Asymptotic stability of kinetic plasmas for general collision potentials

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I. Background

Physical description of a plasma

- Plasma is the 4th state of matter: solid→liquid→gas→plasma
- 99.9% of the universe exists in a plasma state
- Plasma is a gas of charged particles, e.g. electrons and ions
- The motion of plasmas strongly responds to the self-consistent electromagnetic field through the Maxwell equations

$$\frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c} J, \quad \frac{1}{c}\partial_t B + \nabla \times E = 0,$$
$$\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0.$$

- Plasma physics involves the physics of classical mechanics, electromagnetism, and non relativistic statistical mechanics
- Challenge lies in the long-range coulomb interaction

Mathematical description of a plasma

- **microscopic particle model for** $[x_i(t), \xi_i(t)]$
- **mesoscopic kinetic model for** $f(t, x, \xi)$
- **macroscopic fluid model for** $[n(t, x), u(t, x)]$

1st type (Klimontovich)

Microscopic motion equations governing $[x_i(t), \xi_i(t)]$ of all plasma particles $1 \leq i \leq N_0$ of s-species at any time t:

$$
\begin{aligned}
& m_s \frac{d\xi_i}{dt} = q_s[E(t, x_i) + \frac{\xi_i}{c} \times B(t, x_i)], \\
& \frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c} J, \\
& \frac{1}{c}\partial_t B + \nabla \times E = 0, \\
& \nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0,
\end{aligned}
$$

$$
\rho = \sum_s q_s \int_{\mathbb{R}^3} N_s(t, x, \xi)\, d\xi, \quad J = \sum_s q_s \int_{\mathbb{R}^3} \xi N_s(t, x, \xi)\, d\xi,
$$

$$
N_s(t, x, \xi) = \sum_{i=1}^{N_0} \delta(x - x_i(t))\delta(\xi - \xi_i(t)).
$$

2nd type (kinetic plasma equations)

VDFs $f_s = f_s(t,x,\xi) \geq 0$, $t \geq 0, x \in \mathbb{R}^3, \xi \in \mathbb{R}^3$, $s = i, e$

$$\partial_t f_s + \xi \cdot \nabla_x f_s + \frac{q_s}{m_s}(E + \frac{\xi}{c} \times B) \cdot \nabla_\xi f_s = \left(\frac{\partial f_s}{\partial t}\right)_c,$$

$$\frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c} J,$$

$$\frac{1}{c}\partial_t B + \nabla \times E = 0,$$

$$\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0,$$

$$\rho = \sum_s q_s \int_{\mathbb{R}^3} f_s(t,x,\xi)\, d\xi, \quad J = \sum_s q_s \int_{\mathbb{R}^3} \xi f_s(t,x,\xi)\, d\xi.$$

Depending on the collisional feature, the system is called

- **Vlasov-Maxwell-Boltzmann**
- **Vlasov-Maxwell-Landau**

Characterization of collisions

– **Boltzmann collision (Boltzmann, 1872):**

$$\left(\frac{\partial f_s}{\partial t}\right)_c = \sum_{s'} Q(f_s, f_{s'}),$$

$$Q(f_1, f_2)(\xi) = \iint_{\mathbb{R}^3 \times S^2} B(\xi - \xi_*, \omega)\{f_1(\xi')f_2(\xi'_*) - f_1(\xi)f_2(\xi_*)\}\, d\xi_* d\omega,$$

$$\begin{cases} \xi' = \xi \quad - \frac{2m_2}{m_1+m_2}[(\xi - \xi_*)\cdot\omega]\omega, \\ \xi'_* = \xi_* + \frac{2m_1}{m_1+m_2}[(\xi - \xi_*)\cdot\omega]\omega, \end{cases}$$

$$B(\xi - \xi_*, \omega) = \Phi(|\xi - \xi_*|)\, b\left(\frac{\xi - \xi_*}{|\xi - \xi_*|}\cdot\omega\right),$$

$$\Phi(|z|) \sim |z|^\gamma\, (-3 < \gamma \le 1), \quad \sin\theta\, b(\cos\theta) \sim \frac{1}{\theta^{1+2\nu}}\, (0 < \nu < 1)$$

An example: For the inverse power law $U(r) = r^{-(p-1)}$ $(p > 2)$**,** $\gamma = \frac{p-5}{p-1}$**,** $\nu = \frac{1}{p-1}$.

Grad's angular cutoff assumption:

$$\int_0^{\pi/2} \sin\theta\tilde{b}(\cos\theta)\, d\theta < \infty.$$

– **Landau collision (Landau, 1936):**

$$\left(\frac{\partial f_s}{\partial t}\right)_c = \sum_{s'} Q(f_s, f_{s'}),$$

$$Q(f_1, f_2)$$
$$= \frac{1}{m_1}\nabla_\xi \cdot \int_{\mathbb{R}^3} \Phi(\xi - \xi')\{\frac{1}{m_1} f_1(\xi)\nabla_\xi f_2(\xi') - \frac{1}{m_2} f_2(\xi)\nabla_\xi f_1(\xi')\}\, d\xi',$$

$$\Phi(z) = |z|^{\gamma+2}(\mathbf{I} - \frac{z \otimes z}{|z|^2})\ (\gamma \geq -3),$$
$\gamma = -3$: **Coulomb potential**

Remark: Boltzmann⇒Landau (Grazing limit)

3rd type (fluid plasma equations)

