Analysis of Asymptotic Preserving schemes with the modified equation 2D extension on general meshes

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collaboration with
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Model problem (stiff source terms, radiation , friction, ...)

$$\begin{cases} \partial_t u_\varepsilon + \frac{1}{\varepsilon}\partial_x v_\varepsilon = 0, \\ \partial_t v_\varepsilon + \frac{1}{\varepsilon}\partial_x u_\varepsilon + \frac{\sigma}{\varepsilon^2} v_\varepsilon = 0. \end{cases}$$

$\sigma > 0$ is a given coefficient. The small parameter is $0 < \varepsilon \leq 1$.

Hilbert expansion with respect to the small parameter ε

$$u_\varepsilon = u^0 + \cdots , \qquad v_\varepsilon = v^0 + \varepsilon v^1 + \cdots$$

yields $v^0 = 0$ and $\partial_x u^0 + \sigma v^1 = 0$. It justifies the limit diffusion equation

$$\partial_t u^0 - \frac{1}{\sigma}\partial_{xx} u^0 = 0.$$

Jin-Levermore, Numerical methods for hyperbolic conservation laws with stiff relaxation terms, JCP 1996.

Standard discretization

Model problem

Modified equations

Main result

Schemes

Numerical results

The usual F.V scheme (Riemann) writes

$$\begin{cases} \dfrac{\overline{u_j} - u_j}{\Delta t} + \dfrac{v_{j+1} - v_{j-1}}{2\varepsilon \Delta x} - \dfrac{u_{j+1} - 2u_j + u_{j-1}}{2\varepsilon \Delta x} = 0, \\ \dfrac{\overline{v_j} - v_j}{\Delta t} + \dfrac{u_{j+1} - u_{j-1}}{2\varepsilon \Delta x} - \dfrac{v_{j+1} - 2v_j + v_{j-1}}{2\varepsilon \Delta x} + \dfrac{\sigma}{\varepsilon^2} u_j = 0. \end{cases}$$

Hilbert expansion ($u_j = u_j^0 + \cdots$ and $v_j = v_j^0 + \varepsilon v_j^1 + \cdots$) yields

$$v_j^0 = 0, \qquad \frac{u_{j+1}^0 - u_{j-1}^0}{2\Delta x} + \sigma v_j^1 = 0.$$

The limit scheme is

$$\frac{\overline{u_j}^0 - u_j^0}{\Delta t} - \frac{1}{\sigma} \frac{u_{j+2}^0 - 2u_j^0 + u_{j-2}^0}{\Delta x^2} - \frac{\Delta x}{2\varepsilon} \frac{u_{j+1}^0 - 2u_j^0 + u_{j-1}^0}{\Delta x^2} = 0.$$

Summary

- The limit scheme is consistent with

$$\partial_t u - \left(\frac{1}{\sigma} + \frac{\Delta x}{2\varepsilon}\right)\partial_{xx} u = 0.$$

This is not correct in the regime $\Delta x \geq \varepsilon$! !

- The main objective of A.P. methods is to modify the basic scheme.

- There exists different approaches : Jin-Levermore, Gosse-Toscani, . . ., Jin-Pareschi, ... The structure of these methods may seem **strange**, in particular for the most efficient methods.

- First goal of this presentation : can we understand <u>**a priori**</u> the necessary modifications, starting from PDEs ?

• Start from the modified equation

$$\begin{cases} \partial_t u_{\varepsilon,\alpha} + \dfrac{1}{\varepsilon}\left(\partial_x v_{\varepsilon,\alpha} - \alpha\partial_{xx} u_{\varepsilon,\alpha}\right) = 0, \\ \partial_t v_{\varepsilon,\alpha} + \dfrac{1}{\varepsilon}\left(\partial_x u_{\varepsilon,\alpha} - \alpha\partial_{xx} u_{\varepsilon,\alpha}\right) = -\dfrac{\sigma}{\varepsilon^2} v_{\varepsilon,\alpha}, \end{cases}$$

with two small parameters : ε and $\alpha \approx \frac{\Delta x}{2}$.

