Wave-wave interactions of a gasdynamic type

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Abstract

We consider, in two significant gasdynamic contexts [isentropic/anisentropic], a quasilinear hyperbolic system of a gasdynamic type – possibly multidimensional:

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u)\frac{\partial u_j}{\partial x_k} = 0, \quad 1 \le i \le n. \tag{1}$$

Two ***analytic*** approaches are considered for such a system:

- a Burnat type "algebraic" genuinely nonlinear approach [constructed from a duality connection between the hodograph character and the physical character] and,
- a Martin type two-dimensional "differential" approach [associated with a Monge–Ampère type representation].

A pair of ***significant*** classes of solutions are constructed for ***each*** of the two mentioned cases [isentropic/anisentropic].

- In the ***isentropic*** case a Burnat type approach is used to constructively structure:
  - the class of ***simple waves solutions*** – here called ***waves***,
  - the class of ***wave-wave regular interaction solutions***,
  - a ***multidimensional extension*** of the two classes mentioned above – with a *classifying potential*.
- In the ***isentropic*** case we use the pair of classes mentioned above to structure ***two selfsimilar exemples*** of two-dimensional solutions.
  - The two examples are concurrently associated to some cases of a quantifiable "amount" of genuine nonlinearity.
  - The isentropic types of wave-wave regular interactions constructed appear to parallel, from an analytic, local and regular prospect, some details [interactions of simple waves solutions] of the Zhang and Zheng two-dimensional qualitative, global and irregular construction.
  - The two examples mentioned above suggest that a regular character of the wave-wave interaction described essentially reflects facts of a multidimensional and skew construction.

Abstract (continued)

- In the ***anisentropic*** case – and in two independent variables – a Martin type approach is used, as associated with a ***particular*** example, to constructively structure an anisentropic ***analogue*** of the isentropic pair of classes mentioned above: the ***anisentropic pair*** which puts together
  - the class of ***pseudo simple waves solutions***,
  - the class of ***pseudo wave-wave regular interaction solutions***.

  Details concerning the nature of the mentioned analogous character are presented.

A ***classifying parallel*** is finally constructed between the two analogous pairs of classes [isentropic, anisentropic]. In such a parallel the two mentioned constructions [Burnat type, Martin type] show some distinct, complementary valences – making evidence of some ***consonances*** and, respectively, of some ***nontrivial contrasts***.

The isentropic pair of classes

- is used at first *to give a significant structure* to some two-dimensional isentropic solutions,
- is finally *parallelled to an analogous* anisentropic *pair*.

Two significant isentropic solutions

For the system

$$\begin{cases} \dfrac{\partial c}{\partial t} + v_x \dfrac{\partial c}{\partial x} + v_y \dfrac{\partial c}{\partial y} + \dfrac{\gamma - 1}{2} c \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} \right) = 0 \\ \dfrac{\partial v_x}{\partial t} + v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y} + \dfrac{2}{\gamma - 1} c \dfrac{\partial c}{\partial x} = 0 \\ \dfrac{\partial v_y}{\partial t} + v_x \dfrac{\partial v_y}{\partial x} + v_y \dfrac{\partial v_y}{\partial y} + \dfrac{2}{\gamma - 1} c \dfrac{\partial c}{\partial y} = 0 \ . \end{cases}$$

describing an ***isentropic flow*** (in usual notations) considered in a self-similar form

$$\begin{cases} (v_x - \xi) \dfrac{\partial c^2}{\partial \xi} + (v_y - \eta) \dfrac{\partial c^2}{\partial \eta} + (\gamma - 1) c^2 \left(\dfrac{\partial v_x}{\partial \xi} + \dfrac{\partial v_y}{\partial \eta} \right) & = & 0 \\ \dfrac{\partial c^2}{\partial \xi} + (\gamma - 1)(v_x - \xi) \dfrac{\partial v_x}{\partial \xi} + (\gamma - 1)(v_y - \eta) \dfrac{\partial v_x}{\partial \eta} & = & 0 \\ \dfrac{\partial c^2}{\partial \eta} + (\gamma - 1)(v_x - \xi) \dfrac{\partial v_y}{\partial \xi} + (\gamma - 1)(v_y - \eta) \dfrac{\partial v_y}{\partial \eta} & = & 0, \end{cases}$$

$$\xi = \frac{x - x_0}{t - t_0}, \quad \eta = \frac{y - y_0}{t - t_0}$$

we identify two significant examples of local solutions for which

$$v_x = \mathfrak{a} \xi + \mathfrak{b} \eta + \mathfrak{c}, \quad v_y = \overline{\mathfrak{a}} \xi + \overline{\mathfrak{b}} \eta + \overline{\mathfrak{c}}; \qquad \text{real constant } \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \overline{\mathfrak{a}}, \overline{\mathfrak{b}}, \overline{\mathfrak{c}} \ .$$

Two significant isentropic solutions (continued)

A first significant solution

$$v_x = \frac{1}{\gamma}\xi + \mathfrak{c}\,, \;\; v_y = \frac{1}{\gamma}\eta + \bar{\mathfrak{c}}\,; \quad \text{arbitrary} \;\; \mathfrak{c}\,, \bar{\mathfrak{c}}\,,$$

$$c^2 = \frac{1}{2}\left[\left(\frac{\gamma-1}{\gamma}\xi - \mathfrak{c}\right)^2 + \left(\frac{\gamma-1}{\gamma}\eta - \bar{\mathfrak{c}}\right)^2\right]$$

A second significant solution [for $\frac{3-\gamma}{\gamma+1} < \mathfrak{a} < 1$]

$$v_x = \mathfrak{a}\xi \pm \eta\sqrt{(1-\mathfrak{a})\left(\mathfrak{a}-\frac{3-\gamma}{\gamma+1}\right)} + K\sqrt{1-\mathfrak{a}}, \quad K = \frac{\mathfrak{c}}{\sqrt{1-\mathfrak{a}}} = \mp\frac{\bar{\mathfrak{c}}}{\sqrt{\mathfrak{a}-\frac{3-\gamma}{\gamma+1}}}$$

$$v_y = \pm\xi\sqrt{(1-\mathfrak{a})\left(\mathfrak{a}-\frac{3-\gamma}{\gamma+1}\right)} + \eta\left(\frac{4}{\gamma+1} - \mathfrak{a}\right) \mp K\sqrt{\mathfrak{a}-\frac{3-\gamma}{\gamma+1}}$$

$$c = \varepsilon\sqrt{\frac{(3-\gamma)(\gamma-1)}{2(\gamma+1)}}\left(\xi\sqrt{1-\mathfrak{a}} \mp \eta\sqrt{\mathfrak{a}-\frac{3-\gamma}{\gamma+1}} - K\right), \quad \varepsilon = \pm 1$$

Burnat type "algebraic" approach.