Number density n_s and velocity v_s of s-species:

$$
\begin{aligned}
&\partial_t n_s + \nabla \cdot (n_s v_s) = 0,\\
&m_s n_s(\partial_t v_s + v_s \cdot \nabla v_s) + \nabla P_s = q_s n_s\left(E + \frac{v_s}{c} \times B\right)\\
&\qquad\qquad + \sum_{s'} \nu_{ss'} \frac{m_s m_{s'} n_s n_{s'}}{m_s n_s + m_{s'} n_{s'}} (v_s - v_{s'}),\\
&\frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c} J,\\
&\frac{1}{c}\partial_t B + \nabla \times E = 0,\\
&\nabla \cdot E = 4\pi\rho, \qquad \nabla \cdot B = 0,\\
&\rho = \sum_s q_s n_s, \qquad J = \sum_s q_s n_s v_s.
\end{aligned}
$$

Euler-Maxwell system with/without collisions

The Plasma Stability Problem

- Due to the collision AND particle-field interactive mechanism, a plasma usually relaxes to different kinds of profiles such as equilibrium states, periodic states, and wave patterns.
- Both physically and mathematically, it is an important task to understand the stability of those profiles.
- Stability theory addresses the following three questions:
  - Can the initial (small) perturbation of a given profile imply the global-in-time existence of solutions?
  - Will the solution converge to it? How fast for the rate of convergence?
  - If unstable, how to characterize the growth modes?

Remark: Problems without collisions are quite different (nonlinear effect and structure) !

- **Molecule model:** H. Weitzner (CPAM '12)
- **Vlasov-Poisson system:** Lemou-Mehats-Raphael (IM '11), Mouhot-Villani (AM '11), ...
- **Euler-Maxwell system:** Germain-Masmoudi (arXiv '11)
- **...**

Remark: In the presence of the electromagnetic field, one may need to take into account some additional physical effects, such as

- **Relativistic effect:** $v(p) = \frac{p}{\sqrt{1+|p|^2/c^2}}$
- **Quantum effect: Feimi-Dirac/Bose-Einstein**
- **Dynamical screening effect: due to plasma polarization**
- **...**

II. Main results

Time-asymptotic stability of kinetic plasmas for general collision potentials

Boltzmann's celebrated H-theorem

$\partial_t f = Q(f, f) \Rightarrow$ **(Physical) entropy increasing:**

$$\frac{d}{dt}\int_{\mathbb{R}^3} d\xi \left\{-f \log f\right\} \geq 0.$$

This is a manifestation of the second law of thermodynamics.

- **Entropy takes the maximization at the Maxwellian**

$$\mathbf{M} = \mathbf{M}_{[\rho,u,T]}(\xi) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|\xi-u|^2}{2T}}.$$

  ρ : **density,** u : **bulk velocity,** T: **temperature.**

- **L. Boltzmann himself predicted rapid convergence in large time to the Maxwellian due to the H-theorem. The "proof" was however held back by "analytical difficulties".**

- **Goal: prove convergence and convergence rate around the Maxwellian in the spatially non-homogeneous case.**

Degeneration of H-theorem

$$\{\partial_t + \xi \cdot \nabla_x\} f = Q(f, f) \Rightarrow$$

$$\frac{d}{dt} \int_{\Omega} dx \int_{\mathbb{R}^3} d\xi \, \{-f \log f\} \geq 0, \quad \Omega = \mathbb{R}^3 \text{ or } \mathbb{T}^3.$$

- **H-theorem fails at the local Maxwellian**

$$\mathbf{M}_{[\rho(t,x), u(t,x), T(t,x)]}(\xi).$$

- **In $\mathbb{T}^3$ case, a key tool to overcome the degeneration is the Poincare inequality:**

$$\|\rho - \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} \rho \, dx\|_{L^2_x(\mathbb{T}^3)} \leq C \|\nabla \rho\|_{L^2_x(\mathbb{T}^3)}$$

- **In $\mathbb{R}^3$ case, the Poincare inequality fails.**
- **Idea: seek out the enough dissipative mechanisms for the components of the local Maxwellian**

Degeneration of the electromagnetic field

$$\frac{1}{c}\partial_t E - \nabla \times B = -\frac{4\pi}{c} J, \quad \frac{1}{c}\partial_t B + \nabla \times E = 0,$$
$$\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0.$$

- **For the Maxwell system in vacuum ($J = 0, \rho = 0$),**

  $$\partial_t E - \nabla \times B = 0, \quad \partial_t B + \nabla \times E = 0, \quad \nabla \cdot E = \nabla \cdot B = 0,$$

  the total energy is preserved at all time.

- **Can the coupling with the kinetic equation imply a kind of the dissipative mechanism?**

- **Idea: again, seek out the enough dissipative mechanisms for the electromagnetic field with the understanding of the structure of the system**

The Vlasov-Maxwell-Boltzmann/Landau system

$f_\pm = f_\pm(t, x, \xi) \geq 0$ **of two-species:**

$$\partial_t f_+ + \xi \cdot \nabla_x f_+ + (E + \xi \times B) \cdot \nabla_\xi f_+ = Q(f_+, f_+) + Q(f_+, f_-),$$
$$\partial_t f_- + \xi \cdot \nabla_x f_- - (E + \xi \times B) \cdot \nabla_\xi f_- = Q(f_-, f_+) + Q(f_-, f_-).$$

It is coupled with the Maxwell system

$$\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} \xi (f_+ - f_-) d\xi,$$
$$\partial_t B + \nabla_x \times E = 0,$$
$$\nabla_x \cdot E = \int_{\mathbb{R}^3} (f_+ - f_-) d\xi, \quad \nabla_x \cdot B = 0.$$

