Entropy solutions via JKO scheme for a class of degenerate convection-diffusion equations

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Table of contents

Table of contents

Motivation

For some evolutionary PDEs, the following two concept of solutions can be formulated:

- The notion of *Wasserstein gradient flow* (De Giorgi, Otto, Ambrosio - Gigli - Savaré), which can be introduced every time you deal with

$$\frac{\partial \rho}{\partial t} = \mathrm{div}\left(\rho \nabla \frac{\delta \mathcal{F}[\rho]}{\delta \rho}\right)$$

  - $\mathcal{F}[\rho] = \int \rho \log \rho dx$ gives $\rho_t = \Delta \rho$,
  - $\mathcal{F}[\rho] = \int \rho W * \rho dx$ gives $\rho_t = \mathrm{div}(\rho \nabla W * \rho)$,
  - $\mathcal{F}[\rho] = \frac{1}{m-1} \int \rho^m dx$ gives $\rho_t = \Delta \rho^m$, $m > 1$.
- The notion of *Entropy solution* (Oleinik, Kružkov), which classically applies to
  - nonlinear conservation laws $\rho_t + f(\rho)_x = 0$,
  - nonlinear convection diffusion equations $\rho_t = g(\rho)_{xx} + f(\rho)_x$.

Significant examples

- Nonlocal interaction equations with nonlinear diffusion

$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho \nabla G * \rho) = 0$$

with $m > 1$ and $G \in C^2$ and G even. Here, both notions have been used (almost at the same time!) to prove uniqueness of solutions.

- Scalar conservation laws with symmetric inhomogeneous coefficient

$$\partial_t \rho + \partial_x (m(\rho) \nabla V(x)) = 0$$

with m concave, $V \in C^2$ and V even. Here, the interpretation as a gradient flow is still partially open, whereas it is (obviously) clear that you can obtain unique entropy solutions under reasonable assumptions.

Our model

Nonlinear convection diffusion equation with space dependent coefficient:

$$\partial_t u = (u^m)_{yy} + (b(y)u^m)_y, \quad y \in \mathbb{R}, \quad t \geq 0. \tag{1}$$

- Porous medium exponent: $m > 1$.
- Assumptions on b:

$$b \in L^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) \tag{2}$$

- Initial condition $u^0 \in L^m(\mathbb{R})$, nonnegative with finite second moment.

Important remark

Here the gradient flow theory will not give you a unique solution for free, whereas the literature [Karlsen-Risebro 2003] already contains a uniqueness result for entropy solutions.

The scaling

Problem: How to see the gradient flow structure for (1)?
Answer: There exists a function a

(a1) $a(x) \geq \underline{a} > 0$ for all $x \in \mathbb{R}$,

(a2) $a \in W^{2,\infty}(\mathbb{R})$,

and a bijective *change of coordinates* $y = T_a(x)$ on $\mathbb{R}$ such that, for a given weak solution $u(y,t)$ to (1) with initial datum $u_0 \in L^1(\mathbb{R})$, the scaled function

$$\rho(x,t) := T_a'(x)u(T_a(x),t)$$

satisfies $\rho_0(x) := \rho(x,0) = u(x,0)$ and the equation

$$\partial_t \rho = \left(\rho[a(x)\rho^{m-1}]_x\right)_x, \tag{3}$$

Intuitive idea behind the scaling

Start from (3). Substitute $\rho(x,t) := T_a'(x)u(T_a(x),t)$, to get an equation for $u(y,t)$:

$$\partial_t u(y,t) = \Big((m-1)(T_a'(x))^{m+1}a(x)u^{m-1}u_y + (T_a'(x))^m a'(x)u^m\Big)_y\Big|_{x=T^{-1}(y)}.$$

Choose

$$T_a(x) = \int_0^x \left[\frac{m-1}{m}a(\xi)\right]^{-\frac{1}{m+1}} \mathrm{d}\xi. \qquad (4)$$

to make the diffusion coefficient homogeneous. We obtain (1).
Notation: $u(y,t) = \mathbf{S}_a[\rho(\cdot,t)](y)$.