Dual pairs of directions

- **System of equations considered** [its hyperbolic character]

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u)\frac{\partial u_j}{\partial x_k} = 0, \quad 1 \leq i \leq n \tag{1}$$

- **Dual pairs of directions** [their real character]

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u^*)\beta_k\kappa_j = 0, \quad 1 \leq i \leq n. \tag{2}$$

- **Types of gasdynamic dual pairs** [Lax, Peradzyński]
  - one-dimensional general case: *finite* number of pairs with an *one-to-one* structure $\vec{\kappa}_i \equiv \vec{R}_i \rightleftharpoons \vec{\beta}_i \equiv \Theta_i(u)[-\lambda_i(u), 1]$, $i = 1, ..., n$,
  - two-dimensional isentropic case: *infinite* number of pairs, with an *one-to-one* structure $\vec{\kappa} \rightleftharpoons \vec{\beta}$,
  - three-dimensional isentropic case: *infinite* number of pairs, with a *new* structure $\vec{\kappa} \rightleftharpoons (\vec{\beta}_1, ..., \vec{\beta}_k)$.
- **Hodograph characteristic curve**
- **Hodograph characteristic surface**

Genuine nonlinearity

- **Types of gasdynamic dual pairs structures**
  - one-dimensional general case: $\vec{\kappa}_i \equiv \vec{R}_i \rightharpoondown \vec{\beta}_i \equiv \Theta_i(u)[-\lambda_i(u),1]$, $1 \le i \le n$,
  - two-dimensional isentropic case: $\vec{\kappa} \rightharpoondown \vec{\beta}$,
  - three-dimensional isentropic case: $\vec{\kappa} \rightharpoondown (\vec{\beta}_1,\ldots,\vec{\beta}_k)$.
- **Types of genuine nonlinearity (*gnl*)**
  - one-dimensional general case: $\vec{\kappa}_i(u) \diamond \vec{\beta}_i(u) \equiv \vec{R}_i(u) \cdot \mathrm{grad}_u \lambda_i(u) \neq 0$ in a region $\mathcal{R}$ for a *gnl* index i; this condition transcribes the requirement $\frac{\mathrm{d}\vec{\beta}_i}{\mathrm{d}\alpha} \neq 0$ along each characteristic curve $\mathcal{C}_i \subset \mathcal{R}$.
  - two-dimensional isentropic case: $\left|\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\alpha}\right| \neq 0$ along a characteristic curve $\mathcal{C} \subset H$,
  - three-dimensional isentropic case: $\sum_{\mu=1}^{k} \left|\frac{\mathrm{d}\vec{\beta}_\mu}{\mathrm{d}\alpha}\right| \neq 0$ along a characteristic curve $\mathcal{C} \subset H$.
- **Hodograph genuinely nonlinear characteristic curve**
- **Hodograph genuinely nonlinear characteristic surface**
- **Types of simple waves solutions: in implicit form**
  - one-dimensional general case:
    $u = U[\alpha(x,t)]$; where $\alpha = \theta(\xi), \xi = x - \zeta_i(\alpha)t$, with $\zeta_i(\alpha) \equiv \lambda_i[U(\alpha)]$; for a *gnl* index i; $U(\alpha)$ describes a hodograph characteristic $\mathcal{C}_i$;
  - two-dimensional isentropic case: ***planar*** simple waves solutions
    $u = U[\alpha(x,t)]$; where $\alpha = \theta(\xi)$; $\xi = \sum_{\nu=0}^{m} \beta_\nu[U(\alpha)] x_\nu$;
  - three-dimensional isentropic case: ***nonplanar*** simple waves solutions
    $u = U[\alpha(x,t)]$; where $\alpha = \theta(\xi_1, \ldots, \xi_k)$; $\xi_j = \sum_{\nu=0}^{m} \beta_{j\nu}[U(\alpha)] x_\nu$, $1 \le j \le k$.

“Algebraic” nondegeneracy

- **A class of solutions of the system (1)**

  Let $R_1, \ldots, R_p$ be characteristic coordinates on a given p-dimensional characteristic region $\mathcal{R}$ of a hodograph hypersurface $\mathcal{S}$. Solutions of the ***intermediate*** system

$$\frac{\partial u_l}{\partial x_s} = \sum_{k=1}^{p} \eta_k \kappa_{kl}(u) \beta_{ks}(u),\ u \in \mathcal{R};\ 1 \le l \le n,\ 0 \le s \le m \tag{3}$$

  appear to concurrently satisfy the system (1). This indicates an “algebraic” importance of the concept of dual pair [see (2)].

- **Wave-wave “algebraic” regular interactions. Riemann–Burnat invariants**

  A solution of (1) whose hodograph is laid on a characteristic hypersurface is said to correspond to a ***wave-wave regular interaction*** if its hodograph possesses a *gnl* system of coordinates ***and*** there exists a set of ***Riemann–Burnat invariants*** $R(x)$ structuring the dependence on x of the solution u by a ***regular*** interaction representation

$$u_l = u_l[R_1(x_0, \ldots, x_m), \ldots, R_p(x_0, \ldots, x_m)],\ 1 \le l \le n, \tag{4}$$

  We compare (3) and (4) to see that for a wave-wave regular interaction solution of (3) $R_i(x)$ must fulfil an (overdetermined and Pfaff) system

$$\frac{\partial R_k}{\partial x_s} = \eta_k \beta_{ks}[u(R)],\ 1 \le k \le p,\ 0 \le s \le m. \tag{5}$$

- **Extending the *gnl* character.**

$$\frac{\partial u_l}{\partial x_s} = \sum_{k=1}^{p} \eta_k \kappa_{kl}(u) \beta_{ks}(u),$$

$$\frac{\partial u_l}{\partial x_s} = \sum_{k=1}^{p} \frac{\partial u_l}{\partial R_k} \cdot \frac{\partial R_k}{\partial x_s} = \sum_{k=1}^{p} \kappa_{kl}(u) \frac{\partial R_k}{\partial x_s}\ .$$

Genuinely nonlinear "algebraic" approach: a review

- System of equations considered

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u)\frac{\partial u_j}{\partial x_k} = 0, \quad 1 \le i \le n. \tag{1}$$

- Dual pairs of directions

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u^*)\beta_k\kappa_j = 0, \quad 1 \le i \le n. \tag{2}$$

- Genuine nonlinearity requirements
- A class of solutions of the system (1)

$$\frac{\partial u_l}{\partial x_s} = \sum_{k=1}^{p} \eta_k\kappa_{kl}(u)\beta_{ks}(u),\ u \in \mathcal{R};\ 1\le l \le n,\ 0\le s\le m. \tag{3}$$

- Wave-wave "algebraic" regular interactions. Riemann–Burnat invariants

$$u_l = u_l[R_1(x_0, ..., x_m),...,R_p(x_0,...,x_m)],\ 1\le l\le n, \tag{4}$$

- A Pfaff system for the Riemann–Burnat invariants

$$\frac{\partial R_k}{\partial x_s} = \eta_k\beta_{ks}[u(R)],\ 1 \le k \le p,\ 0 \le s \le m. \tag{5}$$

Regular integrability

- Peradzyński restrictions [of a Cartan involution type]

$$\vec\beta_k \wedge \left(\vec\beta_j \wedge \frac{\partial\vec\beta_k}{\partial R_j}\right) = 0,\ \ k, j = 1, \dots, p, \quad k \ne j \quad \text{[no summation]}.$$

- Tsarev–Ferapontov restrictions [of a Darboux type]

$$\left(\vec\beta_k \wedge \vec\beta_j\right) \wedge \left(\vec\beta_k \wedge \frac{\partial\vec\beta_k}{\partial R_j}\right) = 0\ , \quad j \ne k \quad \text{[no summation]}.$$

