Capillarity Approximation of Conservation Laws with Discontinuous Fluxes

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Two–Phase Flow Motion in a one dimensional porous medium

- A one dimensional porous medium.
- Flow of two phases in this porous medium.
- Fluids are immiscible.
- Fluids are incompressible.
- The density of the two fluids is constant.
- Wetting phase **w**.
- Nonwetting phase **n**.
- The mass flow rate per unit volume injected or produced is zero.

The mass conservation for each phase:

$$\begin{aligned} \partial_t(\phi S^{\mathbf{w}}) + \partial_x(v^{\mathbf{w}}) &= 0, \qquad t > 0, \quad x \in \mathbb{R}, \\ \partial_t(\phi S^{\mathbf{n}}) + \partial_x(v^{\mathbf{n}}) &= 0, \qquad t > 0, \quad x \in \mathbb{R}. \end{aligned}$$

- $\phi \equiv$ porosity.
- $S^{\mathbf{w}}, S^{\mathbf{n}} \equiv$ saturations of the two phases.
- $v^{\mathbf{w}}, v^{\mathbf{n}} \equiv$ velocities.

The saturation defines the fraction of the pore volume occupied by a phase.

$$S^l(t,x) = \frac{V^l(t,x)}{V_p}, \quad l = \mathbf{w}, \mathbf{n}.$$

Therefore,

$$S^{\mathbf{w}} + S^{\mathbf{n}} = 1.$$

The porosity denotes the fraction of the volume available for flow. It depends on the pressure.

$$\phi = \phi_0[1 + c_R(p - p_0)].$$

- $c_R \equiv$ rock compressibility.
- $p \equiv$ pressure.
- $\phi_0 \equiv$ the porosity at the pressure p_0.

We assume

$$\phi = 1.$$

The mass conservation, and the saturation conservation give

$$v^{\mathbf{w}} + v^{\mathbf{n}} = q.$$

- $q \equiv$ total flow rate (specified, for instance, by boundary conditions).

We are able to describe the motion of the system.

- unknowns: $S^{\mathbf{w}}, S^{\mathbf{n}}, v^{\mathbf{w}}, v^{\mathbf{n}}$.
- equations: 4.

Darcy's Law

- The phase velocities are modelled on Darcy's Law.

For the motion of the single-flow in a porous medium, in 1856, Darcy discovered the following law:

Darcy's Law

The volumetric flow rate of a homogeneous fluid through a porous medium is proportional to the pressure gradient and to the cross-sectional area normal to the direction of flow and inversely proportional to the viscosity of the fluid.

$$v = -\frac{K}{\mu}(\partial_x p - \rho g).$$

- $K \equiv$ rock permeability. (It can vary (discontinuously) in space)
- $\mu \equiv$ viscosity of fluid
- $g \equiv$ accelaration due to gravity

- Two-Phases Flow.

$$v^{\mathbf{w}} = K\lambda^{\mathbf{w}}(\partial_x p^{\mathbf{w}} - \rho^{\mathbf{w}} g),$$
$$v^{\mathbf{n}} = K\lambda^{\mathbf{n}}(\partial_x p^{\mathbf{n}} - \rho^{\mathbf{n}} g).$$

- $\lambda^l = \frac{k_r^l(S^l)}{\mu^l} \quad l = \mathbf{w}, \mathbf{n}$: phase mobility.
- $k_r^l \equiv$ relative permeability.

$$k_r^l \leq K \quad l = \mathbf{w}, \mathbf{n},$$
$$k_r^l = k_r^l(S^{\mathbf{w}}) \quad l = \mathbf{w}, \mathbf{n}.$$

Adding $v^{\mathbf{w}}$ and $v^{\mathbf{n}}$, we have

$$K\lambda^{\mathbf{w}}(\partial_x p^{\mathbf{w}} - \rho^{\mathbf{w}} g) + K\lambda^{\mathbf{n}}(\partial_x p^{\mathbf{n}} - \rho^{\mathbf{n}} g) = q.$$

Capillary pressure

Capillary Pressure

The capillary pressure is the difference between the pressure $p^{\mathbf{w}}$ and $p^{\mathbf{n}}$, on the contact surface of the two fluids.

$$p^c = p^{\mathbf{w}} - p^{\mathbf{n}}.$$

- p^c is a decreasing function of $S^{\mathbf{w}}$

Following the model proposed in:

R. Helmig, A. Weiss and B. I. Wohlmuth

Dynamic capillary effects in heterogeneous porous media. Comp. Geosci., 11 (2007), 261-274.

we consider

$$p^{dc} = p^{sc}(S^{\mathbf{w}}) + \tau \partial_t S^{\mathbf{w}}.$$

- p^{dc} $\equiv$ dynamic capillary pressure.
- p^{sc} $\equiv$ static capillary pressure.
- $\tau \geq 0$ dynamic factor.

