Analysis of Oscillations and Defect Measures in Plasma Physics

Donatella Donatelli
(joint work with P. Marcati)

Dipartimento di Matematica Pura ed Applicata
Università degli Studi dell'Aquila
67100 L'Aquila, Italy
donatell@univaq.it

Simplified Model for Plasma

Compressible Navier Stokes Poisson System

$$\partial_t \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0$$

$$\partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \nabla(\rho^\lambda)^\gamma = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda$$

$$\lambda^2 \Delta V^\lambda = \rho^\lambda - 1, \qquad x \in \mathbb{R}^3, t \geq 0$$

$\rho^\lambda(x,t)$ is the *negative charge density*

$m^\lambda(x,t) = \rho^\lambda(x,t)u^\lambda(x,t)$ is the *current density*

$u^\lambda(x,t)$ is the *velocity vector density*

$V^\lambda(x,t)$ is the *electrostatic potential*

$\overline{\mu}$ is the *shear viscosity* and $\overline{\nu}$ is the *bulk viscosity*

$\lambda = \lambda_D / L$, λ_D is the *Debye length*

Quasineutral limit

$$\partial_t \rho^\lambda + \mathrm{div}(\rho^\lambda u^\lambda) = 0$$

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Study the limit $\lambda \to 0$

$$\Downarrow$$

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$\Downarrow$

Formally yields to an
INCOMPRESSIBLE DYNAMICS

$$\rho = 1$$

$$\mathrm{div}\, u = 0$$

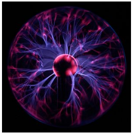
Main Issues

This formal limit will not be in general true

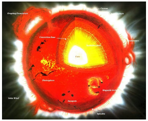
- Control of Acoustic waves
- Control of Space localized, high frequency in time Wave Packets

What is a plasma ?

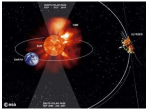
A Plasma is a fluid which contains ions and electrons, such that charge neutrality is mantained



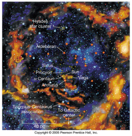
A gas heated up to sufficiently high temperatures so that the atoms ionise



Sun's core. The plasma at the center of the sun, where fusion of hydrogen to form helium generates the suns heat.



Solar wind. The wind of plasma that blows off the sun and outward through the region between the planets.



Interstellar medium. The plasma, in our Galaxy, that fills the region between the stars.

a charged particle inside a plasma attracts particles with opposite charge and repels those with the same charge

$\Downarrow$

creation of a net cloud of opposite charge around itself

$\Downarrow$

the cloud shields the particle's own charge from external view

HOW LARGE IS THIS CLOUD?

$V = V(r)$, n_0=mean density of electrons and protons

n_e=electron density=$n_0 e^{eV/kT}$ n_i=ion density= $n_0 e^{-eV/kT}$

$$\Delta V = -\frac{1}{\epsilon_0}(n_i - n_e) = -\frac{n_0 e}{\epsilon_0}(e^{-eV/kT} - e^{eV/kT})$$

potential energy $eV \ll$ kinetic energy kT

$$\Delta V = -\frac{n_0 e}{\epsilon_0}\left(1 - \frac{eV}{kT} - 1 - \frac{eV}{kT}\right) = \left(\frac{2n_0 e^2}{\epsilon_0 kT}\right) V$$

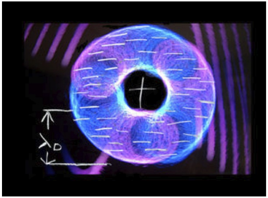
$$\lambda_D = \sqrt{\frac{\epsilon_0 kT}{2n_0 e^2}} = \text{Debye lenght}$$

$$\Delta V - \frac{1}{\lambda_D^2} V = 0 \qquad \text{Debye law}$$

$$\lambda_D = \sqrt{\frac{\epsilon_0 kT}{2n_0 e^2}} \qquad V(r) = \frac{q}{r} e^{-r/\lambda_D}$$

- ⇒ the electric field dies off on distance greater than λ_D
- ⇒ this is the screening effect due to the polarization cloud which screens the field of charge for distances larger than λ_D
- ⇒ charge fluctuation may occur over distances smaller than λ_D
- ⇒ the plasma is quasineutral for a distance $L >> \lambda_D$ (we can define as a parameter the *"plasma density"*)

