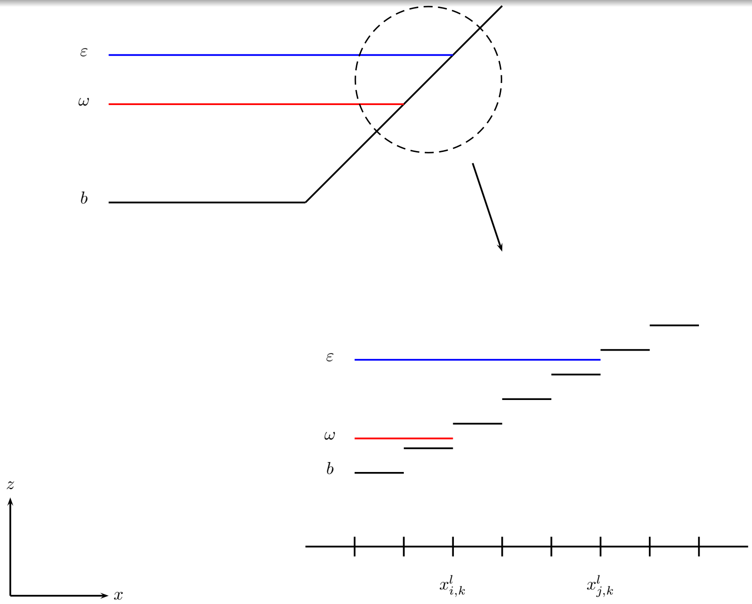
Preserve positivity

consider

$$\underbrace{(h_1 u_1)_t + (h_1 u_1^2)_x + (h_1 u_1 v_1)_y}_{(\partial_t + \vec{u}_1^T \nabla)(h_1 u_1) + h_1 u_1 \operatorname{div}(\vec{u}_1)} = \underbrace{-gh_1(h_1 + h_2 + b)_x + fh_1 v_1}_{source}$$

adaption also suffices for geostrophic equilibrium

Adaption at shoreline



Adaption at shoreline

if $x^l_{i,k}$ is a wet/dry front and $u_s(x^l_{i,k}) = 0$, $s = 1, 2$ $\Longrightarrow$ $\omega_{i+1} = \omega_i$

if $x^l_{j,k}$ is a wet/dry front and $u_s(x^l_{j,k}) = 0$, $s = 1, 2$ $\Longrightarrow$ $\varepsilon_{j+1} = \varepsilon_j$

Dambreak

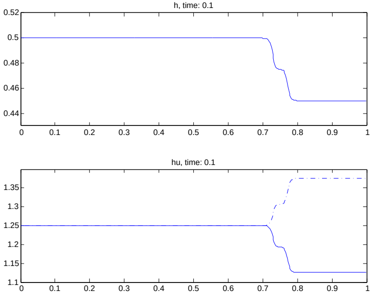
We took $g = 9.81$, $r = 0.98$, $b \equiv 0$, $r = 0.98$ and choose the following initial data

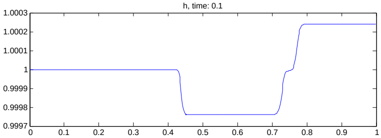
$$h_1(0,x) = \begin{cases} 0.5, & \text{if } x < 0.5, \\ 0.55, & \text{if } x > 0.5, \end{cases}$$

$$h_2(0,x) = \begin{cases} 0.5, & \text{if } x < 0.5, \\ 0.45, & \text{if } x > 0.5, \end{cases}$$

$$u_1(0,x) = u_2(0,x) = 2.5.$$

Dambreak


h, time: 0.1
0.52
0.5
0.48
0.46
0.44
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
hu, time: 0.1
1.35
1.3
1.25
1.2
1.15
1.1


h, time: 0.1
1.0003
1.0002
1.0001
1
0.9999
0.9998
0.9997
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