Scaling: from b to a

The rigorous definition of the new coefficient $a(x)$ for fixed $b \in L^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. Set

$$\alpha(y) = \alpha_0 \exp\left(-\frac{(m-1)}{2m}\int_0^y b(\eta)d\eta\right).$$

Notice that $\alpha \in W^{2,\infty}(\mathbb{R})$ and $\alpha \geq \underline{\alpha} > 0$. Consider the unique solution to the Cauchy problem

$$\begin{cases} T'(x) = \alpha(T(x)), & x \in \mathbb{R} \\ T(0) = 0, \end{cases} \tag{5}$$

Finally, define a as follows

$$a(x) := \frac{m}{m-1}\left(\alpha(T(x))\right)^{-(m+1)}. \tag{6}$$

a satisfies: $a \in W^{2,\infty}(\mathbb{R})$ and $a(x) \geq \underline{a} > 0$.

To fix ideas:

Throughout this talk we shall always think of ρ and u being $1:1$-related via the scaling

$$u(y,t) = \mathbf{S}_a[\rho(\cdot,t)](y).$$

- $\rho(x,t)$ should be thought as a solution to (3)

$$\partial_t \rho = \left(\rho[a(x)\rho^{m-1}]_x\right)_x,$$

  for which we can use a *gradient flow (JKO)* approach,
- u should be thought as a solution to (1)

$$\partial_t u = (u^m)_{yy} + (b(y)u^m)_y, \quad y \in \mathbb{R}, \quad t \geq 0,$$

  for which we can use the *entropy solution* approach.

Entropy solutions for (1)

Following [Carrillo M. - ARMA 1999] and [Karlsen, Risebro - DCDS 2003]:

Definition (Definition of entropy solution)

Let $u^0 \in L^1 \cap L^\infty(\mathbb{R})$, and let $T > 0$. A non-negative measurable function $u :]0, T[\times\mathbb{R} \to \mathbb{R}$ is an *entropy solution* to (1) with initial condition u^0 if $u \in L^1 \cap L^\infty(]0, T[\times\mathbb{R})$, if $u^m \in L^2(]0, T[; H^1(\mathbb{R}))$, if $u(t) \to u^0$ in $L^1(\mathbb{R})$ as $t \downarrow 0$, and if inequality

$$\int_0^T \int_{\mathbb{R}} |u-k|\varphi_t \, \mathrm{d}y \, \mathrm{d}t \geq \int_0^T \int_{\mathbb{R}} \mathrm{Sgn}(u^m - k^m)\Big(\big[(u^m)_y + b(u^m - k^m)\big]\varphi_y - b_y k^m \varphi\Big) \, \mathrm{d}y \, \mathrm{d}t.$$

is satisfied for all nonnegative test functions $\varphi \in C_c^\infty(]0, T[\times\mathbb{R})$, and for all $k \in \mathbb{R}_+$.

Construction of entropy solutions

In the literature there are many methods used to construct entropy solutions. Some of them do not apply to our model. Here is a (possibly incomplete) list:

- Vanishing viscosity approach
- Wave front tracking algorithm
- Glimm's scheme
- Semigroup approach (implicit Euler)

We shall add another method to the list, namely:

- JKO - De Giorgi scheme for ρ $\quad\Rightarrow\quad$ scaling $u = \mathbf{S}_a[\rho]$

The JKO scheme for (3)

$$\mathcal{F}[\rho] := \frac{1}{m}\int_{\mathbb{R}} a(x)\rho^m \, \mathrm{d}x.$$

For a time step $\tau > 0$,

$$\mathcal{F}_\tau(\rho;\sigma) = \frac{1}{2\tau}\mathbf{W}_2(\rho,\sigma)^2 + \mathcal{F}(\rho),$$

where $\mathbf{W}_2(\rho,\sigma)$ is the 2-Wasserstein distance.