Plasma	$T(K)$	$\lambda_D(m)$
Gas discharge	10^4	10^{-4}
Sun's core	10^7	10^{-11}
Solar wind	10^5	10
Interstellar medium	10^4	10



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Plasma Oscillation

$1 - D$ linearized system

$$\sigma_t + u_x = 0$$
$$u_t + c^2\sigma_x = V_x$$
$$\lambda^2 V_{xx} = \sigma$$

Fourier Transform

$$\begin{pmatrix} \hat{\sigma}_t \\ \hat{u}_t \end{pmatrix} + \begin{pmatrix} 0 & i\xi \\ i\xi c^2 + \dfrac{i}{\lambda^2\xi} & 0 \end{pmatrix} \begin{pmatrix} \hat{\sigma} \\ \hat{u} \end{pmatrix} = 0$$

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Solutions

$$\begin{pmatrix} \hat{\sigma}(\xi,t) \\ \hat{u}(\xi,t) \end{pmatrix} = \begin{pmatrix} \hat{\sigma}_+(\xi) \\ \hat{u}_+(\xi) \end{pmatrix} e^{i\theta(\xi)t} + \begin{pmatrix} \hat{\sigma}_-(\xi) \\ \hat{u}_-(\xi) \end{pmatrix} e^{-i\theta(\xi)t}$$

$$\theta(\xi) = \left(c^2\xi^2 + \frac{1}{\lambda^2} \right)^{1/2}$$

Mathematical difficulties - 1

$$u = \underbrace{\mathbf{P}[u]}_{solenoidal\ part} + \underbrace{\mathbf{Q}[u]}_{gradient\ part}$$

$$\operatorname{div} \mathbf{P}[u] = 0 \qquad \mathbf{Q}[u] = \nabla\Psi$$

$$\partial_t \sigma + \operatorname{div} u = 0$$
$$\partial_t u + \nabla\sigma = \nabla V$$
$$\lambda^2 \Delta V = \sigma$$

$$\partial_t \sigma + \Delta\Psi = 0$$
$$\partial_t \nabla\Psi + \nabla\sigma = \frac{1}{\lambda^2}\nabla\Delta^{-1}\sigma$$

Problem: "weak compactness"

Mathematical difficulties - 1

$$u = \underbrace{\mathbf{P}[u]}_{solenoidal\ part} + \underbrace{\mathbf{Q}[u]}_{gradient\ part}$$

$$\operatorname{div} \mathbf{P}[u] = 0 \qquad \mathbf{Q}[u] = \nabla\Psi$$

$$\partial_t\sigma + \operatorname{div} u = 0 \qquad\qquad \partial_t\sigma + \Delta\Psi = 0$$
$$\partial_t u + \nabla\sigma = \nabla V \qquad\qquad \partial_t\nabla\Psi + \nabla\sigma = \frac{1}{\lambda^2}\nabla\Delta^{-1}\sigma$$
$$\lambda^2\Delta V = \sigma$$

Problem: "weak compactness"

$$\begin{aligned}\operatorname{div}(\rho u \otimes u) \approx \operatorname{div}(u \otimes u) &= \operatorname{div}(u \otimes \mathbf{P}[u]) + \operatorname{div}(\mathbf{P}[u] \otimes \nabla\Psi) \\ &+ \frac{1}{2}\nabla|\nabla\Psi|^2 + \Delta\Psi\nabla\Psi\end{aligned}$$

$$\|$$

$$\partial_t(\sigma\nabla\Psi) - \frac{1}{2}\nabla\sigma^2 + \frac{1}{\lambda^2}\sigma\nabla\Delta^{-1}\sigma$$

Mathematical difficulties -2

$$\partial_t \rho^\lambda + \mathrm{div}(\rho^\lambda u^\lambda) = 0$$

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$$\lambda^2 \Delta V^\lambda = \rho^\lambda - 1$$

$L_t^\infty L_x^2$ bound on $\lambda \nabla V^\lambda = \lambda E^\lambda$ $\quad \rho^\lambda \nabla V^\lambda \sim \lambda E^\lambda \otimes \lambda E^\lambda$