Genuine nonlinearity / linear degeneracy: some two-dimensional details

In case of the ***isentropic*** form of system (1) (in usual notations)

$$\begin{cases} \dfrac{\partial c}{\partial t} + v_x \dfrac{\partial c}{\partial x} + v_y \dfrac{\partial c}{\partial y} + \dfrac{\gamma - 1}{2} c \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} \right) = 0 \\ \dfrac{\partial v_x}{\partial t} + v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y} + \dfrac{2}{\gamma - 1} c \dfrac{\partial c}{\partial x} = 0 \\ \dfrac{\partial v_y}{\partial t} + v_x \dfrac{\partial v_y}{\partial x} + v_y \dfrac{\partial v_y}{\partial y} + \dfrac{2}{\gamma - 1} c \dfrac{\partial c}{\partial y} = 0 \ . \end{cases}$$

condition (2) takes, at a point $u^* \in H$, the form [as $n = m + 1$]

$$c^2 \kappa_1 \left[\left(\frac{2}{\gamma - 1} \right)^2 \kappa_1^2 - (\kappa_2^2 + \kappa_3^2) \right] = 0 \qquad (2')$$

which will be connected with [as $n = m + 1$]

$$(\beta_0 + \beta_1 v_x^* + \beta_2 v_y^*)[(\beta_0 + \beta_1 v_x^* + \beta_2 v_y^*)^2 - c^{*2}(\beta_1^2 + \beta_2^2)] = 0 \qquad (2'')$$

in order to determine the ***duality relations***:

- given $\vec{\beta}$ we obtain from (2)

$$\vec{\beta} = (v_x \beta_1 + v_y \beta_2, -\beta_1, -\beta_2) \ \rightharpoondown \ \vec{\kappa} = (0, -\beta_2, \beta_1),$$

$$\vec{\beta} = [-(v_x \beta_1 + v_y \beta_2) - \varepsilon c, \beta_1, \beta_2] \ \rightharpoondown \ \vec{\kappa} = \left[\varepsilon \frac{\gamma - 1}{2}, \beta_1, \beta_2 \right], \quad \varepsilon = \pm 1,$$

- given $\vec{\kappa}$ [cf. (2′)] we get from (2)

$$\vec{\kappa} = (0, \kappa_2, \kappa_3) \ \rightharpoondown \ \vec{\beta} = (v_x \kappa_3 - v_y \kappa_2, -\kappa_3, \kappa_2),$$

$$\vec{\kappa} = \left[\varepsilon \frac{\gamma - 1}{2}, \kappa_2, \kappa_3 \right] \ \rightharpoondown \ \vec{\beta} = [-(v_x \kappa_2 + v_y \kappa_3) - \varepsilon c, \kappa_2, \kappa_3], \quad \varepsilon = \pm 1.$$

- It is easy to show that the hodograph characteristic curves along which, in a gasdynamic construction, $\kappa_1 \neq 0$ have a ***genuinely nonlinear*** character.

- *Any* smooth curve $\overline{\mathcal{C}}$ placed in a plane $c =$ constant $\neq 0$ appears to be a hodograph characteristic curve corresponding to $\kappa_1 = 0$ in (2′).

- It is easy to show that the hodograph characteristic curves corresponding to $\kappa_1 = 0$ are ***linearly degenerate*** only if they are straightlines ***and*** have a ***genuinely nonlinear*** character if they do not include straight-lined arcs.

Genuine nonlinearity / linear degeneracy: some two-dimensional examples (I)

For the self-similar form

$$\left\{\begin{array}{rcl} (v_x-\xi)\dfrac{\partial c^2}{\partial \xi}+(v_y-\eta)\dfrac{\partial c^2}{\partial \eta}+(\gamma-1)c^2\left(\dfrac{\partial v_x}{\partial \xi}+\dfrac{\partial v_y}{\partial \eta}\right) & = & 0 \\ \dfrac{\partial c^2}{\partial \xi}+(\gamma-1)(v_x-\xi)\dfrac{\partial v_x}{\partial \xi}+(\gamma-1)(v_y-\eta)\dfrac{\partial v_x}{\partial \eta} & = & 0 \\ \dfrac{\partial c^2}{\partial \eta}+(\gamma-1)(v_x-\xi)\dfrac{\partial v_y}{\partial \xi}+(\gamma-1)(v_y-\eta)\dfrac{\partial v_y}{\partial \eta} & = & 0, \end{array}\right.$$

$$\xi=\frac{x-x_0}{t-t_0},\quad \eta=\frac{y-y_0}{t-t_0}$$

we considered two significant examples of local solutions for which

$$v_x=\mathfrak{a}\xi+\mathfrak{b}\eta+\mathfrak{c},\quad v_y=\overline{\mathfrak{a}}\xi+\overline{\mathfrak{b}}\eta+\overline{\mathfrak{c}};\qquad \text{real constant } \mathfrak{a},\mathfrak{b},\mathfrak{c},\overline{\mathfrak{a}},\overline{\mathfrak{b}},\overline{\mathfrak{c}}\ .$$

A first significant solution

$$v_x=\frac{1}{\gamma}\xi+\mathfrak{c}\,,\ \ v_y=\frac{1}{\gamma}\eta+\overline{\mathfrak{c}}\ ;\quad \text{arbitrary}\ \ \mathfrak{c}\ ,\overline{\mathfrak{c}}\ ,$$

$$c^2=\frac{1}{2}\left[\left(\frac{\gamma-1}{\gamma}\xi-\mathfrak{c}\right)^2+\left(\frac{\gamma-1}{\gamma}\eta-\overline{\mathfrak{c}}\right)^2\right]$$

A second significant solution [for $\frac{3-\gamma}{\gamma+1}<\mathfrak{a}<1$]

$$v_x=\mathfrak{a}\xi\pm\eta\sqrt{(1-\mathfrak{a})\left(\mathfrak{a}-\frac{3-\gamma}{\gamma+1}\right)}+K\sqrt{1-\mathfrak{a}},\quad K=\frac{\mathfrak{c}}{\sqrt{1-\mathfrak{a}}}=\mp\frac{\overline{\mathfrak{c}}}{\sqrt{\mathfrak{a}-\frac{3-\gamma}{\gamma+1}}}$$

$$v_y=\pm\xi\sqrt{(1-\mathfrak{a})\left(\mathfrak{a}-\frac{3-\gamma}{\gamma+1}\right)}+\eta\left(\frac{4}{\gamma+1}-\mathfrak{a}\right)\mp K\sqrt{\mathfrak{a}-\frac{3-\gamma}{\gamma+1}}$$

$$c=\varepsilon\sqrt{\frac{(3-\gamma)(\gamma-1)}{2(\gamma+1)}}\left(\xi\sqrt{1-\mathfrak{a}}\mp\eta\sqrt{\mathfrak{a}-\frac{3-\gamma}{\gamma+1}}-K\right),\quad \varepsilon=\pm1$$

Genuine nonlinearity / linear degeneracy: some two-dimensional examples (II)

The case of the first significant solution

For the first significant solution we find, around each point c^*, v_x^*, v_y^* of its ***conical*** hodograph, three families of hodograph characteristic fields: two families of ***conical helices***

$$\begin{cases} c = \dfrac{\gamma - 1}{\sqrt{2}} \exp[-(R_+ + R_-)] \\ V_x = \exp[-(R_+ + R_-)] \cos(R_+ - R_-) \\ V_y = \exp[-(R_+ + R_-)] \sin(R_+ - R_-) \end{cases}$$

$$V_x = v_x - v_x^*,\ V_y = v_y - v_y^*\ ;\ v_x^* = \frac{\gamma}{\gamma - 1}\mathfrak{c}\ ,\ v_y^* = \frac{\gamma}{\gamma - 1}\bar{\mathfrak{c}}.$$

and a family of ***horizontal circles***. All these families appear to be ***genuinely nonlinear***.