1. $\rho_\tau^0 := \rho^0$.
2. For $n \geq 1$, let $\rho_\tau^n \in \mathcal{P}_2(\mathbb{R})$ be the (unique global) minimizer of $\mathcal{F}_\tau(\cdot;\rho_\tau^{n-1})$.

Piecewise constant interpolation:

$$\bar{\rho}_\tau(t) = \rho_\tau^n \quad \text{for } (n-1)\tau < t \leq n\tau.$$

Main result

Theorem (DF, Matthes - (almost preprint))

Let $\rho^0 \in \mathcal{P}_2(\mathbb{R}) \cap L^\infty(\mathbb{R})$*, and let a as above. Then every vanishing sequence* $(\tau_k)_{k\in\mathbb{N}}$ *of time steps contains a subsequence (not relabeled) such that the* $\bar{\rho}_{\tau_k}$ *converge to a curve* $\rho_* : [0,\infty[\to \mathcal{P}_2(\mathbb{R})$ *— in* $L^m(]0,T[\times\mathbb{R})$*, and also uniformly in* $\mathbf{W}_2$ *on each time interval* $[0,T]$ *— that is continuous with respect to* $\mathbf{W}_2$*. The rescaled function* $u_* = \mathbf{S}_a[\rho_*]$ *is an entropy solution to* (1) *in the sense of Definition 1, with initial condition* $u^0 = \mathbf{S}_a[\rho^0]$*.*

Corollary

Since the entropy solution $u_* = \mathbf{S}_a[\rho_*]$ *to* (1) *is unique, then the JKO scheme for* $\mathcal{F}$ *has a unique limit point.*

Remarks

- A similar connection has been pointed out for instance by [Gigli-Otto 2010] in the context of the inviscid Burger's equation. The interpretation of the coincidence between entropy solutions and gradient flows is that both types of solutions can be characterized by diminishing an underlying entropy functional "as fast as possible".
- Apart from revealing an interesting connection between two different theories for nonlinear evolution equations, our simple example indicates a possible general strategy to prove existence and uniqueness of certain degenerate parabolic equations. First, identify a variational structure behind the equation and use methods from the calculus of variations to construct a candidate for a solution; a priori, there might be several. Second, show that this candidate is an entropy solution by deriving further a priori estimates in the variational framework. Third, conclude uniqueness of the entropy solution.

Table of contents

A crash course on Wasserstein gradient flows[1]

Let $\mathcal{F}$ be a functional on the space $\mathbb{P}_2$ of probability measures with finite second moment, such that the JKO scheme

$$\rho_\tau^{n+1} = \mathrm{argmin}\left\{\frac{1}{2\tau}\mathbf{W}_2{}^2(\rho_\tau^n,\rho) + \mathcal{F}[\rho],\ \rho\in\mathbb{P}_2\right\}, \quad \rho_\tau^0 \text{ given}$$

is well defined, let $\bar{\rho}_\tau$ be the (left continuous) time piecewise constant interpolation of the sequence ρ_τ^n. If $\mathcal{F}$ satisfies the κ-*convexity* property

- $\mathcal{F}[\rho(t)] \leq (1-t)\mathcal{F}[\rho(0)] + t\mathcal{F}[\rho(1)] - \frac{\kappa}{2}t(1-t)\mathbf{W}_2(\rho(0),\rho(1))^2$ on the $\mathbf{W}_2$-geodesic $\rho(t)$ with constant speed connecting $\rho(0)$ to $\rho(1)$, for some $\lambda\in\mathbb{R}$

(plus some minor extra assumptions), then $\bar{\rho}_\tau$ has a unique limit point which is also an *energy* solution to the PDE

$$\partial_t\rho = \mathrm{div}\left(\rho\nabla\frac{\delta\mathcal{F}}{\delta\rho}\right). \qquad (7)$$

[1] [Jordan-Kinderlehrer-Otto, 1997], [Otto, 2001], and [Ambrosio-Gigli-Savaré, 2005]