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$L^\infty_t L^2_x$ bound on $\lambda \nabla V^\lambda = \lambda E^\lambda$ $\quad \rho^\lambda \nabla V^\lambda \sim \lambda E^\lambda \otimes \lambda E^\lambda$

simplified example: space independent case

$$\lambda^2 \partial_{tt} E^\lambda + E^\lambda = F$$

Fourier transform in time

$$\lambda \hat{E}^\lambda(\tau) = \lambda \frac{1}{1-\lambda^2\tau^2}\hat{F}(\tau)$$

the L^2 mass of $\lambda \hat{E}^\lambda$ concentrates around $\tau = \frac{1}{\lambda}$ as $\lambda \to 0$

$\Longrightarrow$ corrector analysis

State of Art - References

- Quasineutral limit for Euler Poisson system in 1D
  - weak solutions, Gasser and Marcati ('01)('03)
- Quasineutral limit for Euler Poisson system in H^s
  - Cordier and Grenier ('00), Grenier ('96), Cordier, Degond, Markowich and Schmeiser ('96), Loeper ('05), Peng, Wang and Yong ('06)
- Combined quasineutral limit and inviscid limit in $\mathbb{T}^d$
  - smooth solutions, well- prepared initial data ,Wang ('04)
  - weak solutions, general initial data, Jiang and Wang ('06)
- Quasineutral limit for Navier Stokes Poisson system
  - regular solutions, ill-prepared data, Ju,Li and Li ('09)
  - weak solutions, well-prepared data, Ju, Li and Wang ('08)
  - weak solutions, D. Donatelli P.Marcati, A quasineutral type limit for the Navier Stokes Poisson system with large data, *Nonlinearity, 21, (2008), 135-148.*
- Singular Limits
  - E. Feireisl, A. Novotny, *Singular Limits in Thermodynamics of Viscous Fluids*, Birkhäuser Verlag, 2009.
  - L. Saint-Raymond, *Hydrodynamic Limits of the Boltzmann Equation*, Springer, 2009.

Existence for Navier Stokes Poisson

$$\partial_t \rho^\lambda + \mathrm{div}(\rho^\lambda u^\lambda) = 0$$

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$$\lambda^2 \Delta V^\lambda = \rho^\lambda - 1$$

Renormalized pressure:

$$\pi^\lambda = \frac{(\rho^\lambda)^\gamma - 1 - \gamma(\rho^\lambda - 1)}{\gamma(\gamma - 1)}$$

Total Energy:

$$E(t) = \int_{\mathbb{R}^3} \left(\rho^\lambda \frac{|u^\lambda|^2}{2} + \pi^\lambda + \frac{\lambda^2}{2} |\nabla V^\lambda|^2 \right) dx$$

Initial conditions:

$$\int_{\mathbb{R}^3} \left(\pi^\lambda|_{t=0} + \frac{|m_0^\varepsilon|^2}{2\rho_0^\lambda} + \lambda^2 |V_0^\lambda|^2 \right) dx \le C_0, \quad \text{where} \quad \rho^\lambda u^\lambda|_{t=0} = m_0^\lambda.$$

Existence of global weak solution

(B. Ducomet, E. Feireisl, H. Petzeltová, and I. Straškraba, 2004)

$$E(t) + \int_0^t \int_{\mathbb{R}^3} \left(\mu |\nabla u^\lambda|^2 + (\nu + \mu) |\,\mathrm{div}\, u^\lambda|^2 \right) dx ds \le E(0).$$

Strategy

- Uniform bounds
- Strong convergence for $\mathbf{P}u$
- Recover Acoustic equation and control oscillations
  - Strichartz estimates
- Strong convergence for $\mathbf{Q}u$
- Compactness for $\lambda\nabla V^{\lambda}$
  - introduction of correctors
  - construction of microlocal defect measure

Uniform bounds

$$\int_{\mathbb{R}^3}\left(\rho^\lambda\frac{|u^\lambda|^2}{2}+\frac{(\rho^\lambda)^\gamma-1-\gamma(\rho^\lambda-1)}{\gamma(\gamma-1)}+\lambda^2|\nabla V^\lambda|^2\right)dx$$
$$+\int_0^t\int_{\mathbb{R}^3}\left(\mu|\nabla u^\lambda|^2+(\nu+\mu)|\operatorname{div}u^\lambda|^2\right)dxds\le C_0.$$

the convexity of $z\to z^\gamma-1-\gamma(z-1)$

$\Downarrow$

density fluctuation $=\sigma^\lambda=\rho^\lambda-1\in L_t^\infty L_2^k,\ k=\min\{2,\gamma\}$