- We dispose in this case of ***three*** genuinely nonlinear systems of coordinates to present the first significant solution in ***three distinct manners*** as a regular interaction of [multidimensional] simple waves solutions.
- To each manner a pair of Riemann invariants contribute:

$$R_+(\xi,\eta), R_-(\xi,\eta)\ ;\ R_+(\xi,\eta), R_0(\xi,\eta)\ ;\ R_-(\xi,\eta), R_0(\xi,\eta)\ ,$$

which result when the solution is compared with its Riemann invariance structure mentioned above.

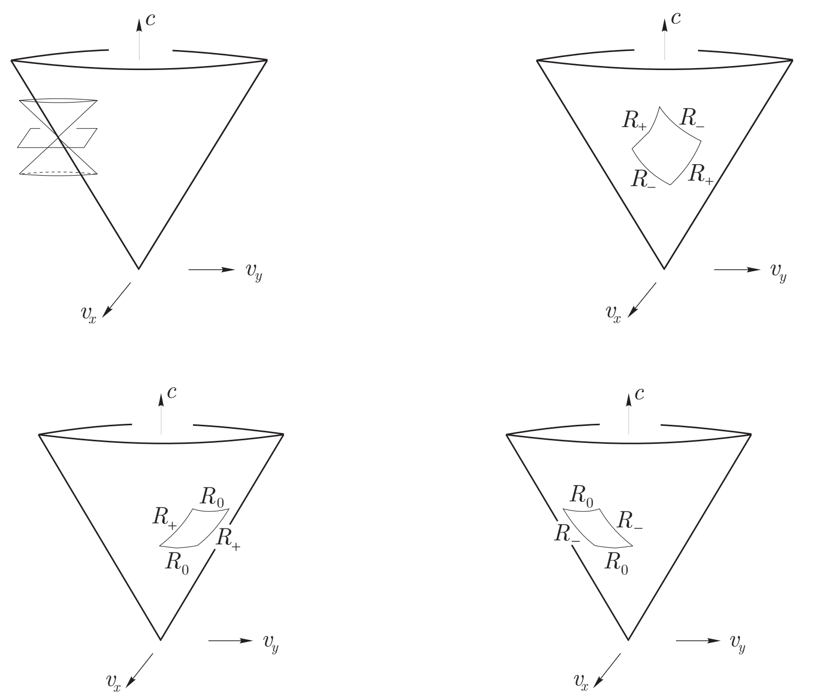
- The possibility of *several* Riemann structures appears to be a ***new fact*** of the multidimensional approach.

The case of the second significant solution

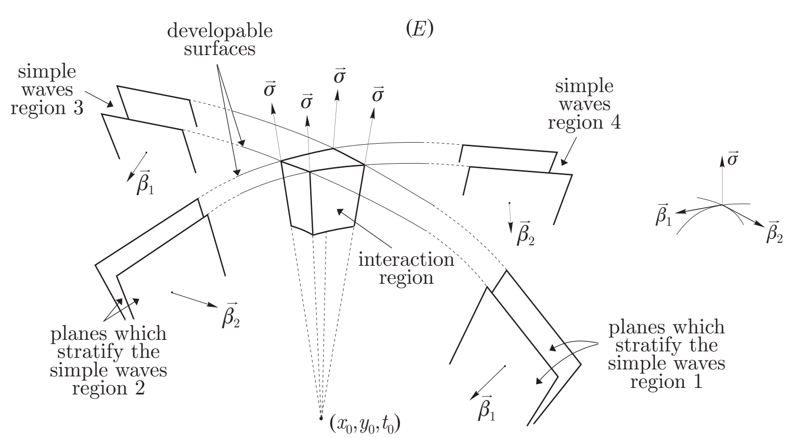
For the second significant solution we find again, around each point c^*, v_x^*, v_y^* of its [double] ***planar*** hodograph, three families of ***straightlined*** hodograph characteristic fields: of which two are ***non-horizontal*** and appear to be ***genuinely nonlinear*** and the third is ***horizontal*** and shows a ***linearly degenerate*** character.

- We dispose in this case of ***a single*** genuinely nonlinear system of coordinates to present the solution as a regular interaction of simple waves solutions. The corresponding pair of Riemann invariants $[R_+(\xi,\eta), R_-(\xi,\eta)]$ results again when the solution is compared with its Riemann invariance structure.

Genuine nonlinearity / linear degeneracy: some two-dimensional examples (III)

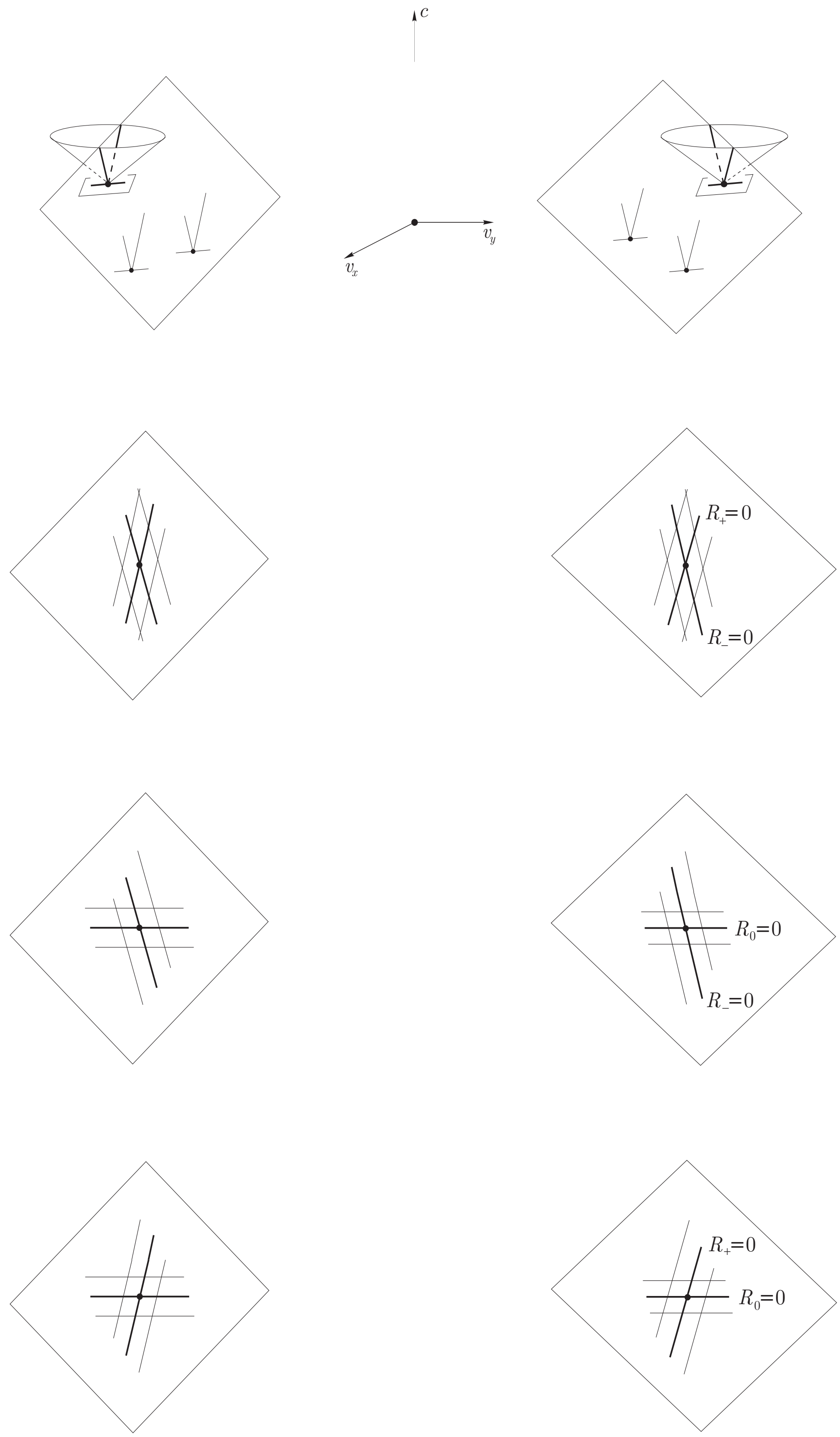


Hodograph details of the first significant solution



Physical details of the first significant solution

Genuine nonlinearity / linear degeneracy: some two-dimensional examples (IV)