- $p^{sc}(S^{\mathbf{w}}) : [0,1] \to [0,\infty)$.
- $p^{sc}(S^{\mathbf{w}})$ is continuously differentiable.
- $p^{sc}(S^{\mathbf{w}})$ is strictly decreasing.

p^{dc} depends on the dynamics of moving fronts.

PDE in $S^{\mathbf{w}}$

- $\tau = 1$.

Thanks to the mass conservation, Darcy's Law, and the dynamic capillary pressure, we have

$$\partial_t S^{\mathbf{w}} + \partial_x\Big(\frac{\lambda^{\mathbf{w}} q}{\lambda^{\mathbf{w}}+\lambda^{\mathbf{n}}} - \frac{\lambda^{\mathbf{w}}\lambda^{\mathbf{n}} K(\rho^{\mathbf{w}}-\rho^{\mathbf{n}})g}{\lambda^{\mathbf{w}}+\lambda^{\mathbf{n}}}\Big) = -\partial_x\Big(K\lambda^{\mathbf{w}}\lambda^{\mathbf{n}}\partial_x p^{sc}\Big) - \partial_x\Big(K\lambda^{\mathbf{w}}\lambda^{\mathbf{n}}\partial^2_{tx} S^{\mathbf{w}}\Big).$$

- The capillary pressure is supposed to be very small.

We can rescale the previous PDE with a small scale ν.

$$\partial_t S^{\mathbf{w}} + \partial_x\Big(\frac{\lambda^{\mathbf{w}} q}{\lambda^{\mathbf{w}}+\lambda^{\mathbf{n}}} - \frac{\lambda^{\mathbf{w}}\lambda^{\mathbf{n}} K(\rho^{\mathbf{w}}-\rho^{\mathbf{n}})g}{\lambda^{\mathbf{w}}+\lambda^{\mathbf{n}}}\Big) = -\nu\partial_x\Big(K\lambda^{\mathbf{w}}\lambda^{\mathbf{n}}\partial_x p^{sc}\Big) - \nu^2\partial_x\Big(K\lambda^{\mathbf{w}}\lambda^{\mathbf{n}}\partial^2_{tx} S^{\mathbf{w}}\Big).$$

PDE in u

- $S^{\mathbf{w}} = u$,

$$0 \leq u(t,x) \leq 1.$$

we denote

$$\begin{aligned} f(k,u) &= \frac{\lambda^{\mathbf{w}} q}{\lambda^{\mathbf{w}} + \lambda^{\mathbf{n}}} - \frac{\lambda^{\mathbf{w}} \lambda^{\mathbf{n}} K(\rho^{\mathbf{w}} - \rho\mathbf{n}) g}{\lambda^{\mathbf{w}} + \lambda^{\mathbf{n}}}, \\ g(l,u) &= -K\lambda^{\mathbf{w}}\lambda^{\mathbf{n}} \frac{dp^{sc}}{du}, \\ h(m,u) &= -K\lambda^{\mathbf{w}}\lambda^{\mathbf{n}}. \end{aligned}$$

- PDE in u.

$$\partial_t u + \partial_x f(k,u) = \nu \partial_x (g(l,u)\partial_x u) + \nu^2 \partial_x (h(m,u)\partial_{tx}^2 u).$$

Cauchy Problem

Let $t \geq 0, x \in \mathbb{R}$.

$$\begin{cases} \partial_t u_\nu + \partial_x f(k_\nu, u_\nu) = \nu \partial_x (g(\ell_\nu, u_\nu)\partial_x u_\nu) + \nu^2 \partial_x (h(m_\nu, u_\nu)\partial^2_{tx} u_\nu), \\ \partial_t k_\nu = \nu \partial^2_{xx} k_\nu, \\ \partial_t \ell_\nu = \nu \partial^2_{xx} \ell_\nu, \\ \partial_t m_\nu = \nu \partial^2_{xx} m_\nu, \\ u_\nu(0,x) = u_{0,\nu}(x), \\ k_\nu(0,x) = k_{0,\nu}(x), \\ \ell_\nu(0,x) = \ell_{0,\nu}(x), \\ m_\nu(0,x) = m_{0,\nu}(x). \end{cases}$$

- $(u_\nu,\ k_\nu,\ \ell_\nu,\ m_\nu)$ is a smooth solution,
- $0 \leq u_\nu \leq 1$.