$\Longrightarrow$

∇u^λ is bounded in $L^2_{t,x}$, $\lambda\nabla V^\lambda$ is bounded in $L_t^\infty L_x^2$,

u^λ is bounded in $L^2_{t,x}\cap L_t^2L_x^6$ $\sigma^\lambda u^\lambda$ is bounded in $L_t^2H_x^{-1}$

Leray Projectors

$$u = \underbrace{\mathbf{P}[u]}_{soleinodal\ \ part} + \underbrace{\mathbf{Q}[u]}_{gradient\ \ part}$$

$$\operatorname{div} \mathbf{P}[u] = 0 \qquad \mathbf{Q}[u] = \nabla \Psi$$

where

$$\mathbf{P}[u] = \mathbf{I} - \mathbf{Q}[u] \qquad \mathbf{Q}[u] = \nabla \Delta^{-1} \operatorname{div}[u]$$

Convergence of $\mathrm{P}u^\lambda$

- convolution techniques
- L^p compactness

$$\|\mathbf{P}u^\lambda(t+h) - \mathbf{P}u^\lambda(t)\|_{L^2([0,T]\times\mathbb{R}^3)} \le C_T h^{1/5}$$

$$\Downarrow$$

$$\mathbf{P}u^\lambda \longrightarrow \mathbf{P}u, \qquad \text{strongly in } L^2(0,T;L^2_{loc}(\mathbb{R}^3))$$

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Acoustic equation

$$\sigma^\lambda = \rho^\lambda - 1 = density\ fluctuation$$

$$\begin{aligned}
\partial_t \sigma^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) &= 0 \\
\partial_t(\rho^\lambda u^\lambda) + \nabla \sigma^\lambda &= \mu \Delta u^\lambda + (\nu + \mu)\nabla \operatorname{div} u^\lambda - \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) \\
&\quad - (\gamma - 1)\nabla \pi^\lambda + \sigma^\lambda \nabla V^\lambda + \nabla V^\lambda, \\
\lambda^2 \Delta V^\lambda &= \sigma^\lambda.
\end{aligned}$$

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Differentiate in t the “density fluctuation equation”, taking the divergence of the second equation

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\end{aligned}$$

Differentiate in t the "density fluctuation equation", taking the divergence of the second equation

$$\begin{aligned}
\partial_{tt}\sigma^\lambda - \Delta\sigma^\lambda &= \operatorname{div}(\mu\Delta u^\lambda + \ldots\ldots) - \operatorname{div}\nabla V^\lambda \\
\lambda^2\Delta V^\lambda &= \sigma^\lambda
\end{aligned}$$

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$$- (\gamma - 1)\nabla\pi^\lambda + \sigma^\lambda \nabla V^\lambda + \nabla V^\lambda,$$

$$\lambda^2 \Delta V^\lambda = \sigma^\lambda.$$

Differentiate in t the "density fluctuation equation", taking the divergence of the second equation

KLEIN GORDON EQUATION

$$\partial_{tt}\sigma^\lambda - \Delta\sigma^\lambda + \frac{\sigma^\lambda}{\lambda^2} = \operatorname{div}(\mu\Delta u^\lambda + (\nu+\mu)\nabla \operatorname{div} u^\lambda)$$

$$+ \operatorname{div}(\operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + (\gamma - 1)\nabla\pi^\lambda - \sigma^\lambda \nabla V^\lambda)$$

Scaling: mass renormalization

changing the time and space scale:

$$t = \lambda \tau \qquad x = \lambda y$$

$$\begin{aligned}\partial_{\tau\tau}\tilde{\sigma} - \Delta\tilde{\sigma} + \tilde{\sigma} = &-\frac{1}{\lambda}\operatorname{div}(\mu\Delta\tilde{u} + (\nu+\mu)\nabla\operatorname{div}\tilde{u}) \\ &+ \operatorname{div}(\operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + (\gamma-1)\nabla\tilde{\pi} + \tilde{\sigma}\nabla\tilde{V}) \\ &+ \operatorname{div}(\tilde{\sigma}\nabla\tilde{V}).\end{aligned}$$