• Reminder : Hilbert expansion with respect to ε ($\alpha > 0$ fixed) is not correct

$$\partial_t u - \left(\frac{1}{\sigma} + \frac{\alpha}{\varepsilon}\right)\partial_{xx} u = 0$$

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Modified equation of the first kind

• Introduce the **Magic coefficient** $M = \frac{\varepsilon}{\varepsilon+\sigma\alpha} \in]0,1]$ together with

$$\begin{cases} \partial_t \overline{u}_{\varepsilon,\alpha} + \dfrac{M}{\varepsilon}\left(\partial_x \overline{v}_{\varepsilon,\alpha} - \alpha \partial_{xx} \overline{u}_{\varepsilon,\alpha}\right) = 0, \\ \partial_t \overline{v}_{\varepsilon,\alpha} + \dfrac{1}{\varepsilon}\left(\partial_x \overline{u}_{\varepsilon,\alpha} - \alpha \partial_{xx} \overline{v}_{\varepsilon,\alpha}\right) = -\dfrac{\sigma}{\varepsilon^2} \overline{v}_{\varepsilon,\alpha}, \end{cases}$$

• Hilbert expansion with respect to ε ($\alpha > 0$ fixed) yields

$$\partial_t \overline{u}_{0,\alpha} - M\left(\frac{1}{\sigma} + \frac{\alpha}{\varepsilon}\right)\partial_{xx} \overline{u}_{0,\alpha} = 0.$$

This is correct now since : $M\left(\frac{1}{\sigma} + \frac{\alpha}{\varepsilon}\right) = \frac{\varepsilon}{\varepsilon+\alpha\sigma}\frac{\varepsilon+\sigma\alpha}{\sigma\varepsilon} = \frac{1}{\sigma}$.

Modified equation of the second kind

- The symetry in the equations is better using the **Magic coefficient** in both equations

$$\begin{cases} \partial_t \widehat{u}_{\alpha,\varepsilon} + \dfrac{M}{\varepsilon}\left(\partial_x \widehat{v}_{\alpha,\varepsilon} - \alpha \partial_{xx} \widehat{u}_{\alpha,\varepsilon}\right) = 0, \\ \partial_t \widehat{v}_{\alpha,\varepsilon} + \dfrac{M}{\varepsilon}\left(\partial_x \widehat{u}_{\alpha,\varepsilon} - \alpha \partial_{xx} \widehat{v}_{\alpha,\varepsilon}\right) = -\dfrac{\sigma M}{\varepsilon^2}\widehat{v}_{\alpha,\varepsilon}, \end{cases}$$

- Once again, Hilbert expansion is correct

$$\partial_t \widehat{u}_{\alpha,0} - \frac{1}{\sigma}\partial_{xx}\widehat{u}_{\alpha,0} = 0.$$

- The identity energy is correct in the sense

$$\int_{\mathbb{R}} \frac{\widehat{u}_{\alpha,\varepsilon}(t)^2 + \widehat{v}_{\alpha,\varepsilon}(t)^2}{2} dx \leq \int_{\mathbb{R}} \frac{\widehat{u}_{\alpha,\varepsilon}(0)^2 + \widehat{v}_{\alpha,\varepsilon}(0)^2}{2} dx.$$

Main result

An extension of a result proved for numerical methods in **Design of asymptotic preserving finite volume schemes for the hyperbolic heat equation on unstructured meshes**, Numer. Math., 2012, with C. Buet and E. Franck yields

Theor : Consider well prepared data

$$u_0 \in H^3(\mathbb{R}), \quad v_0 = -\frac{\varepsilon}{\sigma}\partial_x u_0 + \varepsilon^2 v_2 \qquad v_2 \in H^1(\mathbb{R}).$$

There exists $C > 0$ indépendent of ε such that $(t \leq T)$

$$\|u_\varepsilon(t) - \widehat{u}_{\alpha,\varepsilon}(t)\|_{L^2(\mathbb{R})} + \|v_\varepsilon(t) - \widehat{v}_{\alpha,\varepsilon}(t)\|_{L^2(\mathbb{R})} \leq C \min(\alpha, \varepsilon).$$

Sketch of the proof

Define $e = u_\varepsilon - \widehat{u}_{\alpha,\varepsilon}$ and $f = v_\varepsilon - \widehat{v}_{\alpha,\varepsilon}$ One has

$$\begin{cases} \partial_t e + \dfrac{M}{\varepsilon}(\partial_x f - \alpha \partial_{xx} e) = r, \\ \partial_t f + \dfrac{M}{\varepsilon}(\partial_x f - \alpha \partial_{xx} f) + \dfrac{\sigma M}{\varepsilon^2} f = s, \end{cases}$$

with

$$r = \partial_t u_\varepsilon + \frac{M}{\varepsilon}(\partial_x v_\varepsilon - \alpha \partial_{xx} u_\varepsilon) = (1-M)\partial_t u_\varepsilon - \frac{M\alpha}{\varepsilon}\partial_{xx} u_\varepsilon,$$

$$s = \partial_t v_\varepsilon + \frac{M}{\varepsilon}(\partial_x u_\varepsilon - \alpha \partial_{xx} v_\varepsilon) + \frac{\sigma M}{\varepsilon^2} v_\varepsilon = (1-M)\partial_t v_\varepsilon - \tfrac{M\alpha}{\varepsilon}\partial_{xx} v_\varepsilon.$$

Main intermediate result

$$\|r\|_{L^\infty([0,T]:L^2(\mathbb{R}))} + \|s\|_{L^2([0,T]:L^2(\mathbb{R}))} \leq C \min(\alpha, \varepsilon)$$

which proves the claim.