Hodograph details of the second significant solution

Two-dimensional Riemann problem (I)

- The isentropic wave-wave interactions constructed parallel, from an ***analytic***, ***local*** and ***regular*** prospect, some details [interactions of simple waves solutions] of the Zhang and Zheng two-dimensional ***qualitative***, ***global*** and ***irregular*** construction.

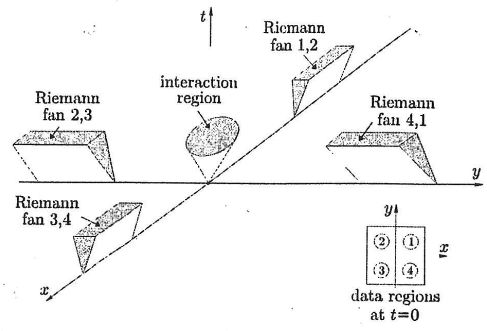
- The regular character reflects a *multidimensional* and *skew* construction generally. In Zhang and Zheng approach the contributing waves are one-dimensional and the interaction structure is *orthogonal*.

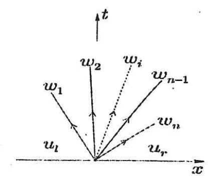
- To the self-similar form of the system $(1')$ we associate the *self-similar Mach number*:

$$\widetilde{M} = \frac{1}{c}\sqrt{(v_x - \xi)^2 + (v_y - \eta)^2}.$$

- For the two significant solutions mentioned above we compute $\widetilde{M} \equiv \sqrt{2} > 1$ and respectively $\widetilde{M} \equiv \text{constant} = \frac{2}{\sqrt{3-\gamma}} > 1$.

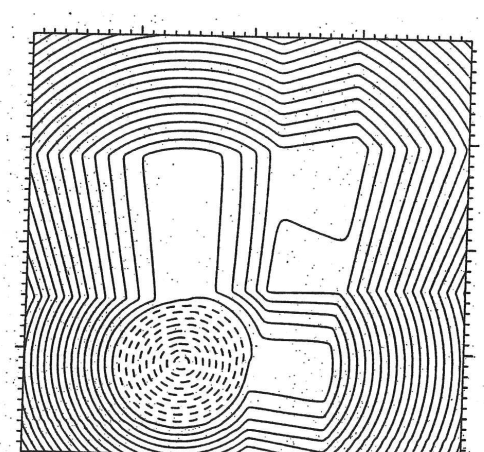
Two-dimensional Riemann problem (II)



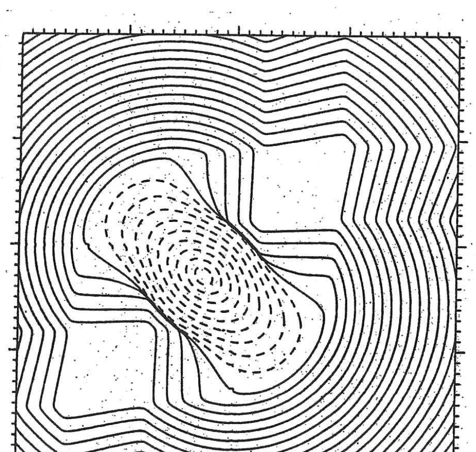


Configuration 1: $\vec{R}_{21}\vec{R}_{32}\vec{R}_{34}\vec{R}_{41}$

Configuration 2: $\vec{R}_{21}\overleftarrow{R}_{32}\overleftarrow{R}_{34}\vec{R}_{41}$



Config. 1: Self–sim. Mach no. (0.01 to 2.99)
29 contour lines: 0.10 to 2.90 step 0.10



Config. 2: Self–sim. Mach no. (0.01 to 2.95)
29 contour lines: 0.10 to 2.90 step 0.10

Martin type "differential" approach. An unsteady one-dimensional version

A particular anisentropic flow

A continuous [smooth] strictly adiabatic [anisentropic] flow results behind a shock discontinuity of non-constant continuous [smooth] velocity which penetrates into a region of uniform flow.

A class of solutions to describe such an anisentropic flow

We seek for solutions which fulfil the [natural] requirement [Stanyukovich]

$$\frac{\partial p}{\partial t}\frac{\partial \psi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \psi}{\partial t} \neq 0. \qquad (6)_{x,t}$$

A Monge–Ampère type representation

In the particular anisentropic context considered the entropy appears to be a prescribed function of ψ - which induce an algebraic relation between p, ρ, ψ throughout the adiabatic flow region. We select [Martin] p and ψ as new independent variables in place of x and t and find

$$\begin{gathered} v_x = v_x(p,\psi) = \frac{\partial \xi}{\partial \psi}, \;\; t = t(p,\psi) = \frac{\partial \xi}{\partial p}, \\ x = x(p,\psi) = \int \left(\frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p \partial \psi} + \frac{1}{\rho} \right) \mathrm{d}\psi + \left(\frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p^2} \right) \mathrm{d}p \end{gathered} \qquad (7)$$

where ξ must fulfil the hyperbolic Monge–Ampère equation

$$\frac{\partial^2 \xi}{\partial p^2}\frac{\partial^2 \xi}{\partial \psi^2} - \left(\frac{\partial^2 \xi}{\partial p \partial \psi} \right)^2 = -\zeta^2(p,\psi) \equiv \frac{\partial}{\partial p}\left(\frac{1}{\rho} \right) \equiv -\frac{1}{\rho^2 c^2} \qquad (8)$$

with $\rho = \rho(p,\psi)$ and $c(p,\psi) = \sqrt{\left(\frac{\partial \rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed.

On reversing $(7)_{2,3}$ into $p = p(x,t)$, $\psi = \psi(x,t)$ and carrying this into $(7)_1$ we get a form $p(x,t)$, $v_x(x,t)$, $\psi(x,t)$ of the corresponding anisentropic solution.

A pseudo isentropic behaviour. Correspondence of characteristics

For such a flow, entropy $S(p,\rho)$ is a function of ψ alone, $F(\psi)$, determined by the shock conditions. This results in a ***noncharacteristic*** character of the particle lines.

Martin type "differential" approach.
An unsteady one-dimensional version: appendix

Importance of a conservation form

We consider the gasdynamic system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho v_x)}{\partial x} = 0 \\ \dfrac{\partial(\rho v_x)}{\partial t} + \dfrac{\partial}{\partial x}\left(\rho v_x^2 + p\right) = 0 \\ \dfrac{\partial(\rho S)}{\partial t} + \dfrac{\partial(\rho v_x S)}{\partial x} = 0 \ , \quad S = S(p, \rho) \end{cases} \tag{1'}$$

(in usual notations: ρ, v_x, p, S are respectively the mass density, fluid velocity, pressure and entropy density).