Scaling: mass renormalization

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$$+ \operatorname{div}(\operatorname{div}\underbrace{(\tilde{\rho}\tilde{u}\otimes\tilde{u})}_{L_t^\infty L_x^1} + (\gamma-1)\nabla\underbrace{\ \tilde{\pi}\ }_{L_t^\infty L_x^1})$$

$$+ \operatorname{div}\underbrace{\left(\tilde{\sigma}\nabla\tilde{V}\right)}_{L_t^\infty L_x^1}$$

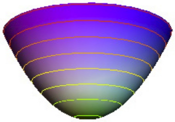
Strichartz estimates for Klein-Gordon equations

$$w_{tt} - \Delta w + w = F \quad w(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g, \quad (x,t) \in \mathbb{R}^d \times [0,T]$$

$$\|w\|_{L^q_{t,x}} + \|\partial_t w\|_{L^q_t W^{-1,q}_x} \lesssim \|f\|_{\dot{H}^\gamma_x} + \|g\|_{\dot{H}^{\gamma-1}_x} + \|F\|_{L^p_{t,x}}$$

(p, q), are *admissible pairs in* $3 - D$ if

$$\frac{4}{3} \le p \le \frac{10}{7} \quad \frac{10}{3} \le q \le 4$$



By choosing $p = 4/3$ and $q = 4$ and by using Duhamel's principle we get this "non standard estimate "

$$\|w\|_{L^4_{\tau,x}} + \|\partial_\tau w\|_{L^4_\tau W^{-1,4}_x} \lesssim \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{-1/2}_x} + \|F\|_{L^1_\tau L^2_x}$$

Scaling: mass renormalization

changing the time and space scale:

$$t = \lambda \tau \qquad x = \lambda y$$

$$\partial_{\tau\tau}\tilde{\sigma} - \Delta\tilde{\sigma} + \tilde{\sigma} = -\frac{1}{\lambda}\operatorname{div}\underbrace{\left(\mu\Delta\tilde{u} + (\nu+\mu)\nabla\operatorname{div}\tilde{u}\right)}_{L^2_t H^{-1}_x}$$
$$+\operatorname{div}(\operatorname{div}\underbrace{(\tilde{\rho}\tilde{u}\otimes\tilde{u})}_{L^\infty_t L^1_x} + (\gamma-1)\nabla\underbrace{\tilde{\pi}}_{L^\infty_t L^1_x})$$
$$+\operatorname{div}\underbrace{(\tilde{\sigma}\nabla\tilde{V})}_{L^\infty_t L^1_x}$$

...we end up with the estimate

$$\lambda^{-\frac{1}{2}}\|\sigma^{\lambda}\|_{L_t^4 W_x^{-s_0-2,4}} + \lambda^{-\frac{1}{2}}\|\partial_t \sigma^{\lambda}\|_{L_t^4 W_x^{-s_0-3,4}}$$

$$\lesssim \lambda^{s_0-\frac{1}{2}}\|\sigma_0^{\lambda}\|_{H_x^{-3/2}} + \lambda^{s_0-\frac{1}{2}}\|m_0^{\lambda}\|_{H_x^{-5/2}}$$

$$+ T\|\operatorname{div}(\operatorname{div}(\sigma^{\lambda} u^{\lambda} \otimes u^{\lambda}) - (\gamma-1)\nabla \pi^{\lambda})\|_{L_t^{\infty} H_x^{-s_0-2}}$$

$$+ \lambda^{s_0}\|\operatorname{div}\Delta u^{\varepsilon} + \nabla \operatorname{div} u^{\varepsilon}\|_{L_t^2 H_x^{-2}} + T\|\operatorname{div}(\sigma^{\lambda}\nabla V^{\lambda})\|_{L_t^{\infty} H_x^{-s_0-1}}$$

Strong convergence of $\mathrm{Q}u^\lambda$

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$$\mathbf{Q}u^\lambda = \mathbf{Q}\underbrace{(\rho^\lambda u^\lambda)}_{?} - \mathbf{Q}\underbrace{(\sigma^\lambda u^\lambda)}_{L^2_t H^{-1}_x}$$

but......

$$\mathbf{Q}(\rho^\lambda u^\lambda) = \nabla\Delta^{-1}\,\mathrm{div}(\rho^\lambda u^\lambda)$$

$$\partial_t \sigma^\lambda = -\,\mathrm{div}(\rho^\lambda u^\lambda)$$

Strong convergence of Qu^λ

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but......