The initial data in this system is given as

$$f_\pm(0, x, \xi) = f_{0,\pm}(x, \xi), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x).$$

Previous results on VMB

Boltzmann collision term Q takes the hard sphere model:

$$B(\xi - \xi_*, \omega) = |(\xi - \xi_*) \cdot \omega|.$$

- $\Omega = \mathbb{T}^3$
  - Global existence: Guo (IM, '03) (Energy method)
  - Large-time behavior of solutions: Jang (ARMA, '09)
- $\Omega = \mathbb{R}^3$
  - Global existence: Strain (CMP, '06) (Use two-species' cancelation property to control E and pure time derivatives)
  - Large-time behavior of solutions: D.-Strain ('10) (Linearized analysis + bootstrap to the nonlinear equation)

! Unknown for non hard-sphere model !

Previous results on VML

For the Coulomb potential, the only existing results concern the case of the absence of the variable magnetic field, i.e.

Vlasov-Poisson-Landau instead of VML

- $\Omega = \mathbb{T}^3$: Guo (JAMS, '12)
- $\Omega = \mathbb{R}^3$:
  - D.-Yang-Zhao (arXiv '11): use a new exponential weight
  - Strain-Zhu (arXiv '12) and Yu (preprint '12): approach by Guo
  - Wang (arXiv '12): pure energy method without linearized analysis

! Unknown in the case of VML !

Our main results:

Global classical solutions near a global Maxwellian uniquely exist and time asymptotically tend to the Maxwellian with some rates for the cases of

- D.-Yang-Zhao ('11): Vlasov-Poisson-Boltzmann, angular cutoff with $-2 \leq \gamma \leq 1$
- D.-Liu ('11): Vlasov-Poisson-Boltzmann, angular non cutoff with $-3 < \gamma < -2s$ and $1/2 \leq s < 1$
- D.-Liu-Yang-Zhao ('12): Vlasov-Maxwell-Boltzmann, angular non cutoff with $\max\{-3, -\frac{3}{2} - 2s\} < \gamma < -2s$ and $1/2 \leq s < 1$
- D. (arXiv '12): Vlasov-Maxwell-Landau, soft potentials $-3 \leq \gamma < -2$ including the Coulomb $\gamma = -3$

Remarks

- Mathematically, when there is an external force, it is highly nontrivial to generalize existing results to the case of non hard-sphere model, which is also of physical importance!

- A key observation was used by Guo (JAMS, '11):

  - the potential force $E = -\nabla\phi$ with vanishing $B = 0$:

    $$(\xi \cdot \nabla_x u + \nabla_x \phi \cdot \xi u) e^{\phi} = \xi \cdot \nabla_x (e^{\phi} u).$$

    But, it is difficult to deal with the non-potential force !!!

  - the dissipation of the Landau operator is anisotropic containing the velocity differentiation:

    $$|u|^2_{\mathbf{D}} \sim |\langle\xi\rangle^{\gamma} P_\xi \nabla_\xi u|^2 + |\langle\xi\rangle^{\gamma+2}\{I - P_\xi\}\nabla_\xi u|^2 + |\langle\xi\rangle^{\gamma+2} u|^2.$$

    What happens to the cutoff Boltzmann operator for which no velocity diffusion is available?

- A completely different approach was developed by D.-Yang-Zhao (arXiv ’11)
  - A new dissipative mechanism due to the introduction of the time-velocity dependent weight

$$\boxed{\exp\{\lambda\langle\xi\rangle^q/(1+t)^\theta\}}$$

$$\Rightarrow \partial_t e^{\frac{\lambda\langle\xi\rangle^q}{(1+t)^\theta}} = -\lambda\theta\frac{\langle\xi\rangle^q}{(1+t)^{1+\theta}}e^{\frac{\lambda\langle\xi\rangle^q}{(1+t)^\theta}}.$$

  - The approach that we developed can apply to

    • Landau or Boltzmann
    • $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$
    • For the Boltzmann with most of values of γ: angular cutoff or non-cutoff
    • Maxwell system (non-potential force) can be included!

III. Proof

Linearized analysis

From now on, let us focus on the Cauchy problem on the VML system:

$$\partial_t f_+ + \xi \cdot \nabla_x f_+ + (E + \xi \times B) \cdot \nabla_\xi f_+ = Q(f_+, f_+) + Q(f_+, f_-),$$
$$\partial_t f_- + \xi \cdot \nabla_x f_- - (E + \xi \times B) \cdot \nabla_\xi f_- = Q(f_-, f_+) + Q(f_-, f_-).$$

It is coupled with the Maxwell system

$$\begin{aligned} &\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} \xi (f_+ - f_-) d\xi, \\ &\partial_t B + \nabla_x \times E = 0, \\ &\nabla_x \cdot E = \int_{\mathbb{R}^3} (f_+ - f_-) d\xi, \quad \nabla_x \cdot B = 0. \end{aligned}$$

The initial data in this system is given as

$$f_\pm(0, x, \xi) = f_{0,\pm}(x, \xi), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x).$$

Linearization (Carleman, Grad, ...)