2nd Dambreak

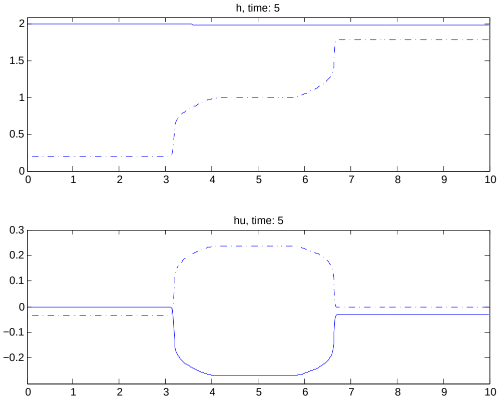
We took $g = 9.81$, $r = 0.98$, $b \equiv 0$, $r = 0.98$ and choose the following initial data

$$h_1(0,x) = \begin{cases} 0.2, & \text{if } x < 5, \\ 1.8, & \text{if } x > 5, \end{cases}$$

$$h_2(0,x) = \begin{cases} 1.8, & \text{if } x < 5, \\ 0.2, & \text{if } x > 5, \end{cases}$$

$$u_1(0,x) = u_2(0,x) = 0.$$

2nd Dambreak



Experimental order of convergence

$$
\begin{aligned}
h_1(0,x,y) &= 10 + e^{\sin(2\pi x)} \cdot \cos(2\pi y),\\
h_1 u_1(0,x,y) &= \sin(\cos(2\pi x)) \cdot \sin(2\pi y),\\
h_1 v_1(0,x,y) &= \cos(2\pi x) \cdot \cos(\sin(2\pi y)),\\
h_2(0,x,y) &= 2,\\
u_2(0,x,y) &= v_2(0,x,y) = 0,\\
b(x,y) &= \sin(2\pi x) + \cos(2\pi y)\\
r &= 0.98,\; g = 9.812,\; f = 0
\end{aligned}
$$

Experimental order of convergence

No. cells	h_1	EOC	h_2	EOC
20 × 20	2.5155e-02		9.5881e-03	
40 × 40	5.7887e-03	2.1195	2.5155e-03	1.9304
80 × 80	1.2775e-03	2.1800	6.3063e-04	1.9960
160 × 160	2.8574e-04	2.1605	1.5581e-04	2.0170
320 × 320	6.1849e-05	2.2079	3.8325e-05	2.0234

No. cells	u_1	EOC	u_2	EOC
20 × 20	7.8012e-02		2.0035e-02	
40 × 40	1.5628e-02	2.3196	4.3540e-03	2.2022
80 × 80	3.3066e-03	2.2407	1.0165e-03	2.0987
160 × 160	7.0742e-04	2.2247	2.4500e-04	2.0528
320 × 320	1.5367e-04	2.2027	5.5853e-05	2.1331

No. cells	v_1	EOC	v_2	EOC
20 × 20	1.6859e-01		3.1914e-02	
40 × 40	3.3968e-02	2.3113	6.4839e-03	2.2992
80 × 80	6.9749e-03	2.2839	1.3617e-03	2.2515
160 × 160	1.5272e-03	2.1913	3.0344e-04	2.1659
320 × 320	3.6685e-04	2.0576	7.2801e-05	2.0594

Lake at rest

$$h_1(0,x,y) = \frac{K_2 - K_1}{r - 1}$$

$$h_2(0,x,y) = \frac{1}{1-r}\left(K_2 - r \cdot K_1\right) - b$$

$$b(x,y) = \begin{cases} 0.2, & \text{if } \|(x,y)\|_\infty < 0.5, \\ 0.1, & \text{else}, \end{cases}$$

$$u_1(0,x,y) = u_2(0,x,y) = v_1(0,x,y) = v_2(0,x,y) = 0,$$

where $\|.\|_\infty$ denotes the maximum norm and $r := \frac{\rho_1}{\rho_2} = 0.5$. The parameters are $K_1 = 1.0$, $K_2 = 0.7$, $g = 9.81$ and $f = 0$. Thus we have

$$h_1 + h_2 + b = K_1, \qquad r \cdot h_1 + h_2 + b = K_2.$$

Experimental tests were done for several grids using $5 \times 5, 10 \times 10, 20 \times 20, \ldots, 500 \times 500$ mesh cells. Here we have used different CFL numbers from $(0,1]$. The results always yield