Evolution variational inequality (EVI)

In [Daneri-Savaré, 2008], the κ convexity is proven to be 'basically' equivalent to the following property:

$\mathcal{F}$ admits a uniquely determined κ-flow $\mathfrak{S}^s_{\mathcal{F}}$, i. e. a semigroup $\mathfrak{S}_{\mathcal{F}} : [0, +\infty) \times \mathbb{P}_2$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\sigma}\Big|_{\sigma=s} \mathbf{W}_2\left(\mathfrak{S}^{\sigma}_{\mathcal{F}}[\rho], \eta\right)^2 + \frac{\kappa}{2}\mathbf{W}_2(\mathfrak{S}^s_{\mathcal{F}}[\rho], \eta)^2 \leq \mathcal{F}(\eta) - \mathcal{F}(\mathfrak{S}^s_{\mathcal{F}}[\rho]) \quad (8)$$

for all $\rho, \eta \in \mathbb{P}_2$.

- The candidate κ-flow is the PDE (7),
- By a density argument, it is sufficient to uniquely determine the *k*-flow and to prove the above inequality (8) for *smooth* initial conditions.

k-convexity for inhomogeneous free energies

Consider a general *inhomogeneous* free energy functional of the form

$$\Psi(\eta) = \int_{\mathbb{R}} F(x, \eta(x))\,\mathrm{d}x \tag{9}$$

for a given function $F : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$.

Problem: We aim to determine a reasonable sufficient condition on F such that Ψ admits a κ-flow for some $\kappa \in \mathbb{R}$.

For a concise formulation of that condition, we introduce the adjoint function $H : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$H(x, \xi) = \xi F(x, 1/\xi). \tag{10}$$

Remark: In dimension d, when F is independent of x, $\mathcal{F}$ is k-convex if and only if the McCann condition $m > (d-1)/d$ is satisfied. If this is the case, then $k \leq 0$.

k-convexity for inhomogeneous free energies

Lemma

(Con) *Assume that there is some* $\kappa \in \mathbb{R}$ *such that*

$$(x,\xi) \mapsto H(x,\xi) - \frac{\kappa}{2}x^2 \tag{11}$$

is (jointly) convex on $\mathbb{R} \times \mathbb{R}_+$.

(Reg) *Assume further that there exist two positive constants* c *and* C *such that* $\eta F_{\eta\eta}(x,\eta) \geq c$, $F_{\eta x}(x,\eta) \leq C$, *and* $\eta F_{\eta\eta}(x,\eta) \leq C$ *for all* $(x,\eta) \in \mathbb{R} \times [0,+\infty)$.

Then, the semigroup $\mathfrak{S}_\Psi$ *of the evolution equation*

$$\partial_t \eta = \mathrm{D}_x(\eta \mathrm{D}_x[F_\eta(x,\eta)]). \tag{12}$$

is a κ*-flow for the functional* Ψ *from* (9).

Sketch of the proof

We need to prove the EVI

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathbf{W}_2(\mathfrak{S}_\Psi^t\eta_0,\tilde{\eta})^2+\frac{\kappa}{2}\mathbf{W}_2(\eta_0,\tilde{\eta})^2\le\Psi(\tilde{\eta})-\Psi(\eta_0) \tag{13}$$

for all smooth and strictly positive $\eta_0,\tilde{\eta}\in\mathcal{P}_2(\mathbb{R})$. $\eta(t;x):=\mathfrak{S}_\Psi^t\eta_0(x)$.