- **Boltzmann's H-theorem implies:** $f_\pm \to \mathbf{M}$, $[E, B] \to 0$.
- **Define the perturbation** u **as** $u = \mathbf{M}^{-1/2}(f - \mathbf{M})$, $u = [u_+, u_-]$, $f = [f_+, f_-]$, $\mathbf{M} = \mathbf{M}_{[1,0,1]}(\xi)$.
- **The linearized system**

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u - E \cdot \xi \mathbf{M}^{1/2}[1, -1] = \mathbf{L}u + S, \\ \partial_t E - \nabla_x \times B = -\langle [\xi, -\xi]\mathbf{M}^{1/2}, \{\mathbf{I} - \mathbf{P}\}u \rangle, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = \langle \mathbf{M}^{1/2}, u_+ - u_- \rangle, \quad \nabla_x \cdot B = 0, \\ [u, E, B]|_{t=0} = [u_0, E_0, B_0], \end{cases}$$

$$\ker \mathbf{L} = \mathrm{span}\left\{[1,0]\mathbf{M}^{\frac{1}{2}}, [0,1]\mathbf{M}^{\frac{1}{2}}, [\xi_i, \xi_i]\mathbf{M}^{\frac{1}{2}} (1 \le i \le 3), [|\xi|^2, |\xi|^2]\mathbf{M}^{\frac{1}{2}}\right\}.$$

- **The local Maxwellian**

$$\mathbf{P}_\pm u = \{a_\pm(t,x) + b(t,x) \cdot \xi + c(t,x)(|\xi|^2 - 3)\}\mathbf{M}^{\frac{1}{2}}.$$

Dissipation from $\mathbf{L}$:

$$\int_{\mathbb{R}^3} u \cdot \mathbf{L}u \, d\xi \lesssim -\left\{ \int_{\mathbb{R}^3} \nu(\xi) |\{\mathbf{I} - \mathbf{P}\}u|^2 \, d\xi + \cdots \right\}$$

- **Collision frequency:** $\nu(\xi) = \langle \xi \rangle^{\gamma+2}$; $\mathbf{P}$: **projection from** L^2_ξ **to** $\ker \mathbf{L}$

- **A summary of possible difficulties:**
  - **The dissipation of $\mathbf{P}u$ is missing: The local Maxwellian is dispersive in the whole space due to the degeneration of $\mathbf{L}$! (Hypocoercivity: Villani)**
  - **The dissipation of (E, B) is missing: a feature of hyperbolic system with anti-symmetric relaxations! (Ueda-D.-Kawashima '11)**
  - **If $\gamma + 2 < 0$ then $\nu(\xi)$ is degenerate in large velocity! (Use velocity weight!)**

Dissipation of $\mathbf{P}u = \{a_\pm(t,x) + b(t,x)\cdot\xi + c(t,x)(|\xi|^2-3)\}\mathbf{M}^{\frac{1}{2}}$

! System of moment equations !

Observation (Grad, Kawashima, Liu-Yu, Guo, D.-Strain '10):

- **Find dissipation from the dynamics of the local Maxwellian?**

$$
\begin{aligned}
&\partial_t a_\pm + \nabla_x\cdot b + \nabla_x\cdot\langle \xi \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}u\rangle = 0,\\
&\partial_t[b_i + \langle \xi_i \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}u\rangle] + \partial_i(a_\pm + 2c) \mp E_i\\
&\qquad + \nabla_x\cdot\langle \xi\xi_i \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}u\rangle = 0,\\
&\partial_t\left[c + \frac{1}{6}\langle(|\xi|^2-3)\mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}u\rangle\right] + \frac{1}{3}\nabla_x\cdot b\\
&\qquad + \frac{1}{6}\nabla_x\cdot\langle(|\xi|^2-3)\xi \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}u\rangle = 0,
\end{aligned}
$$

These equations are NOT closed!

- **To close the system of the local Maxwellian, we also need to study the high-order moment equations:**

$$
\begin{aligned}
&\partial_t[\Theta_{ii}(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}u) + 2c] + 2\partial_i b_i = \Theta_{ii}(l_\pm),\\
&\partial_t\Theta_{ij}(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}u) + \partial_j b_i + \partial_i b_j + \nabla_x \cdot \langle \xi \mathbf{M}^{1/2}, \{\mathbf{I}_\pm - \mathbf{P}_\pm\}u\rangle \\
&\qquad\qquad\qquad\qquad = \Theta_{ij}(l_\pm), \quad i \neq j,\\
&\partial_t\Lambda_i(\{\mathbf{I}_\pm - \mathbf{P}_\pm\}u) + \partial_i c = \Lambda_i(l_\pm).
\end{aligned}
$$

Here, the high-order moment functions are defined by

$$
\Theta_{ij}(u_\pm) = \langle(\xi_i\xi_j - 1)\mathbf{M}^{1/2}, u_\pm\rangle, \Lambda_i(u_\pm) = \frac{1}{10}\langle(|\xi|^2 - 5)\xi_i\mathbf{M}^{1/2}, u_\pm\rangle,
$$

and $l_\pm$ are defined in terms of $\{\mathbf{I} - \mathbf{P}\}u$.