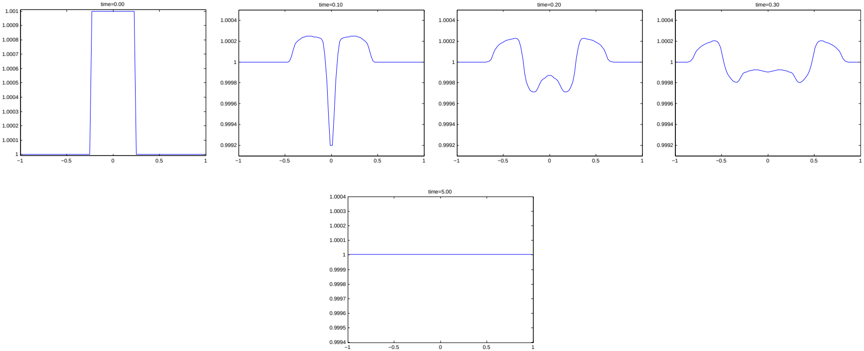
$$\|h_1 + h_2 + b - K_1\|_{L^1} = 0, \qquad \|r \cdot h_1 + h_2 + b - K_2\|_{L^1} = 0.$$

In order to calculate the L^1-norms the double precision arithmetic was applied.

Lake at rest

Now we perturb h_1

$$h_1(0,x,y) = \frac{K_2 - K_1}{r - 1} + \begin{cases} 10^{-3}, & \text{if } \|(x,y)\|_\infty < 0.25, \\ 0.0, & \text{else.} \end{cases}$$