Details on the representation (7)

We use, to begin with, the first two equations above to introduce [Martin], the functions $\psi, \tilde{\xi}$ and ξ cf.

$$\rho = \frac{\partial \psi}{\partial x},\ \rho v_x = -\frac{\partial \psi}{\partial t};\ \rho v_x = \frac{\partial \tilde{\xi}}{\partial x},\ \rho v_x^2 + p = -\frac{\partial \tilde{\xi}}{\partial t};\ \xi = \tilde{\xi} + pt.$$

We get

$$\mathrm{d}x = \frac{1}{\rho}\mathrm{d}\psi + v_x \mathrm{d}t, \quad \mathrm{d}\tilde{\xi} = v_x \mathrm{d}\psi - \rho \mathrm{d}t, \quad \mathrm{d}\xi = v_x \mathrm{d}\psi + t\mathrm{d}p.$$

which leads to

$$\frac{\partial x}{\partial \psi} = v_x \frac{\partial t}{\partial \psi} + \frac{1}{\rho},\ \frac{\partial x}{\partial p} = v_x \frac{\partial t}{\partial p},\ v_x = \frac{\partial \xi}{\partial \psi},\ t = \frac{\partial \xi}{\partial p}. \tag{7'}$$

On eliminating x from $(7')_{1,2}$ and taking $(7')_{3,4}$ into account it results that ξ must fulfil the hyperbolic Monge–Ampère equation (8).

Martin linearization. Riemann−Martin invariants

Some “differential” ***analogues*** of the simple waves solutions or wave-wave regular interaction solutions are available for $\zeta = P(p)\Psi(\psi)$ in (8) *in a few cases only* [Martin, Ludford]. For each of these cases to the equation (8) two intermediate integrals $\mathcal{F}_\pm\left(p,\psi,\xi,\frac{\partial\xi}{\partial p},\frac{\partial\xi}{\partial\psi}\right)$, ***linear*** in ξ, can be associated for which $\mathcal{F}_\pm$=constant$_\pm$=$R_\pm$ along a characteristic $\overline{\mathcal{C}}_\pm$. We have to distiguish, for each of the mentioned cases, between the circumstances (a) when $R_\pm$ depend on the characteristic $\overline{\mathcal{C}}_\pm$, and ($b$) when R_+ or R_- are overall constants.

• In the case (a) we may use $R_\pm$ as new independent variables. It can be shown, in this construction (a), that the entities p^{-1}, v_x, ψ^{-1}, t fulfil [Martin] various Euler−Poisson−Darboux ***linear*** equations to which well-known representations are associated; we present these representations by

$$p=p(R_+,R_-),\ \psi=\psi(R_+,R_-),\ v_x=v_x(R_+,R_-),\ t=t(R_+,R_-),\ x=x(R_+,R_-) \quad (9)$$

where $x(R_+,R_-)$ results by quadratures. Reversing $(9)_{4,5}$ into $R_\pm=R_\pm(x,t)$ will induce a form of solution (9), parallel to (5) [as $R_\pm$ have a characteristic nature]. We call $R_\pm(x,t)$ ***Riemann−Martin invariants***.

• In the case (b) we notice that a solution $\xi(p,\psi)$ of the ***linear*** equation $\mathcal{F}_+\equiv R_+$ or $\mathcal{F}_-\equiv R_-$ will automatically fulfil (8). We call such solution a ***pseudo simple waves solution***.

• A pseudo wave-wave interaction results by glueing together an element of construction (a) and four suitable elements of construction (b).

“Algebraic” approach and “differential” approach: first details of a parallel

At each point of the hodograph (9), the following relation is computed [Dinu−Dinu] between the Burnat characteristic directions $\vec{\kappa}$ and the Martin characteristic directions $\vec{\mu}$

$$\vec{\mu}_{\pm} = \left(\frac{\partial p}{\partial R_{\pm}}, \frac{\partial v_x}{\partial R_{\pm}}, \frac{\partial S}{\partial R_{\pm}} \right)^t = \eta_{\mp} \vec{\kappa}_{\mp} + \eta_0 \vec{\kappa}_0 ; \quad S(R_+, R_-) \equiv F[\psi(R_+, R_-)],$$

$$\eta_{\mp} = \frac{1}{\Lambda_{\mp}} \frac{\partial v_x}{\partial R_{\pm}}, \; \eta_0 = \frac{\partial S}{\partial R_{\pm}}.$$

• Representation (9) corresponds to an example of hodograph surface which ***is not*** a Burnat characteristic surface.

• Still, incidentally and essentially for the anisentropic linearized approach, this representation is associated with an example of hodograph surface for which a characteristic character ***persists in a Martin sense***.

Pseudo simple waves solution

In contrast with a *sws*:

- a *pseudo sws* has a two-dimensional hodograph [cf. (6)] and
- for it none of the characteristic fields $\mathcal{C}_\pm$ in the physical plane x,t is made of straightlines generally.

Example

To $\zeta=\frac{\psi^{\nu-1}}{p^{\nu+1}}$ two intermediate integrals $\mathcal{F}_\pm \equiv p\frac{\partial\xi}{\partial p}+\psi\frac{\partial\xi}{\partial\psi}-\xi\pm\frac{1}{\nu}\left(\frac{\psi}{p}\right)^\nu$ correspond. We satisfy $\mathcal{F}_+\equiv R_+=0$ by $\xi=\frac{1}{\nu}\left(\frac{\psi}{p}\right)^\nu$, $\nu=-\frac{\gamma-1}{2\gamma}$, integral ν, $\nu\neq 0,1$. We get ($\mathcal{C}_0$ are the particle lines)

$$|x|=K_-|t|^{k_-};\ K_-=\log\frac{|x^*|}{|t^*|^{k_-}} \text{ along } \mathcal{C}_-$$

$$|x|=K_+|t|^{k_+};\ K_+=\log\frac{|x^*|}{|t^*|^{k_+}} \text{ along } \mathcal{C}_+$$

$$|x|=K_0|t|^{k_0},\ K_0=\log\frac{|x^*|}{|t^*|^{k_0}} \text{ along } \mathcal{C}_0.$$