- **Coupling with the electromagnetic field:**

$$\begin{aligned}
&\partial_t(a_+ - a_-) + \nabla_x \cdot H = 0,\\
&\partial_t H + \nabla_x(a_+ - a_-) - 2E + \nabla_x \cdot \Theta(\{I-P\}u \cdot q_1)\\
&\qquad\qquad\qquad\qquad = \langle [\xi, -\xi]\mu^{1/2}, L\{I-P\}u\rangle,\\
&\partial_t E - \nabla_x \times B = -H, \quad \nabla_x \cdot E = a_+ - a_-,\\
&\partial_t B + \nabla_x \times E = 0,
\end{aligned}$$

with

$$H = \langle [\xi, -\xi]\mu^{1/2}, \{I-P\}u\rangle = \langle \xi\mu^{1/2}, \{I-P\}u \cdot q_1\rangle.$$

- For $\gamma + 2 \geq 0$ **(D.-Strain '10), there is a time-frequency functional** $\mathcal{E}(t,k)$ **such that**

$$\boxed{\mathcal{E}(t,k) \sim \|\hat{u}\|_{L^2_\xi}^2 + |\hat{E}|^2 + |\hat{B}|^2,}$$

and

$$\boxed{\partial_t \mathcal{E}(t,k) + \frac{\lambda |k|^2}{(1+|k|^2)^2}\mathcal{E}(t,k) \leq 0, \quad \forall\, t \geq 0,\ k \in \mathbb{R}^3.}$$

Remark:

- **The inequality seems terrible to prove decay rates because the "dissipative term" goes to zero as** $|k| \to \infty$**.**
- **It is an essential "regularity-loss" feature for the VMB system, not a deficiency of our approach; see D. (**Eigenvalue analysis of damped Euler-Maxwell**, '11), Hosono-Kawashima (M3AS '06), Ueda-D.-Kawashima ('11):**

$$\lambda(ik) \sim -\frac{1}{|k|^2} \pm i|k| \quad (|k| \to \infty).$$

Case when $\gamma+2<0$: D. (arXiv '12)

The situation becomes more subtle for $\gamma+2<0$; see Strain ('11) and D.-Yang-Zhao ('11)

- **Let $w=w(\xi)=\langle\xi\rangle^{\frac{\gamma+2}{2}}$ for Landau.**
- **Derive**

$$\partial_t M_\ell(t,k)+\kappa D_\ell(t,k)\leq 0,$$

with

$$\begin{aligned} M_\ell(t,k) &= \|\hat{u}\|_{L^2}^2+|[\hat{E},\hat{B}]|^2+\kappa_0\,\Re\,\mathcal{E}^{\rm int}(t,k)\\ &\quad +\kappa_2\left|w^\ell\{\mathbf{I}-\mathbf{P}\}\hat{u}\right|_{L^2}\chi_{|k|\leq 1}+\frac{\kappa_1}{1+|k|^2}\left|w^\ell\hat{u}\right|_{L^2}^2\chi_{|k|\geq 1},\end{aligned}$$

$$\begin{aligned} D_\ell(t,k) &= |\{\mathbf{I}-\mathbf{P}\}\hat{u}|_{\mathbf{D}}^2+\frac{1}{1+|k|^2}\left|w^\ell\{\mathbf{I}-\mathbf{P}\}\hat{u}\right|_{\mathbf{D}}^2\\ &\quad +\frac{|k|^2}{1+|k|^2}(|\widehat{a_+ + a_-}|^2+|\hat{b}|^2+|c|^2)+|\widehat{a_+ - a_-}|^2\\ &\quad +\frac{1}{1+|k|^2}|\hat{E}|^2+\frac{|k|^2}{(1+|k|^2)^2}|\hat{B}|^2.\end{aligned}$$

Set

$$\rho(k) = \frac{|k|^2}{(1+|k|^2)^2}.$$

It is NOT true to have as in $\gamma + 2 \geq 0$

$$D_\ell(t,k) \geq \kappa\rho(k)M_\ell(t,k).$$

However, through the analysis of ODE

$$\boxed{\partial_t M_\ell(t,k) + \kappa D_\ell(t,k) \leq 0,}$$

one can make the time weighted estimate

$$\boxed{[1+\epsilon\rho(k)t]^J M_\ell(t,k) \lesssim M_{\ell+J+p-1}(0,k).}$$

holds true for any $t \geq 0$, $k \in \mathbb{R}^3$, where the parameters p, ϵ and J with

$$p > 1,\ \ 0 < \epsilon \leq 1,\ \ J > 0,\ \ C_1\epsilon J \leq \frac{\kappa}{4}.$$

Theorem (D., arXiv '12)

Let $[u, E, B]$ be the solution to the Cauchy problem of the linearized homogeneous VML system for $-3 \leq \gamma < -2$. Define the velocity weight function $w = w(\xi)$ by

$$w(\xi) = \langle \xi \rangle^{-\frac{\gamma+2}{2}}$$

Then, for $\ell \geq 0$ and $\alpha \geq 0$ with $m = |\alpha|$,

$$\begin{aligned} &\|w^\ell \partial^\alpha u\| + \|\partial^\alpha (E, B)\| \\ &\lesssim (1+t)^{-\sigma_m} (\|w^{\ell+\ell_*^{low}} u_0\|_{Z_1} + \|(E_0, B_0)\|_{L^1_x}) \\ &\qquad + (1+t)^{-j} (\|w^{\ell+\ell_*^{high}} \nabla_x^{j+1} \partial^\alpha u_0\| + \|\nabla_x^{j+1} \partial^\alpha (E_0, B_0)\|), \end{aligned}$$

where

$$\sigma_m = \frac{3}{4} + \frac{m}{2}, \;\; \ell_*^{low} > 2\sigma_m, \;\; \ell_*^{high} > 0, \;\; 0 \leq j < \ell_*^{high}.$$

IV. Proof continued

Nonlinear energy estimates – Difficulties and ideas

Ideas in the proof:

- **Find an energy functional $\mathcal{E}(t)$ and its time-weighted norm $X(t)$ such that**

$$\boxed{X(t) \lesssim Y_0 + [X(t)]^2.}$$

- **To control the term $E \cdot \nabla_\xi u$ and $E \cdot \xi u$, the time-decay of E is needed. Thus, Y_0 generally includes L^1-norm of initial data. Note that Y_0 needs to be small enough to ensure the global-in-time bound by the continuity argument.**
- **To balance an estimate on both $E \cdot \nabla_\xi u$ and $\xi \cdot \nabla_x u$, γ can NOT be too small in the cutoff case (VMB). However, in the non cutoff case, since**

$$\int u\mathbf{L}u\, d\xi \lesssim -\int \langle \xi \rangle^{\gamma+2s} |\{\mathbf{I}-\mathbf{P}\}u|^2\, d\xi - \{\cdots\},$$

we may require that $\gamma + 2s$ need not be too small.