- Assumption (Reg) is technical (approximation).
- Define the distribution function

$$U(t;x)=\int_{-\infty}^{x}\eta(t;y)\,\mathrm{d}y$$

  which is a smooth diffeomorphism from $\mathbb{R}$ to $]0,1[$, for every $t\in\mathbb{R}_+$ with *inverse* $G:[0,\infty[\times]0,1[\to\mathbb{R}$.
- A tricky calculation gives

$$\partial_t G=\mathrm{D}_\omega[H_\xi(G,\partial_\omega G)]-H_x(G,\partial_\omega G). \tag{14}$$

- Observe that a change of variables $\omega=U(x)$ leads to

$$\Psi(\eta)=\int_0^1 H(G(\omega),\partial_\omega G(\omega))\,\mathrm{d}\omega.$$

Sketch of the proof

Recall the 1-d representation of $\mathbf{W}_2(\eta,\tilde{\eta})^2$ in terms of the respective inverse distribution functions G, $\tilde{G}$:

$$\mathbf{W}_2(\eta,\tilde{\eta})^2 = \int_0^1 [G(\omega) - \tilde{G}(\omega)]^2 \,\mathrm{d}\omega. \tag{15}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathbf{W}_2(\eta,\tilde{\eta})^2 + \frac{\kappa}{2}\mathbf{W}_2(\eta,\tilde{\eta})^2 \\
&= \int_0^1 G_t[G - \tilde{G}]\,\mathrm{d}\omega + \frac{\kappa}{2}\int_0^1 [G - \tilde{G}]^2\,\mathrm{d}\omega \\
&= \int_0^1 H_\xi(G, G_\omega)[\tilde{G}_\omega - G_\omega]\,\mathrm{d}\omega \\
&\quad + \int_0^1 H_x(G, G_\omega)[\tilde{G} - G]\,\mathrm{d}\omega + \frac{\kappa}{2}\int_0^1 [\tilde{G} - G]^2\,\mathrm{d}\omega \\
&\overset{(Con)}{\leq} \int_0^1 \left[H(\tilde{G}, \tilde{G}_\omega) - H(G, G_\omega)\right]\mathrm{d}\omega = \Psi(\tilde{\eta}) - \Psi(\eta).
\end{aligned}$$

$\mathcal{F}$ is not κ-convex

Lemma 4 gives an indication that the functional $\mathcal{F}$ is *not* geodesically κ-convex. Indeed, with $F(x,\eta) = a(x)\eta^m$, we have $H(x,\xi) = a(x)\xi^{1-m}$, and thus

$$\mathrm{D}^2 H(x,\xi) = \begin{pmatrix} a''(x)\xi^{1-m} - \kappa & -(m-1)a'(x)\xi^{-m} \\ -(m-1)a'(x)\xi^{-m} & m(m-1)a(x)\xi^{-(m+1)}. \end{pmatrix}$$

Unless a is a constant, there must exist an $\bar{x} \in \mathbb{R}$ with $a''(\bar{x}) < 0$. Thus, no matter how κ is chosen, there exists further a sufficiently small $\bar{\xi} \in \mathbb{R}_+$ such that the top left element of $\mathrm{D}^2 H(\bar{x},\bar{\xi})$ is negative, that is $a''(\bar{x})\bar{\xi}^{-(m-1)} - \kappa < 0$.

Table of contents

Motivation

MAIN IDEA TO CONSTRUCT ENTROPY SOLUTIONS:

- For a given test function $\varphi \geq 0$, we somewhat need to estimate $\Psi = \int_{\mathbb{R}} |u - k| \varphi dx$ along solutions to (1), in a discrete sense (JKO approximation).
- To fix ideas, assume first φ does not depend on time.
- Consider the 'smoothed' version $\Psi = \int_{\mathbb{R}} |u - k|_\epsilon \varphi dx + \nu \mathbf{H}(u)$ with $\mathbf{H}(u) = \int_{\mathbb{R}} u \log u dx$.
- For technical reasons $|u - k|_\epsilon$ should be replaced by $S_\epsilon(u) := \int_k^u \mathrm{Sgn}_\epsilon(r^m - k^m)\, \mathrm{d}r$.
- Put everything into the $\rho(x, t)$ variables (apply the scaling!):

$$\Psi_{\nu,\epsilon}(\rho) := \int_{\mathbb{R}} S_\epsilon\Big(\frac{\eta(x)}{T_a'(x)}\Big) \varphi \circ T_a(x)\, T_a'(x)\, \mathrm{d}x + \nu \mathbf{H}(\eta),$$