$$\partial_t \sigma^\lambda = -\operatorname{div}(\rho^\lambda u^\lambda)$$

$$\mathbf{Q}(\rho^\lambda u^\lambda) = \nabla\Delta^{-1}\operatorname{div}(\rho^\lambda u^\lambda)$$

$$\mathbf{Q}(\rho^\lambda u^\lambda) = \lambda^{1/2}\nabla\Delta^{-1}\underbrace{\lambda^{-1/2}\partial_t\sigma^\lambda}_{L^4_t W^{-s_0-2,3}_x}$$

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$$\mathbf{Q}u^\lambda = \mathbf{Q}\underbrace{(\rho^\lambda u^\lambda)}_{?} - \mathbf{Q}\underbrace{(\sigma^\lambda u^\lambda)}_{L^2_t H^{-1}_x}$$

but......

$$\mathbf{Q}(\rho^\lambda u^\lambda) = \nabla\Delta^{-1}\operatorname{div}(\rho^\lambda u^\lambda) \qquad \partial_t\sigma^\lambda = -\operatorname{div}(\rho^\lambda u^\lambda)$$

$$\mathbf{Q}(\rho^\lambda u^\lambda) = \lambda^{1/2}\nabla\Delta^{-1}\underbrace{\lambda^{-1/2}\partial_t\sigma^\lambda}_{L^4_t W^{-s_0-2,3}_x}$$

- convolution techniques (Young type estimates)
- interpolation

$$\|\mathbf{Q}u^\lambda\|_{L^2_t L^p_x} \le C_T \lambda^{\frac{6-p}{p(17+s_0)}} \quad \text{for any } p \in [4,6).$$

$$\Downarrow$$

$$\mathbf{Q}u^\lambda \longrightarrow 0 \quad \text{strongly in } L^2_t L^p_x, \text{ for any } p \in [4,6).$$

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- Strong convergence for $\mathbf{Q}u$
- Compactness for $\lambda \nabla V^{\lambda}$
  - introduction of correctors
  - construction of microlocal defect measure

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$$\mathbf{P}\Big(\partial_t u + (u\cdot\nabla)u - \Delta u = \textbf{?}\Big)$$

We know

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Our setting

We want to study the weak continuity of quadratic forms in L^2

$$(Aw_k, w_k)$$

when A belongs to a more refined class of "testing operators"

Defect measures

(e.g. Di Perna, Majda)

Defect measure

$$w_k \in L^2_{loc}(\Omega), \qquad w_k \to w \quad \text{in } \mathcal{D}'(\Omega)$$

$$\nu_k = |w_k - w|^2 \rightharpoonup \nu = \text{defect measure of } w_k$$

$$w_k(x) = e^{ikx\cdot\xi_0},\ \xi_0 \neq 0 \qquad \nu = dx = Lebesque\ measure$$

$$A \in \psi_0^c(\Omega, \mathcal{K}(H))$$

class of pseudodifferential operators

$$A(x,D)f(x) = \int a(x,\xi)\mathcal{F}f(\xi)e^{ix\xi}d\xi := OP(a(x,\xi))$$

polihomogeneous

$$p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$$

$$p_{m-j}(x,r\xi) = r^{m-j}p_{m-j}(x,\xi) \text{for } |\xi| \geq 1$$

whose kernel has compact support

Microlocal defect measures

(L. Tartar 1990, P. Gérard 1991)

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Microlocal defect measure

μ is the *microlocal defect measure* if for any $A \in \psi_0^c(\Omega, \mathcal{K}(H))$

$$\lim_{k\to\infty} (A(w_k - w), (w_k - w)) = \int_{S^{n-1}\times\Omega} tr(a(x,\xi)\mu(dxd\xi))$$

$$w_k(x) = e^{ikx\cdot\xi_0},\ \xi_0 \neq 0 \qquad \nu = dx \otimes \delta_{\xi_0/|\xi_0|}$$

Our setting

$$\lambda E^{\lambda} = \lambda \nabla V^{\lambda} \rightharpoonup 0 \quad \text{weakly in } L^2(0,T,L^2(\mathbb{R}^3))$$