Ideas in the proof (cont.):

- **To deal with the degeneration of $\nu(\xi)$ for soft potentials, choose $\mathcal{E}(t)$ in the way that**
  - **higher the differentiation order is, the order of velocity weights is lower, for instance, consider ($\partial_\alpha^\beta = \partial_x^\beta \partial_\xi^\alpha$, $|\alpha| + |\beta| = N$)**

$$\iint \partial_\alpha^\beta Q(u,u) \cdot w_{\alpha,\beta,\ell}^2(t,\xi) \partial_\alpha^\beta u \, dx d\xi.$$

- **To deal with the degeneration of the Maxwell equations, choose $X(t)$ in the way that**
  - **higher the order of $\mathcal{E}_N(t)$ is, the rate of its time weights is lower;**
  - **the highest-order energy norm $\mathcal{E}_N(t)$ may increase in time! For instance, consider**

$$\iint \partial_\alpha^\beta [(B \times \xi) \cdot \nabla_\xi u] \cdot w_{\alpha,\beta,\ell}^2(t,\xi) \partial_\alpha^\beta u \, dx d\xi.$$

Idea in the proofs (cont.):

- **A trouble occurs to the estimate on**

$$\iint \partial_\beta^\alpha (E\cdot \xi \mathbf{M}^{1/2})\cdot w^2_{\alpha,\beta,\ell}(t,\xi)\partial_\beta^\alpha u\,dxd\xi \quad (|\alpha|+|\beta|=N)$$

No dissipation for $\int |\nabla_x^N E|^2$!

Use an idea from Hosono-Kawashima (M3AS '06) to deduce

$$\frac{d}{dt}[(1+t)^{-\epsilon_0}\mathcal{E}_{N_1}(t)] + \kappa(1+t)^{-\epsilon_0}\mathcal{D}_{N_1}(t) + \frac{\epsilon_0}{(1+t)^{1+\epsilon_0}}\mathcal{E}_{N_1}(t) \lesssim (1+t)^{-\epsilon_0}\times\{h.o.t.\}.$$

from the basic energy inequality without any weight

$$\frac{d}{dt}\mathcal{E}_{N_1}(t)+\kappa\mathcal{D}_{N_1}(t)\lesssim h.o.t..$$

Ideas in the proof (cont.):

- **VMB without angular cutoff**
  - **Exponential weighted estimates are necessary due to the general Lorentz forcing, compared to the potential force in the VPB case**
  - **Taylor expansion of the weight $w_{\ell,\lambda}(\xi)$ takes the form**

$$w_{\ell,\lambda}(\xi') - w_{\ell,\lambda}(\xi) = (\xi' - \xi) \cdot \nabla_\xi w_{\ell,\lambda}(\xi) + \frac{1}{2!}(\xi' - \xi) \otimes (\xi' - \xi) : \nabla_\xi^2 w_{\ell,\lambda}(t,\xi) + \cdots .$$

The velocity growth can be compensated by the time decay factor if $w_{\ell,\lambda}$ contains a factor $\exp\{\langle\xi\rangle/(1+t)^\theta\}$

V. Proof continued

Nonlinear energy estimates – Details for the VML

Norms:

Fix ϑ **with** $0 < \vartheta \leq 1/4$. **Define**

$$w_{\tau,\lambda} = w_{\tau,\lambda}(t,\xi) = \langle\xi\rangle^{(\gamma+2)\tau} \exp\left\{\frac{\lambda}{(1+t)^{\vartheta}}\langle\xi\rangle^2\right\},$$

where $\tau \in \mathbb{R}$ **and** $\lambda \geq 0$. **For** $f = f(t,x,\xi)$, **define**

$$|f(x)|^2_{\tau,\lambda} = \int_{\mathbb{R}^3} w^2_{\tau,\lambda}(t,\xi)|f|^2\,d\xi, \quad \|f\|^2_{\tau,\lambda} = \int_{\mathbb{R}^3} |f(x)|^2_{\tau,\lambda}\,dx,$$

and

$$|f(x)|^2_{\mathbf{D},\tau,\lambda} = \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} w^2_{\tau,\lambda}(t,\xi)\left\{\sigma^{ij}\partial_i f\partial_j f + \sigma^{ij}\frac{\xi_i}{2}\frac{\xi_j}{2}|f|^2\right\}d\xi,$$

$$\|f\|^2_{\mathbf{D},\tau,\lambda} = \int_{\mathbb{R}^3} |f(x)|^2_{\mathbf{D},\tau,\lambda}\,dx,$$

where $\partial_i = \partial_{\xi_i}$ **denotes the velocity derivative with respect to** ξ_i, **and** $\sigma^{ij} = \sigma^{ij}(\xi)$ **is the Landau collision frequency given by**

$$\sigma^{ij}(\xi) = \phi^{ij} * \mu(\xi) = \int_{\mathbb{R}^3} \phi^{ij}(\xi-\xi_*)\mu(\xi_*)\,d\xi_*.$$