- $0 < \nu \leq 1$.
- $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions.
- $0 < \alpha \leq g(\cdot, \cdot), h(\cdot, \cdot)$.
- $f(k, \cdot)$ is genuinely nonlinear for every $k \in \mathbb{R}$.
- $u \in [0, 1] \to f(k, u)$ is not affine on any nontrivial interval for every $k \in \mathbb{R}$.

Moreover, we assume

$$k_{0,\nu}, \ell_{0,\nu}, m_{0,\nu} \in C^\infty(\mathbb{R}) \cap W^{1,1}(\mathbb{R}),$$
$$u_{0,\nu} \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 0 \leq u_{0,\nu} \leq 1,$$

and that there exist

$$k,\ \ell,\ m \in BV(\mathbb{R}) \cap L^1(\mathbb{R}), \quad u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 0 \leq u_0 \leq 1,$$

such that

$$
\begin{aligned}
& u_{0,\nu} \to u_0, \quad k_{0,\nu} \to k, \quad \ell_{0,\nu} \to \ell, \quad m_{0,\nu} \to m, \quad \text{a.e. and in } L^p(\mathbb{R}) \\
& \|u_{\nu,0}\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}, \quad \nu, \\
& \|k_{0,\nu}\|_{L^\infty(\mathbb{R})} \le \|k\|_{L^\infty(\mathbb{R})}, \quad \|k_{0,\nu}\|_{L^2(\mathbb{R})} \le \|k\|_{L^2(\mathbb{R})}, \\
& \|\partial_x k_{0,\nu}\|_{L^1(\mathbb{R})} \le TV(k), \quad \nu, \\
& \|\ell_{0,\nu}\|_{L^\infty(\mathbb{R})} \le \|\ell\|_{L^\infty(\mathbb{R})}, \quad \|\ell_{0,\nu}\|_{L^2(\mathbb{R})} \le \|\ell\|_{L^2(\mathbb{R})}, \\
& \|\partial_x \ell_{0,\nu}\|_{L^1(\mathbb{R})} \le TV(\ell), \quad \nu, \\
& \|m_{0,\nu}\|_{L^\infty(\mathbb{R})} \le \|m\|_{L^\infty(\mathbb{R})}, \|m_{0,\nu}\|_{L^2(\mathbb{R})} \le \|m\|_{L^2(\mathbb{R})}, \\
& \|\partial_x m_{0,\nu}\|_{L^1(\mathbb{R})} \le TV(m), \ \nu,
\end{aligned}
$$

- $1 \le p < \infty$.

Theorem

There exist a sequence $\{\nu_n\}_n \subset (0,1]$, $\nu_n \to 0$, a sequence $\{u_{\nu_n}\}_{n\in\mathbb{N}}$ of solutions of the previous Cauchy problem, and a distributional solution u of

$$\begin{cases} \partial_t u + \partial_x f(k(x), u) = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

such that

$$u_{\nu_n} \to u, \quad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty, \text{ and a.e. in } (0,\infty)\times\mathbb{R}.$$

Proof

Murat's Lemma

Let Ω be a bounded open subset of $\mathbb{R}^2$. Suppose the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

G. M. Coclite, K. H. Karlsen, S. Mishra, and N. H. Risebro

Convergence of vanishing viscosity approximations of 2×2 triangular systems of multi-dimensional conservation laws. (2009).

Let us show that the following family

$$\{\partial_t |u_\nu - c| + \partial_x(\operatorname{sign}(u_\nu - c)(f(k, u_\nu) - f(k, c)))\}_{\nu>0} \qquad (1)$$

is compact in $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$.

- c is a fixed real number.