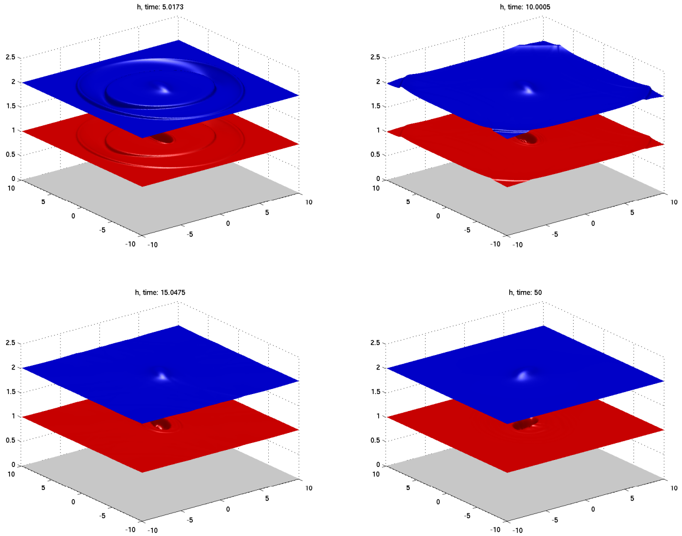


Geostrophic adjustment

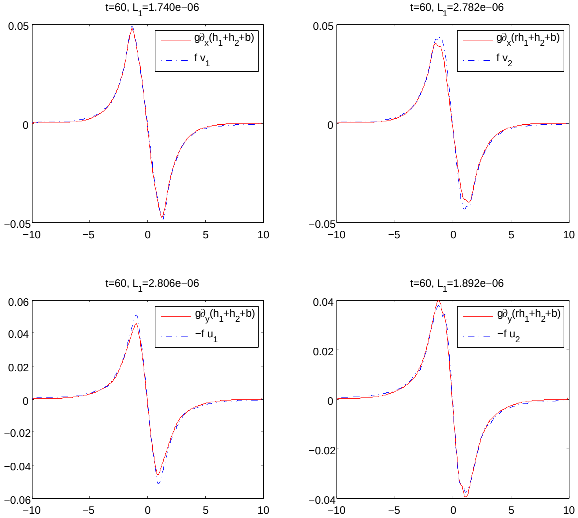
$$
\begin{aligned}
h_1(0,x,y) &= 1 + \frac{A_0}{2}\left(1 - \tanh\left(\frac{\sqrt{(\sqrt{\lambda}x)^2 + (y/\sqrt{\lambda})^2} - R_i}{R_E}\right)\right), \\
h_2(0,x,y) &= 1, \qquad u_1(0,x,y) = v_1(0,x,y) = u_2(0,x,y) = v_2(0,x,y) = 0,
\end{aligned}
$$

- $A_0 = 0.5$, $\lambda = 2.5$, $R_E = 0.1$ and $R_i = 1$
- $g = f = 1$, $r = 0.98$

Geostrophic adjustment



Geostrophic adjustment



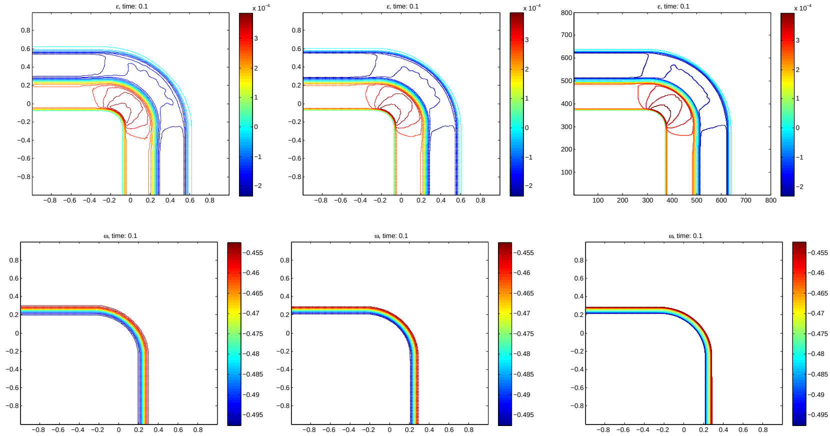
Interface propagation

$$
\begin{aligned}
h_1(0,x,y) &= \begin{cases} 0.5, & \text{if } x \in \Omega, \\ 0.45, & \text{else}, \end{cases} \\
h_2(0,x,y) &= \begin{cases} 0.5, & \text{if } x \in \Omega, \\ 0.55, & \text{else}, \end{cases} \\
u_1(0,x,y) &= u_2(0,x,y) = v_1(0,x,y) = v_2(0,x,y) = 2.5, \\
b(x,y) &= 0, \\
g &= 10, \; r = 0.98, \; f = 0
\end{aligned}
$$

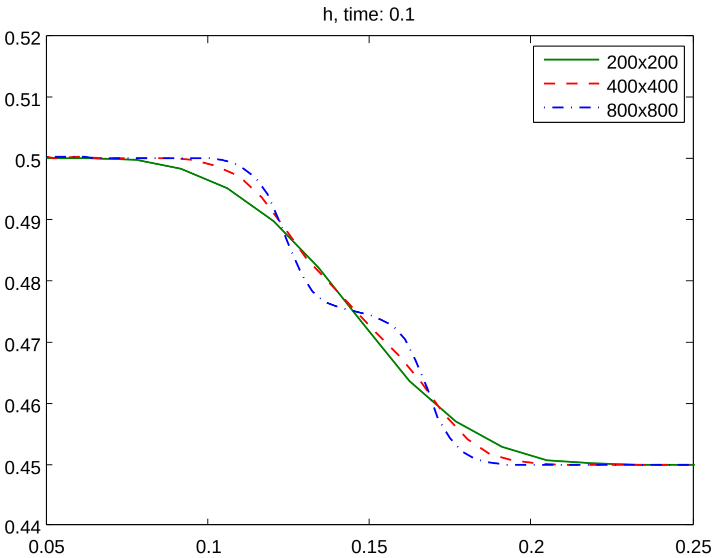
where Ω is given by

$$
\Omega := \{(x+0.5)^2+(y+0.5)^2 < 0.25\} \cup \{x < -0.5, y < 0.0\} \cup \{x < 0.0, y < -0.5\}.
$$