Energy functional and its dissipation rate:

$$\mathcal{E}_{N,\ell,\lambda}(t) \sim \sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_\beta f(t)\|^2_{|\beta|-\ell,\lambda} + \|(E,B)\|^2_{H^N},$$

and

$$\begin{aligned}\mathcal{D}_{N,\ell,\lambda}(t) = & \sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_\beta \{I-P\} f(t)\|^2_{\mathbf{D},|\beta|-\ell,\lambda} \\ & + \sum_{|\alpha|\leq N-1} \|\nabla_x \partial^\alpha (a_\pm, b, c)\|^2 \\ & + \|a_+ - a_-\|^2 + \|E\|^2_{H^{N-1}} + \|\nabla_x B\|^2_{H^{N-2}} \\ & + \frac{\lambda}{(1+t)^{1+\vartheta}} \sum_{|\alpha|+|\beta|\leq N} \|\langle \xi \rangle \partial^\alpha_\beta \{I-P\} f(t)\|^2_{|\beta|-\ell,\lambda},\end{aligned}$$

$X(t)$-norm:

Let N_0 and ℓ_0 be fixed properly large with $\ell_0 - 3N_0$ being also properly large., and let $\lambda_0 > 0$ and $\epsilon_0 > 0$ be fixed properly small; their choice can be seen in the proof. Set

$$N_1 = \frac{3}{2}N_0, \quad \ell_1 = \frac{1}{2}\ell_0.$$

Define

$$\begin{aligned}
X(t) = & \sup_{0\leq s\leq t} \{\mathcal{E}_{N_1}(s) + (1+s)^{\frac{3}{2}}\mathcal{E}_{N_1-2}(s)\} \\
& + \sup_{0\leq s\leq t} \{(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(s) + \mathcal{E}_{N_1-1,\ell_1,\lambda_0}(s) \\
& \qquad + (1+s)^{\frac{3}{2}}\mathcal{E}_{N_1-3,\ell_1-1,\lambda_0}(s)\} \\
& + \sup_{0\leq s\leq t} \{\mathcal{E}_{N_0,\ell_0,\lambda_0}(s) + (1+s)^{\frac{3}{2}}\mathcal{E}_{N_0,\ell_0-1,\lambda_0}(s)\} \\
& + \sup_{0\leq s\leq t} \{(1+s)^{2(1+\vartheta)}\|\nabla_x(E,B)(s)\|^2_{H^{N_0-1}}\}.
\end{aligned}$$

Theorem (D., arXiv '12)

Assume $-3 \leq \gamma < -2$. *Take* $0 < \vartheta \leq 1/4$, *and also take* N_0, ℓ_0 *properly large with* $\ell_0 - 3N_0$ *properly large, and take* $\lambda_0 > 0$, $\epsilon_0 > 0$ *properly small. Fix* $\ell_2 > \frac{5}{4} + \frac{N_0}{2}$. *Let* $f_0 = [f_{0,+}, f_{0,-}]$ *satisfy* $F_\pm(0, x, \xi) = \mu(\xi) + \mu^{1/2}(\xi) f_{0,\pm}(x, \xi) \geq 0$. *If*

$$Y_0 = \sum_{|\alpha|+|\beta|\leq N_0} \|\partial^\alpha_\beta f_0\|_{|\beta|-\ell_0,\lambda_0} + \sum_{|\alpha|+|\beta|\leq N_1} \|\partial^\alpha_\beta f_0\|_{|\beta|-\ell_1,\lambda_0} + \|(E_0, B_0)\|_{H^{N_1}\cap Z_1} + \|w_{-\ell_2} f_0\|_{Z_1}$$

is sufficiently small, then there is $X(t)$ *such that the Cauchy problem of the Vlasov-Maxwell-Landau system admits a unique global solution* $\big(f(t, x, \xi), E(t, x), B(t, x)\big)$ *satisfying* $F_\pm(t, x, \xi) = \mu(\xi) + \mu^{1/2}(\xi) f_\pm(t, x, \xi) \geq 0$ *and*

$$X(t) \lesssim Y_0^2,$$

for all time $t \geq 0$.

Step 0. A priori assumption

$$
\begin{aligned}
X(t) = &\sup_{0\le s\le t}\{\mathcal{E}_{N_1}(s) + (1+s)^{\frac{3}{2}}\mathcal{E}_{N_1-2}(s)\} \\
&+ \sup_{0\le s\le t}\{(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(s) + \mathcal{E}_{N_1-1,\ell_1,\lambda_0}(s) \\
&\qquad\qquad + (1+s)^{\frac{3}{2}}\mathcal{E}_{N_1-3,\ell_1-1,\lambda_0}(s)\} \\
&+ \sup_{0\le s\le t}\{\mathcal{E}_{N_0,\ell_0,\lambda_0}(s) + (1+s)^{\frac{3}{2}}\mathcal{E}_{N_0,\ell_0-1,\lambda_0}(s)\} \\
&+ \sup_{0\le s\le t}\{(1+s)^{2(1+\vartheta)}\|\nabla_x(E,B)(s)\|^2_{H^{N_0-1}}\} \le \delta
\end{aligned}
$$

for $\delta > 0$ **small enough.**

Step 1. Macro dissipation

Define

$$\mathcal{D}_{N,mac}(t) = \sum_{|\alpha|\le N-1} \|\nabla_x \partial^\alpha (a_\pm, b, c)\|^2 + \|a_+ - a_-\|^2 + \|E\|^2_{H^{N-1}} + \|\nabla_x B\|^2_{H^{N-2}}.$$