Philosophy

- An A.P. scheme with good properties introduces the **Magic coefficient** M at the right place.

- The design principle of the A.P. scheme is less important.

  It may be a full Riemann solver, a simplified one, coming from W.B. (well balanced) techniques, a direct finite difference scheme, . . .

Jin-Levermore

Riemann invariants $v \pm u$ are modified

$$\begin{cases} v_j^n + u_j^n & = v_{j+\frac{1}{2}}^n + u_{j+\frac{1}{2}}^n \quad + \frac{\sigma \Delta x}{2\varepsilon} v_{j+\frac{1}{2}}^n , \\ v_{j+1}^n - v_{j+1}^n & = v_{j+\frac{1}{2}}^n - u_{j+\frac{1}{2}}^n \quad + \frac{\sigma \Delta x}{2\varepsilon} v_{j+\frac{1}{2}}^n . \end{cases}$$

The solution of this linear system is

$$\begin{cases} v_{j+\frac{1}{2}}^n = \frac{M}{2} \left(v_j^n + v_{j+1}^n + u_j^n - u_{j+1}^n \right) , \\ u_{j+\frac{1}{2}}^n = \frac{1}{2} \left(u_j^n + u_{j+1}^n + v_j^n - v_{j+1}^n \right) . \end{cases}$$

where $M = \frac{\varepsilon}{\varepsilon + \frac{\Delta x}{2} \sigma}$ is the correct value.

Final scheme

Since $v^n_{j+\frac{1}{2}} = M v^{\text{Classique}}_{j+\frac{1}{2}}$, one obtains the Jin-Levermore scheme

$$\begin{cases} \dfrac{\overline{u_j} - u_j}{\Delta t} + M\dfrac{v_{j+1} - v_{j-1}}{2\varepsilon\Delta x} & -M\dfrac{u_{j+1} - 2u_j + u_{j-1}}{2\varepsilon\Delta x} = 0, \\ \dfrac{\overline{v_j} - v_j}{\Delta t} + \dfrac{u_{j+1} - u_{j-1}}{2\varepsilon\Delta x} & -\dfrac{v_{j+1} - 2v_j + v_{j-1}}{2\varepsilon\Delta x} + \dfrac{\sigma}{\varepsilon^2}v_j = 0. \end{cases}$$

It corresponds to the modified equation of the first kind.

Gosse-Toscani scheme

Moreover replace the source term $\frac{\sigma}{\varepsilon^2} v_j$ by

$$\frac{\sigma}{\varepsilon^2} \frac{v_{j-\frac{1}{2}} + v_{j+\frac{1}{2}}}{2}.$$

One obtains exactly

$$\begin{cases} \frac{\overline{u_j} - u_j}{\Delta t} + M \frac{v_{j+1} - v_{j-1}}{2\varepsilon \Delta x} - M \frac{u_{j+1} - 2u_j + u_{j-1}}{2\varepsilon \Delta x} = 0, \\ \frac{\overline{v_j} - v_j}{\Delta t} + M \frac{u_{j+1} - u_{j-1}}{2\varepsilon \Delta x} - M \frac{v_{j+1} - 2v_j + v_{j-1}}{2\varepsilon \Delta x} + M \frac{\sigma}{\varepsilon^2} v_j = 0, \end{cases}$$

It corresponds to the modified equation of the second kind.

MultiD : $u_\varepsilon \in \mathbb{R}$, $\mathbf{v}_\varepsilon \in \mathbb{R}^2$