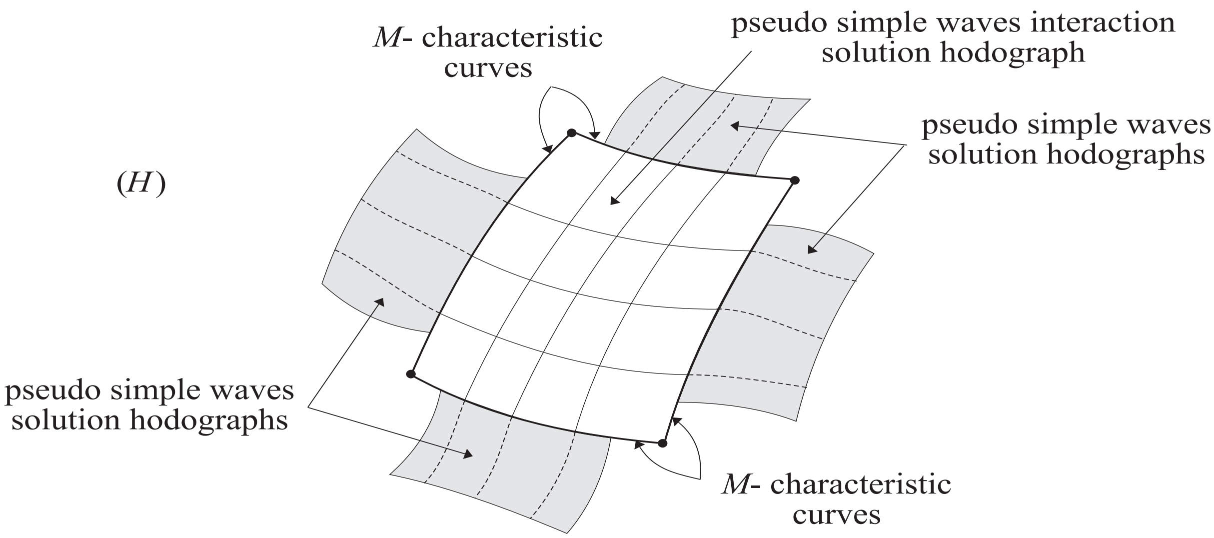
with

$$\begin{cases} k_-=\dfrac{2\nu}{\nu+1}=-2\dfrac{\gamma-1}{\gamma+1};\ \ -\dfrac{1}{2}<k_-<0 \ \text{ for } 1<\gamma<\dfrac{5}{3} \\ k_+=2; \\ k_0=\dfrac{2\nu+1}{\nu+1}=\dfrac{2}{\gamma+1};\ \ 0<k_0<1 \ \text{ for } 1<\gamma<\dfrac{5}{3}\ . \end{cases}$$

“Differential” linearized [pseudo] nondegeneracy

The image of a characteristic $\mathcal{C} \subset E$ on the hodograph of a solution of $(1')$ will be said to be a *M*- characteristic. The hodograph of a formal regular interaction of pseudo *sws* will be then made by glueing, along suitable *M*-characteristics, a hodograph (9) with some suitable hodographs of pseudo *sws*.

(*H*)



Martin type "differential" approach.
A steady supersonic version (I)

A particular anisentropic flow

A continuous [smooth] strictly adiabatic [anisentropic] *rotational* flow results behind a curved shock discontinuity from a region of uniform flow ahead.

A class of solutions to describe such an anisentropic flow

For such a particular flow we seek for solutions which fulfil the (natural) requirement

$$\frac{\partial p}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial \psi}{\partial x} \neq 0 \qquad (6)_{x,y}$$

A Monge–Ampère type representation of this solution

We select [Martin] p and ψ as new independent variables in place of x and y, and find

$$x = \frac{\partial \eta}{\partial p},\ y = -\frac{\partial \xi}{\partial p},\ v_x = \frac{\partial \xi}{\partial \psi},\ v_y = \frac{\partial \eta}{\partial \psi}, \qquad (10)$$

where ξ and η must fulfil the same Monge–Ampère equation

$$4\mathcal{F}\left[\left(\frac{\partial^2 \xi}{\partial p \partial \psi}\right)^2 - \frac{\partial^2 \xi}{\partial p^2}\frac{\partial^2 \xi}{\partial \psi^2}\right] - 4\left(\frac{\partial \xi}{\partial \psi}\frac{\partial \mathcal{F}}{\partial p}\right)\frac{\partial^2 \xi}{\partial p \partial \psi} + 2\left(\frac{\partial \xi}{\partial \psi}\frac{\partial \mathcal{F}}{\partial \psi}\right)\frac{\partial^2 \xi}{\partial p^2}$$

$$+\left\{\left(\frac{\partial \mathcal{F}}{\partial p}\right)^2 - 2\left[\mathcal{F} - \left(\frac{\partial \xi}{\partial \psi}\right)^2\right]\frac{\partial^2 \mathcal{F}}{\partial p^2}\right\} = 0 \qquad (11)$$

with $\rho = \rho(p,\psi)$ and $c(p,\psi) = \sqrt{\left(\frac{\partial \rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed, and we notice that ξ and η are connected by

$$v_x \frac{\partial y}{\partial \psi} - v_y \frac{\partial x}{\partial \psi} = \frac{1}{\rho},\quad v_x \frac{\partial y}{\partial p} - v_y \frac{\partial x}{\partial p} = 0. \qquad (10')$$

On reversing $(10)_{2,3}$ into $p = p(x,t)$, $\psi = \psi(x,t)$ and carrying this into $(10)_{3,4}$ we get a form $p(x,y)$, $v_x(x,y)$, $v_y(x,y)$, $\psi(x,y)$ of the corresponding anisentropic solution.

A pseudo isentropic behaviour

For such a flow, entropy $S(p,\rho)$ and the Bernoulli type function $e + \frac{p}{\rho} + \frac{1}{2}V^2$ [e is the density of the internal energy, $V^2 = v_x^2 + v_y^2$] are functions of ψ alone, $F(\psi)$, respectively $H(\psi)$, determined by the shock conditions. This results in a ***noncharacteristic*** character of the streamlines.

Martin type “differential” approach.
A steady two-dimensional version: appendix

Importance of a conservation form

We consider the gasdynamic system

$$\begin{cases} \dfrac{\partial(\rho v_x)}{\partial x}+\dfrac{\partial(\rho v_y)}{\partial y}=0 \\ \dfrac{\partial}{\partial x}\left(\rho v_x^2+p\right)+\dfrac{\partial}{\partial y}\left(\rho v_x v_y\right)=0 \\ \dfrac{\partial}{\partial x}\left(\rho v_x v_y\right)+\dfrac{\partial}{\partial y}\left(\rho v_y^2+p\right)=0 \\ \dfrac{\partial(\rho v_x S)}{\partial x}+\dfrac{\partial(\rho v_y S)}{\partial y}=0\ , \quad S=S(p,\rho) \end{cases}$$

Details on the representation (10)

We use the first three equations above to introduce [thus parallelling the unsteady one-dimensional case] the functions $\psi, \tilde{\xi}, \tilde{\eta}$ cf.

$$\rho v_x=\frac{\partial\psi}{\partial y},\ \rho v_y=-\frac{\partial\psi}{\partial x};\ \rho v_x v_y=-\frac{\partial\tilde{\xi}}{\partial x},\ \rho v_x^2+p=\frac{\partial\tilde{\xi}}{\partial y};\ \rho v_x v_y=\frac{\partial\tilde{\eta}}{\partial y},\ \rho v_y^2+p=-\frac{\partial\tilde{\eta}}{\partial x}$$

and the functions

$$\xi=\tilde{\xi}-py,\ \ \eta=\tilde{\eta}+px$$

to get

$$\mathrm{d}\psi=-(\rho v_y)\mathrm{d}x+(\rho v_x)\mathrm{d}y,\quad \mathrm{d}\xi=v_x\mathrm{d}\psi-y\mathrm{d}p,\quad \mathrm{d}\eta=v_y\mathrm{d}\psi+x\mathrm{d}p. \qquad (10^*)$$

use $(6)_{x,y}$ to select [thus parallelling the unsteady one-dimensional case] p and ψ as new independent variables in place of x and y, and compute (10) from $(10^*)_{2,3}$ and $(10')$ from $(10^*)_1$. We then transcribe $(10')$ via (10) cf.

$$\frac{\partial\xi}{\partial\psi}\frac{\partial^2\xi}{\partial p\partial\psi}+\frac{\partial\eta}{\partial\psi}\frac{\partial^2\eta}{\partial p\partial\psi}+\frac{1}{\rho(p,\psi)}=0,\quad \frac{\partial\xi}{\partial\psi}\frac{\partial^2\xi}{\partial p^2}+\frac{\partial\eta}{\partial\psi}\frac{\partial^2\eta}{\partial p^2}=0\ . \qquad (10^{**})$$