Lemma

For any integer N with $8 \le N \le N_1$, there is an interactive energy functional $\mathcal{E}_N^{int}(t)$ such that

$$|\mathcal{E}_N^{int}(t)| \lesssim \sum_{|\alpha|\le N} (\|\partial^\alpha f\|^2 + \|\partial^\alpha (E,B)\|^2)$$

and

$$\frac{d}{dt}\mathcal{E}_N^{int}(t) + \kappa \mathcal{D}_{N,mac}(t) \lesssim \sum_{|\alpha|\le N} \|\partial^\alpha \{I-P\} f\|^2_{\mathbf{D}} + \mathcal{E}_N(t)\mathcal{D}_N(t),$$

for $0 \le t \le T$.

Step 2. Lyapunov inequality for $\mathcal{E}_{N_1}(t)$

Lemma

There is an energy functional $\mathcal{E}_{N_1}(t)$ *such that*

$$\frac{d}{dt}\mathcal{E}_{N_1}(t)+\kappa\mathcal{D}_{N_1}(t) \lesssim \frac{\delta}{(1+t)^{1+\vartheta}}\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)+\mathcal{E}_{N_1}(t)\mathcal{E}_{N_0,\ell_0-1,\lambda_0}(t),$$

for $0 \leq t \leq T$.

Step 3. Lyapunov inequality for $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$

Lemma

There is an energy functional $\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)$ *such that*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{N_1,\ell_1,\lambda_0}(t)+\kappa\mathcal{D}_{N_1,\ell_1,\lambda_0}(t)&\lesssim \mathcal{E}_{N_1,\ell_1,\lambda_0}(t)\mathcal{E}_{N_0,\ell_0-1,\lambda_0}(t)\\&\quad+\sum_{|\alpha|=N_1}\langle\partial^\alpha E\cdot\xi\mu^{1/2},w^2_{-\ell_1,\lambda_0}\partial^\alpha f\rangle,\end{aligned}$$

for $0\le t\le T$.

Using the time-weighted method, one can obtain closed estimate on the first portion of the time-weighted energy norm $X(t)$.

Step 2 + Step 3 $\Rightarrow$

Lemma

It holds that

$$\sup_{0\leq s\leq t}\left\{\mathcal{E}_{N_1}(s)+(1+s)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{N_1,\ell_1,\lambda_0}(s)\right\} + \int_0^t \mathcal{D}_{N_1}(s)\,ds \lesssim Y_0^2 + X^2(t),$$

for $0\leq t\leq T$.

Step 4. Lyapunov inequality for $\mathcal{E}_{N_0,\ell_0,\lambda_0}(t)$

Lemma

For any ℓ with $\ell_0 - 1 \leq \ell \leq \ell_0$, there is an energy functional $\mathcal{E}_{N_0,\ell,\lambda_0}(t)$ such that

$$\frac{d}{dt}\mathcal{E}_{N_0,\ell,\lambda_0}(t) + \mathcal{D}_{N_0,\ell,\lambda_0}(t) \lesssim \sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2,$$

for $0 \leq t \leq T$.

Step 5. Decay of electromagnetic fields and macro components

Lemma

It holds that

$$\sup_{0\leq s\leq t}\{(1+s)^{\frac{5}{2}}\|\nabla_x(E,B)\|^2_{H^{N_0-1}} + (1+s)^{\frac{3}{2}}\|(a,b,c,E,B)\|^2\} \lesssim Y_0^2 + X^2(t),$$

for $0\leq t\leq T$.

Step 6. Bound of $\mathcal{E}_{N_0,\ell_0,\lambda_0}(t)$ and decay of $\mathcal{E}_{N_0,\ell_0-1,\lambda_0}(t)$

Lemma

It holds that

$$\sup_{0\leq s\leq t}\{\mathcal{E}_{N_0,\ell_0,\lambda_0}(s)+(1+s)^{\frac{3}{2}}\mathcal{E}_{N_0,\ell_0-1,\lambda_0}(s)\}\lesssim Y_0^2+X^2(t),$$

for $0\leq t\leq T$.

Step 7. Bound of $\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(t)$

Lemma

It holds that

$$\sup_{0\le s\le t}\mathcal{E}_{N_1-1,\ell_1,\lambda_0}(s)+\int_0^t\mathcal{D}_{N_1-1,\ell_1,\lambda_0}(s)\,ds\lesssim Y_0^2+X^2(t),$$

for $0\le t\le T$.

Step 8. Decay of $\mathcal{E}_{N_1-3,\ell_1-1,\lambda_0}(t)$ and $\mathcal{E}_{N_1-2}(t)$

Lemma

It holds that

$$\sup_{0\le s\le t}\{(1+s)^{\frac{3}{2}}[\mathcal{E}_{N_1-3,\ell_1-1,\lambda_0}(s)+\mathcal{E}_{N_1-2}(s)]\}\lesssim Y_0^2+X^2(t),$$

for $0\le t\le T$.

Step 9.

From previous lemmas,

$$X(t) \lesssim Y_0^2 + X^2(t).$$

Since Y_0 is sufficiently small, the global existence follows.

Problems for the future:

- **Bounded domain**
- **Eigenvalue analysis and the spectrum**

Thank you !