Interface propagation



Interface propagation



Runup on canonical island

$$b(x,y) := \begin{cases} 0.625, & (x-12.5)^2+(y-15)^2 \le 1.21, \\ \frac{3.6-\sqrt{(x-12.5)^2+(y-15)^2}}{4}, & (x-12.5)^2+(y-15)^2 \le 12.96, \\ 0, & \text{else}. \end{cases}$$

Layer depths are chosen as

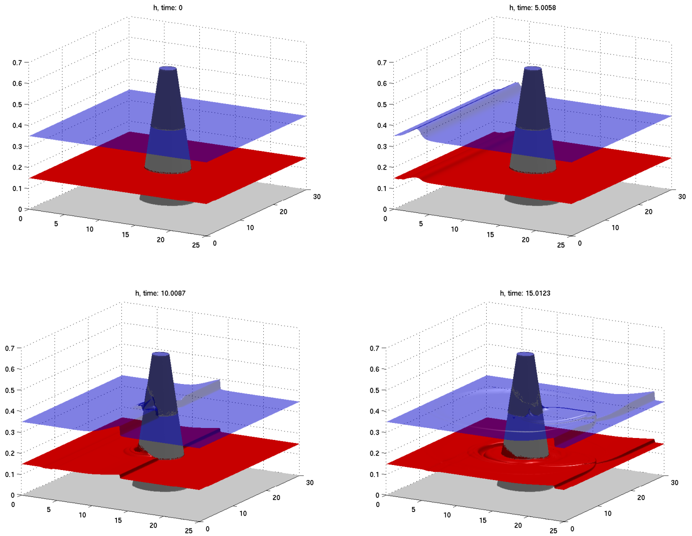
$$h_2(0,x,y) = \max(0, H_2 - b(x,y)),$$
$$h_1(0,x,y) = \max(0, H_1 - h_2(0,x,y) - b(x,y)),$$

where $H_1 = 0.35$ and $H_2 = 0.15$ and the velocities vanish initially. The height of solitary wave entering the domain at time $t = 0$ through the left boundary in the i-th layer is given by

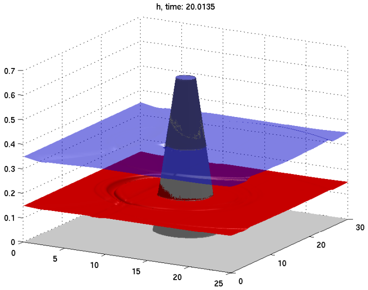
$$w_i(t,0,y) = \alpha H_i \left(\frac{1}{\cosh((t-3.5)\xi_i \sqrt{gH_i}/L)} \right)^2$$

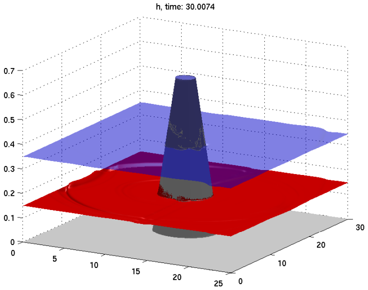
with $L = 15$, $\alpha = 0.1$ and $\xi_i = \sqrt{\frac{3\alpha(1+\alpha)L^2}{4H_i^2}}$. The density ratio is taken as $r = 0.7$.

Runup on canonical island



Runup on canonical island





Lake at rest

Thank you for your attention