Finally we integrate $(10^{**})_1$ with respect to p and obtain

$$\left(\frac{\partial\xi}{\partial\psi}\right)^2+\left(\frac{\partial\eta}{\partial\psi}\right)^2=\mathcal{F}(p,\psi),\quad \frac{\partial\xi}{\partial\psi}\frac{\partial^2\xi}{\partial p^2}+\frac{\partial\eta}{\partial\psi}\frac{\partial^2\eta}{\partial p^2}=0\,;\ \ \mathcal{F}(p,\psi)=2\mathcal{G}(\psi)-2\int^p\frac{\mathrm{d}p}{\rho} \qquad (10^{***})$$

where $\mathcal{F}$ is determined by the shock conditions. We solve simultaneously for $\frac{\partial\eta}{\partial\psi}$ and $\frac{\partial^2\eta}{\partial p^2}$ in (10^{***}) and carry the result into $\frac{\partial^2}{\partial p^2}\left(\frac{\partial\eta}{\partial\psi}\right)=\frac{\partial}{\partial\psi}\left(\frac{\partial^2\eta}{\partial p^2}\right)$ in order to eliminate η in favor of ξ. We are led to (11) for ξ. A given solution ξ of (11) is paired by a computed [cf. (10^{***})] solution η of (11).

Martin type "differential" approach.
A steady two-dimensional supersonic version (II)

The hodograph characteristics. The physical characteristics

The characteristic directions for (11) in the plane p, ψ are given by

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\psi}\right)_{\pm} = \frac{2\mathcal{F}\frac{\partial^2 \xi}{\partial p \partial \psi} - \frac{\partial \mathcal{F}}{\partial p}\frac{\partial \xi}{\partial \psi} \pm \sqrt{\Delta}}{-2\mathcal{F}\frac{\partial^2 \xi}{\partial p^2}}, \quad \Delta = \frac{4}{\rho^2 c^2} v_y^2 \left(V^2 - c^2\right), \quad V^2 = v_x^2 + v_y^2 \, . \tag{12}$$

We have

$$\begin{aligned} -V^2 \mathrm{d}y + \frac{1}{\rho c}\left[c v_x \pm v_y \sqrt{V^2 - c^2}\right] \mathrm{d}\psi &= 0 \\ v_y \left[v_y \mathrm{d}v_x - v_x \mathrm{d}v_y \mp \frac{1}{\rho c}\sqrt{V^2 - c^2}\mathrm{d}p\right] &= 0 \end{aligned} \quad \text{along the characteristics } \overline{\mathcal{C}}_{\pm}. \tag{13}$$

In contrast with the unsteady one-dimensional case, the steady anisentropic description and the Monge−Ampère equation (11) show an ***elliptic-hyperbolic*** character generally. Still, cf. (12), they show both a ***hyperbolic*** character for a ***supersonic*** flow.

The Mach lines $\mathcal{C}_{\pm}$ of the steady anisentropic description in the physical plane and the characteristics $\overline{\mathcal{C}}_{\pm}$ [(12)] of the Monge−Ampère equation (11) are in correspondence. In fact, we get from $(13)_1$ and $(10)_1$

$$-V^2 \mathrm{d}y + \frac{1}{\rho c}\left[c v_x \pm v_y \sqrt{V^2 - c^2}\right] (v_x \mathrm{d}y + v_y \mathrm{d}x) = 0 \quad \text{along the characteristics } \overline{\mathcal{C}}_{\pm}$$

which results in

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c v_x \pm v_y \sqrt{V^2 - c^2}}{c v_y \mp v_x \sqrt{V^2 - c^2}} = \frac{v_x v_y \pm c\sqrt{V^2 - c^2}}{v_x^2 - c^2} = \lambda_{\pm} \quad \text{along the characteristics } \overline{\mathcal{C}}_{\pm}$$

where λ_+ and λ_- are the Mach eigenvalues. This aspect pairs an one-dimensional detail. □

A pseudo isentropic behaviour. Correspondence of characteristics

Martin linearization

Details of the steady Martin linearization are in progress (Dinu-Dinu).

Final remarks

- Finding, or even structuring, a solution to the hyperbolic system (1) or, alternatively, to the hyperbolic Monge–Ampère type equations (8) / (11) is a ***hard task*** generally.

- This suggests to constructively considering ***suitable classes*** of solutions to this system or, alternatively, to the mentioned Monge–Ampère type equations.

- The two compared constructions [“algebraic”, “differential”] are essentially initiated by ***intermediate*** elements [the system (3) and, respectively, the Martin linearized structure].

- The regular passage [which uses the two mentioned pairs of classes] from an isentropic description to an anisentropic description appears to be ***fragile***. This aspect is suggested by noticing “few cases of Martin linearization”, reporting “complementary restrictions concerning the Martin linearization”], indicating a *particular* character of the anisentropic flow considered, or considering dimensional restrictions (two independent variables) and a conservative form – in contrast with Burnat’s availability].

References

- M. Burnat, "The method of characteristics and Riemann's invariants for multidimensional hyperbolic systems", *Sibirsk. Math. J.*, vol. 11, pp. 279−309, 1970.
- L.F. Dinu, "Multidimensional wave-wave regular interactions and genuine nonlinearity: some remarks", *Preprint Series of Newton Institute for Math. Sci.*, Cambridge UK, No.29, 2006.
- L.F. Dinu and M.I. Dinu, "Nondegeneracy, from the prospect of wave-wave regular interactions of a gasdynamic type: some remarks", *Preprint Series of Newton Institute for Math. Sci.*, Cambridge UK, No.55, 2007.
- L.F. Dinu and M.I. Dinu, "Martin's "differential" approach: some classifying remarks", *Preprint Series of Newton Institute for Math. Sci.*, Cambridge UK, No.21, 2009.
- L. F. Dinu and M. I. Dinu, "Gasdynamic regularity: some classifying geometrical remarks", *Balkan Journal of Geometry and Its Applications*, vol. 15, pp. 41-52, 2010.
- L. F. Dinu, "Wave-wave interactions of a gasdynamic type", preprint (2012), to appear on *Communications in Math. Analysis.*
- E.V. Ferapontov and K.R. Khusnutdinova, "On integrability of (2+1)-dimensional quasilinear systems", *Commun. Math. Phys.*, vol. 248, pp. 187−206, 2004.
- P.D. Lax, "Hyperbolic systems of conservation laws (II)", *Communications on Pure Appl. Math.*, vol. 10, pp. 537−566, 1957.
- M.H. Martin, "The Monge–Ampère partial differential equation $rt - s^2 + \lambda^2 = 0$", *Pacific J. Math.*, vol. 3, pp. 165−187, 1953.
- Z. Peradzyński, "Nonlinear plane k-waves and Riemann invariants", *Bull. Acad. Polon. Sci., Ser. Sci. Tech.*, vol. 19, pp. 625−631, 1971.
- Z. Peradzyński, "Riemann invariants for the nonplanar k-waves", *Bull. Acad. Polon. Sci., Ser. Sci. Tech.*, vol. 19, pp. 717−724, 1971.
- K.P. Stanyukovich, Unsteady flows of continua, Gostechizdat, 1955 [in Russian].
- T. Zhang and Y.-X. Zheng, "Conjecture on structure of solutions of Riemann problem for two-dimensional gas dynamics", *SIAM J. Math. Anal.*, vol. 21, pp. 593−630, 1990.