Let $\{(\eta_\nu, Q_\nu)\}_{\nu>0}$ be a family of maps such that

$$\begin{aligned}
&\eta_\nu \in C^\infty(\mathbb{R}), \quad Q_\nu \in C^\infty(\mathbb{R}^2),\\
&\partial_u Q_\nu(k,u) = \partial_u f(k,u)\eta'_\nu(u), \quad \eta''_\nu \geq 0,\\
&\|\eta_\nu - \eta_0\|_{L^\infty([0,1])} \leq \nu, \quad \|\eta'_\nu - \eta'_0\|_{L^1([0,1])} \leq \nu,\\
&\|\eta'_\nu\|_{L^\infty([0,1])} \leq 1, \quad \eta_\nu(0) = Q_\nu(k,0) = 0.
\end{aligned}$$

- It is an approximation of (1).

From the equation, we obtain

$$\begin{aligned}\partial_t|u_\nu - c| + \partial_x(\mathrm{sign}\,(u_\nu - c)\,(f(k,u_\nu) - f(k,c)))\\ = I_{1,\nu} + I_{2,\nu} + I_{3,\nu} + I_{4,\nu} + I_{5,\nu} + I_{6,\nu} + I_{7,\nu} + I_{8,\nu} + I_9,\end{aligned}$$

where

$$\begin{aligned}
I_{1,\nu} &= \nu\partial_x(\eta'_\nu(u_\nu)g(\ell_\nu,u_\nu)\partial_x u_\nu),\\
I_{2,\nu} &= -\nu\eta''_\nu(u_\nu)g(\ell_\nu,u_\nu)(\partial_x u_\nu)^2,\\
I_{3,\nu} &= \nu^2\partial_x(\eta'_\nu(u_\nu)h(m_\nu,u_\nu)\partial^2_{tx}u_\nu),\\
I_{4,\nu} &= -\nu^2\eta''_\nu(u_\nu)h(m_\nu,u_\nu)\partial_x u_\nu\partial^2_{tx}u_\nu,\\
I_{5,\nu} &= -(\eta'_\nu(u_\nu)\partial_u f(k,u_\nu) - \partial_k Q(k,u_\nu))\partial_x k_\nu,\\
I_{6,\nu} &= \partial_t(\eta_0(u_\nu) - \eta_\nu(u_\nu)),\\
I_{7,\nu} &= \partial_x(Q_0(k,u_\nu) - Q_\nu(k,u_\nu)),\\
I_{8,\nu} &= \partial_x(Q_\nu(k,u_\nu) - Q_\nu(k_\nu,u_\nu)),\\
I_9 &= \mathrm{sign}\,(-c)\,\partial_x(f(k,0) - f(k,c)).
\end{aligned}$$

The paper shows that

$$
\begin{aligned}
&\{I_{5,\nu}\}_{\nu>0} \quad \text{is bounded in} \quad L^1((0,T)\times\mathbb{R}) \quad \text{for each} \quad T,\\
&I_{6,\nu}\to 0, \quad I_{7,\nu}\to 0 \quad \text{in } H^{-1}_{loc}((0,\infty)\times\mathbb{R}),\\
&\{I_{8,\nu}\}_{\nu>0} \quad \text{is compact in} \quad H^{-1}_{loc}((0,\infty)\times\mathbb{R}),\\
&I_9\in\mathcal{M}_{loc}((0,\infty)\times\mathbb{R}).
\end{aligned}
$$

We have to prove that

$$
\begin{aligned}
&I_{1,\nu}\to 0, \quad I_{3,\nu}\to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}), \quad \text{for each} \quad T,\\
&\{I_{2,\nu}\}_{\nu>0}, \quad \{I_{4,\nu}\}_{\nu>0} \quad \text{are bounded in} \quad L^1((0,T)\times\mathbb{R}) \quad \text{for each} \quad T.
\end{aligned}
$$

We need the following result:

L^2- estimate

$$\|u_\nu(t,\cdot)\|^2_{L^2(\mathbb{R})} + \nu\mu_1 \int_0^t \|\partial_x u_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}\, ds + \nu^3\mu_2 \int_0^t \left\|\partial^2_{tx} u_\nu(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + \nu\mu_3 \int_0^t \|\partial_t u_\nu(s,\cdot)\|^2_{L^2(\mathbb{R})}\, ds \le Ct + A\|k\|^2_{L^2(\mathbb{R})} + \|u_0\|^2_{L^2(\mathbb{R})}.$$

- $\mu_1, \mu_2, \mu_3, C, A$ are positive constants.

STEPS OF THE PROOF

- The equation is multiplied by $u_\nu + B\partial_t u_\nu$ (B is a positive constant).
- Integrate by parts on $\mathbb{R}$.
- Use the Young's inequality.
- Choose the constant B in a suitable way.

Thank you for your attention.