Heuristics

We then want to find a way to control the auxiliary functional $\Psi_{\nu,\epsilon}$ along the solution ρ to the (3). We explain the idea with the following toy model: consider two gradient flows in $\mathbb{R}^N$

$$\dot{X}(t) = -\nabla F(X(t))$$
$$\dot{Y}(s) = -\nabla G(Y(s))$$

The evolution of F along $Y(s)$ is given by

$$\frac{d}{ds}F(Y(s)) = \nabla F(Y(s)) \cdot \dot{Y}(s) = -\nabla F(Y(s)) \cdot \nabla G(Y(s)), \qquad (16)$$

and the evolution of G along $X(t)$ is given by

$$\frac{d}{dt}G(X(s)) = \nabla G(X(t)) \cdot \dot{X}(t) = -\nabla G(X(t)) \cdot \nabla F(X(t)), \qquad (17)$$

and the two functionals on the r.h.s. of (16) and (17) coincide.

Flow interchange lemma[2]

Lemma

Let $\Psi : \mathcal{P}_2(\mathbb{R}) \to]-\infty, +\infty]$ be a lower semi-continuous functional on $\mathcal{P}_2(\mathbb{R})$ with associated κ-flow $\mathfrak{S}_\Psi$. Define further the dissipation of $\mathcal{F}$ along $\mathfrak{S}_\Psi$ by

$$\mathfrak{D}_\Psi(\rho) := \limsup_{s\downarrow 0} \frac{1}{s}\left[\mathcal{F}(\rho) - \mathcal{F}\left(\mathfrak{S}_\Psi^s \rho\right)\right]$$

for every $\rho \in \mathcal{P}_2(\mathbb{R})$. If ρ_τ^{n-1} and ρ_τ^n are two consecutive steps of the JKO scheme for $\mathcal{F}$, then

$$\Psi(\rho_\tau^{n-1}) - \Psi(\rho_\tau^n) \geq \tau \mathfrak{D}_\Psi(\rho_\tau^n) + \frac{\kappa}{2}\mathbf{W}_2(\rho_\tau^n, \rho_\tau^{n-1})^2. \tag{18}$$

In particular, $\Psi(\rho_\tau^{n-1}) < \infty$ implies $\mathfrak{D}_\Psi(\rho_\tau^n) < \infty$.

[2] [Matthes-McCann-Savaré, 2009]

Flow interchange: corollary

Corollary

Let the κ-flow $\mathfrak{S}_\Psi$ be such that for every $n \in \mathbb{N}$, the curve $s \mapsto \mathfrak{S}_\Psi^s \rho_\tau^n$ lies in $L^m(\mathbb{R})$, where it is differentiable for $s > 0$ and continuous at $s = 0$. And moreover, let a functional $\mathfrak{K} : \mathcal{P}_2(\mathbb{R}) \to]-\infty, \infty]$ satisfy

$$\liminf_{s \downarrow 0} \left(- \frac{\mathrm{d}}{\mathrm{d}\sigma}\Big|_{\sigma=s} \mathcal{F}(\mathfrak{S}_\Psi^\sigma \rho_\tau^n) \right) \geq \mathfrak{K}(\rho_\tau^n). \tag{19}$$

Then the following two estimates hold.

$$\text{For every } n \in \mathbb{N}: \quad \Psi(\rho_\tau^{n-1}) - \Psi(\rho_\tau^n) \geq \tau \mathfrak{K}(\rho_\tau^n) + \frac{\kappa}{2} \mathbf{W}_2(\rho_\tau^n, \rho_\tau^{n-1})^2; \tag{20}$$

$$\text{for every } N \in \mathbb{N}: \quad \Psi(\rho_\tau^N) \leq \Psi(\rho^0) - \tau \sum_{n=1}^{N} \mathfrak{K}(\rho_\tau^n) + \tau \max(0, -\kappa) \mathcal{F}(\rho^0). \tag{21}$$