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homogenous only with respect to the x and we cannot extend it to
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we have to work on λE^λ in order to
isolate the components that oscillates fast in time
$\Downarrow$
we introduce correctors of the electric field

Electric field equation

$$\lambda^2 \partial_{tt} E^\lambda + E^\lambda = \operatorname{div} \Delta^{-1} \nabla \operatorname{div} \left(\rho^\lambda u^\lambda \otimes u^\lambda + (\rho^\lambda)^\gamma \mathbb{I} - \lambda^2 E^\lambda \otimes E^\lambda \right) + \frac{\lambda^2}{2} \operatorname{div} \left(|E^\lambda|^2 \mathbb{I} \right) - 2 \nabla \operatorname{div} u^\lambda = F^\lambda,$$

By using Duhamel's formula

$$E^\lambda(t,x) = \int_0^t \frac{F^\lambda(s,x)}{2i\lambda} \left(e^{i\frac{t-s}{\lambda}} - e^{-i\frac{t-s}{\lambda}} \right) ds + \frac{\mathcal{E}_1^\lambda(x)}{\lambda} e^{it/\lambda} + \frac{\mathcal{E}_2^\lambda(x)}{\lambda} e^{-it/\lambda},$$

$\mathcal{E}_1^\lambda$ and $\mathcal{E}_2^\lambda$ are two functions in L^2_x defined by the initial data of E^λ.

The L^2-mass of λE^λ concentrates around $t = \frac{1}{\lambda}$

Definition of the correctors

$$E_+^\lambda = \lambda e^{-it/\lambda} E^\lambda \qquad E_-^\lambda = \lambda e^{it/\lambda} E^\lambda$$

They take into account of the L^2-mass of λE^λ around $1/\lambda$.

$$E_+^\lambda \rightharpoonup E^+, \qquad E_-^\lambda \rightharpoonup E^- \quad \text{weakly in } L^2$$

So if we look at the limit

$$\lambda E^\lambda - e^{it/\lambda} E^+ - e^{-it/\lambda} E^- \text{ as } \lambda \to 0$$

we take away the L^2-mass of λE^λ which concentrates around $1/\lambda$.

E^+ and E^- are the correctors

Microlocal defect measure for $\lambda\widetilde{E^{\lambda}}$
(isolating space oscillations)

$$\widetilde{E^{\lambda}} = E^{\lambda} - e^{it/\lambda}\frac{E^{+}}{\lambda} - e^{-it/\lambda}\frac{E^{-}}{\lambda}$$

$$\lambda\widetilde{E^{\lambda}} \rightharpoonup 0 \qquad \text{weakly in } L^2(0,T,L^2(\mathbb{R}^3)).$$

The weak convergence of $\lambda\widetilde{E^{\lambda}}$ is caused only by *spatial oscillations*

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$$\Downarrow$$

we can introduce the *microlocal defect measure* in space for $\lambda\widetilde{E^\lambda}$

Construction of the microlocal defect measure

- since $\widetilde{E^\lambda}$ is defined only in $(0,T)$, we need to extend it to 0 out of this interval
- cut-off the frequencies greater than a certain quantity

$$w^\lambda = \mathcal{T}_R[\lambda\widetilde{E^\lambda}] = \lambda\mathcal{F}^{-1}\chi_{B(0,R)}\mathcal{F}[\lambda\widetilde{E^\lambda}]$$

- $\lim_{\lambda\to 0} \Im\int dt\phi(t)(Aw^\lambda, w^\lambda) = 0 \qquad \lim_{\lambda\to 0} \Re\int dt\phi(t)(Aw^\lambda, w^\lambda) \geq 0$
- $\lim_{\lambda\to 0} \int dt\phi(t)(A\lambda\widetilde{E^\lambda}, \lambda\widetilde{E^\lambda}) = \langle \nu^{\widetilde{E^\lambda}}(dt, dx, d\xi), \phi(t)a(x,\xi)\rangle.$

Where we are?