Table of contents

Compactness of the scheme

Lemma

There is a constant A depending only on ρ^0 (and in particular not on τ) such that the piecewise constant interpolations $\bar{\rho}_\tau$ satisfy

$$\left\|\bar{\rho}_\tau^{m/2}\right\|_{L^2(0,T;H^1(\mathbb{R}))} \leq A(1+T) \quad \text{for all } T>0. \tag{22}$$

Proof: use the flow interchange lemma with auxiliary functional

$$\mathbf{H}(\rho) = \int \rho(x)\log\rho(x)\,\mathrm{d}x.$$

$$\partial_s \mathcal{F}(\mathfrak{S}^s_{\mathbf{H}}\eta_0) \leq -\frac{4(m-1)\underline{a}}{m^2}\int_{\mathbb{R}}\left[\partial_x\left(\eta(s,x)^{m/2}\right)\right]^2\mathrm{d}x + \frac{\overline{a_{xx}}}{m}\int_{\mathbb{R}}\eta(s,x)^m\,\mathrm{d}x.$$

Arguments from [Ambrosio-Gigli-Savaré] combined with arguments taken from [Rossi-Savaré, 2003] give *strong* $L^m([0,T]\times\mathbb{R})$ *compactness*.

Derivation of the entropy inequality

The functional $\Psi_{\nu,\epsilon}$ generates a unique κ-flow for some $\kappa \in \mathbb{R}$, as it satisfies the assumptions (Con) and (Reg) above. Moreover, the k-flow is given by (3). Therefore, the above corollary implies that we can estimate $\Psi_{\nu,\epsilon}$ along the interpolation $\bar{\rho}_\tau$. By sending $\tau \to 0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}(\mathfrak{S}^s_{\Psi_\nu}\eta_0) \le \int_{\mathbb{R}} \Big(\big[(v^m)_y + b(v^m - k^m)\big]\phi_y - k^m b_y \phi \Big) \operatorname{Sgn}_\epsilon(v^m - k^m)\,\mathrm{d}y$$
$$- \int_{\mathbb{R}} \big[\big(H_\epsilon(v)\big)_y\big]^2 \phi\,\mathrm{d}y - \int_{\mathbb{R}} b\big[R_\epsilon(v^m - k^m)\big]_y \phi\,\mathrm{d}y - \nu \mathfrak{K}_{\mathbf{H}}(\eta),$$

- we changed variables $(t,x) \mapsto (t,y)$,
- $H_\epsilon, R_\epsilon : \mathbb{R}_+ \to \mathbb{R}$ are smooth and such that $H_\epsilon'(s)^2 = \operatorname{Sgn}_\epsilon'(s)$ and $R_\epsilon'(s) = s \operatorname{Sgn}_\epsilon'(s)$,
- $$\mathfrak{K}^{\mathbf{H}}(\rho) := \frac{4(m-1)\underline{a}}{m^2} \int_{\mathbb{R}} \big[\partial_x\big(\rho^{m/2}\big)\big]^2\,\mathrm{d}x - \frac{\overline{a_{xx}}}{m} \int_{\mathbb{R}} \rho^m\,\mathrm{d}x$$

Time dependent version

Consider now the time dependent test function $\varphi(t,x) = \theta(t)\phi(x)$ with $\theta \in C_c^\infty(]0,T[)$. Multiply (20) by $\theta(n\tau)$ and sum over n to find

$$\tau \sum_{n=1}^{\infty} \Psi_\nu(\rho_\tau^n) \frac{\theta(n\tau) - \theta((n+1)\tau)}{\tau} \geq \tau \sum_{n=1}^{\infty} \psi(n\tau)\mathfrak{K}(\rho_\tau^n) + \kappa_\nu \tau \mathcal{F}(\rho^0).$$

Thanks to the strong convergence of $\bar{\rho}_\tau$ to ρ_* in $L^m(\mathbb{R})$ and the lower semi-continuity of $\mathfrak{K}$, we can pass to the time-continuous limit and find

$$\int_0^T \Psi_\nu(\rho_*(t))\theta'(t)\,\mathrm{d}t \geq \int_0^T \theta(t)\mathfrak{K}(\rho_*(t))\,\mathrm{d}t.$$

As we can prove that $\mathbf{H}(\rho_*(t))$ remains bounded for all times, we can now pass to the inviscid limit $\nu \downarrow 0$ and find the following inequality:

Entropy inequality

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} S_\epsilon((u_*)^m - k^m)\partial_t\varphi \,\mathrm{d}y\,\mathrm{d}t \\
&\geq \int_0^T \int_{\mathbb{R}} \Big(\big[(u_*^m)_y + b(u_*^m - k^m) \big] \varphi_y - k^m b_y \varphi \Big) \mathrm{Sgn}_\epsilon(u_*^m - k^m) \,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}} \big[\big(H_\epsilon(u_*) \big)_y \big]^2 \varphi \,\mathrm{d}y\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}} (b\varphi)_y R_\epsilon(u_*^m - k^m) \,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}
$$

In the final step, we pass to the limit $\epsilon \downarrow 0$. By the uniform convergence of $S_\epsilon(s)$ to $|s-k|$ and the uniform convergence of R_ϵ to zero, we obtain

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} |u_* - k| \varphi_t \,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int_0^T \int_{\mathbb{R}} \mathrm{Sgn}(u_*^m - k^m) \Big(\big[(u_*^m)_y + b(u_*^m - k^m) \big] \varphi_y - b_y k^m \varphi \Big) \,\mathrm{d}y\,\mathrm{d}t \\
&\quad\quad \geq \limsup_{\epsilon \downarrow 0} \int_0^T \int_{\mathbb{R}} \big[(u_*^m)_y \big]^2 \mathrm{Sgn}_\epsilon'(u_*^m - k^m) \varphi \,\mathrm{d}y\,\mathrm{d}t \qquad (23)
\end{aligned}
$$

Table of contents

Uniqueness

We use a uniqueness result by Karlsen and Risebro for the equation (1). In order to apply such result, we need a uniform L^∞, that we obtain by testing the auxiliary functional

$$\Psi_\nu(\eta) = \int_{\mathbb{R}} \mathrm{PosPar}_\epsilon \left(a(x)^{\frac{1}{m-1}} \eta(x) - k \right) a(x)^{\frac{-1}{m-1}} \, \mathrm{d}x + \nu \mathbf{H}(\eta),$$

which provides the comparison property

$$\int_{\mathbb{R}} \left[\rho_*(T,x) - k a(x)^{\frac{-1}{m-1}} \right]_+ \mathrm{d}x \leq \int_{\mathbb{R}} \left[\rho^0(x) - k a(x)^{\frac{-1}{m-1}} \right]_+ \mathrm{d}x.$$

Note that the augmented entropy inequality (23) is needed in the uniqueness proof by [Karlsen-Risebro 2003].

Table of contents

- The strategy can be applied to several other contexts, as it mostly relies on the fact that the functional providing the entropy condition is κ-convex.
- The only condition to check is that the JKO of the given functional provides enough estimates in the limit that allow to fall into a uniqueness theorem for entropy solutions.
- Our project is to tackle gradient flows with *nonlinear mobility*

$$\partial_t \rho = \mathrm{div}(m(\rho)\nabla V(x))$$

  since in this case we would fall into the theory of scalar conservation laws. The JKO framework is provided by [Dolbeault-Nazareth-Savaré]. The functional here is *never* κ convex.
- We are also interested in nonlocal models

$$\partial_t \rho = \mathrm{div}(m(\rho)\nabla W * \rho).$$

  Here the additional difficulty in order to get unique entropy solutions is given by the BV estimate.

End of the talk

THANK YOU