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$$\mathbf{P}\Big(\partial_t u + (u\cdot\nabla)u - \Delta u = \operatorname{div}(E^+ \otimes E^+ + E^- \otimes E^-) + \operatorname{div}\langle \nu^E, \frac{\xi\otimes\xi}{|\xi|^2}\rangle\Big)$$

Theorem

Let $(\rho^\lambda, u^\lambda, V^\lambda)$ be weak solutions of the *NSP* system, then

- $\rho^\lambda \rightharpoonup 1$ weakly in $L^\infty([0,T]; L^k_2(\mathbb{R}^3))$.
- There exists $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$, s.t. $u^\lambda \rightharpoonup u$ weakly in $L^2_t H^1_x$
- $\mathbf{Q}u^\lambda \longrightarrow 0$ stronlgy in $L^2_x L^p_x$, for any $p \in [4,6)$.
- $\mathbf{P}u^\lambda \longrightarrow \mathbf{P}u = u$ strongly in $L^2_t L^2_{loc,x}$
- There exist correctors E^+, E^- and a defect measure ν^E, associated to $E^\lambda = \lambda \nabla V^\lambda$ s.t. $u = \mathbf{P}u$ satisfies in $\mathcal{D}'([0,T] \times \mathbb{R}^3)$

$$\mathbf{P}\Big(\partial_t u - \Delta u + (u\cdot\nabla)u - \\ \mathrm{div}(E^+ \otimes E^+ + E^- \otimes E^-) - \mathrm{div}\,\langle \nu^E, \frac{\xi\otimes\xi}{|\xi|^2}\rangle\Big) = 0$$

Who are the correctors?

- ☞ The correctors E^+, E^- remain important as $\lambda \to 0$ and are not vanishing.
- ☞ They correspond to the physical phenomenon of the high frequency *plasma oscillation*.
- ☞ The effect of ill prepared initial data appears through E^+, E^- and remains important for all times.

Who are the correctors?

If $(\rho^\lambda, u^\lambda, V^\lambda)$ satisfy for s large enough

$$\|\rho^\lambda - 1\|_{L^\infty(0,T;H^s(\mathbb{R}^3))} \leq C \qquad \|\lambda E^\lambda\|_{L^\infty(0,T;H^s(\mathbb{R}^3))} \leq C$$

then for all $s' < s - 2$

$$u^\lambda - \frac{1}{i} e^{-it/\lambda} E^+ - \frac{1}{i} e^{it/\lambda} E^- \longrightarrow v \quad \text{strongly in } C^0(0,T,H^{s'-1}_{loc}(\mathbb{R}^3))$$

$$\lambda(E^\lambda - e^{-it/\lambda} E^+ - e^{it/\lambda} E^-) \longrightarrow 0 \quad \text{strongly in } C^0(0,T,H^{s'-1}_{loc}(\mathbb{R}^3))$$

and $E^\pm$ satisfy

$$\partial_t E^\pm - \Delta E^\pm + \mathbf{Q}\,\mathrm{div}(v \otimes E^\pm) = 0, \qquad \mathbf{P} E^\pm = 0.$$

Remark 1: Well prepared data

$$\int_{\mathbb{R}^3} |\rho_0^\lambda - 1|^2 \chi_{(|\rho_0^\lambda - 1| \le \delta)} dx + \int_{\mathbb{R}^3} |\rho_0^\lambda - 1|^\gamma \chi_{(|\rho_0^\lambda - 1 > \delta)} dx \le M\lambda$$

$$\operatorname{div} u_0 = 0$$

$$\|\sqrt{\rho_0^\lambda} u_0 - u_0\|_{L^2}^2 \le M\lambda \qquad \|\lambda \nabla V_0^\lambda\|_{L^2}^2 \le M\lambda$$

$$\Downarrow$$

No oscillations $\Rightarrow$ Strong Convergence

$$\int_{\mathbb{R}^3} |\rho^\lambda - 1|^2 \chi_{(|\rho^\lambda - 1| \le \delta)} dx + \int_{\mathbb{R}^3} |\rho^\lambda - 1|^\gamma \chi_{(|\rho^\lambda - 1 > \delta)} dx \le M\lambda$$

$$\|\sqrt{\rho^\lambda} u^\lambda - u\|_{L^\infty(0,T;L^2)}^2 + \|\lambda V^\lambda\|_{L^\infty(0,T;L^2)}^2 \le M(T)\lambda^{min\{1/2, 1/\gamma\}}$$

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$\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ is the 3-dimensional torus

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- it is related to the acoustic equation

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in $\mathbb{T}^3$ there are **NO dispersive effects!!**

Great difficulty: small divisors problem