Model problem

Modified equations

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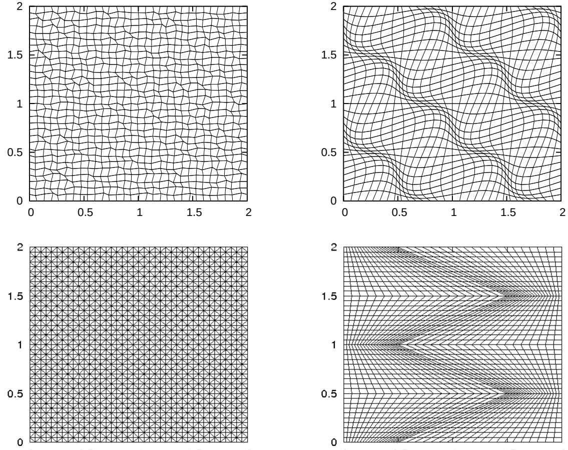
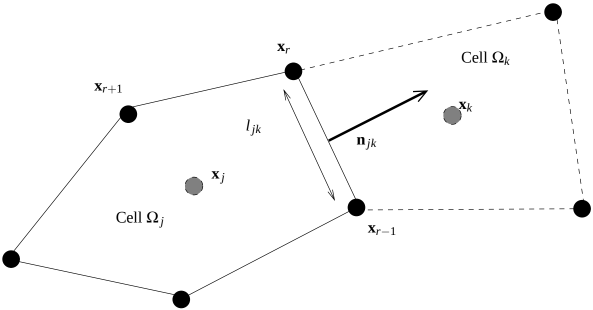
Numerical results

Solve : $\partial_t u_\varepsilon + \frac{1}{\varepsilon}\nabla\cdot\mathbf{v}_\varepsilon = 0, \qquad \partial_t \mathbf{v}_\varepsilon + \frac{1}{\varepsilon}\nabla u_\varepsilon = -\frac{\sigma}{\varepsilon^2}\mathbf{v}_\varepsilon.$



Jin-Levermore on edges



First problem : $\mathbf{x}_j\mathbf{x}_k$ is not parallel to normal vector $\mathbf{n}_{jk}$.

Jin-Levermore on edges

The natural F.V. generalisation of the 1D A.P. scheme is

$$\begin{cases} |\Omega_j|\,\partial_t u_j(t) + \frac{1}{\varepsilon}\sum_k l_{jk}(\mathbf{v}_{jk}, \mathbf{n}_{jk}) = 0, \\ |\Omega_j|\,\partial_t \mathbf{v}_j(t) + \frac{1}{\varepsilon}\sum_k l_{jk} u_{jk}\mathbf{n}_{jk} = -\,|\Omega_j|\,\frac{\sigma}{\varepsilon^2}\mathbf{v}_j, \end{cases}$$

with Jin-Levermore's procedure in the normal direction $\mathbf{n}_{jk}$

$$\begin{cases} u_{jk} - u_j + (\mathbf{v}_{jk} - \mathbf{v}_j, \mathbf{n}_{jk}) = -\frac{\sigma}{\varepsilon} d_{jk}(\mathbf{v}_{jk}, \mathbf{n}_{jk}), \\ u_{jk} - u_k - (\mathbf{v}_{jk} - \mathbf{v}_k, \mathbf{n}_{jk}) = \frac{\sigma}{\varepsilon} d_{kj}(\mathbf{v}_{jk}, \mathbf{n}_{jk}). \end{cases}$$

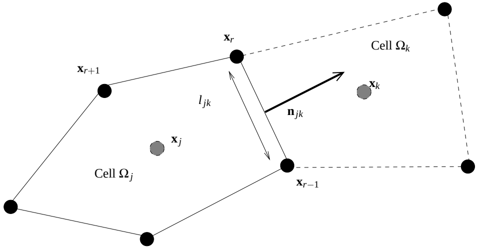
Here $d_{jk} = \mathrm{dist}\,(\mathbf{x}_j, \mathbf{x}_{jk})$ is the distance to the edge.

Formal limit

The limit $\varepsilon \to 0$ is the FV5 scheme

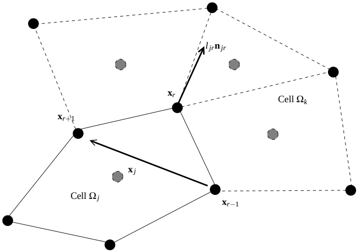
$$|\Omega_j|\,\partial_t u_j(t) - \sum_k l_{jk} \frac{1}{\sigma} \frac{u_k - u_j}{d_{jk} + d_{kj}} = 0, \quad \text{with } d_{jk} + d_{kj} = d(\mathbf{x}_j, \mathbf{x}_k).$$

This scheme is correct for Delaunay meshes.
If $\mathbf{x}_j\mathbf{x}_k$ is parallel to normal vector $\mathbf{n}_{jk}$: OK.



This mesh is not Delaunay.
Edge based Jin-Lervermore not A.P. on general meshes.

Nodal scheme : r is the index of a vertex



Definition : $l_{jr}\mathbf{n}_{jr} = \frac{(\mathbf{x}_{r-1}\mathbf{x}_{r+1})^\perp}{2}$; $\mathbf{n}_{jr}$ is the **corner normal**, l_{jr} is the **length of the corner**.

More general definition : $l_{jr}\mathbf{n}_{jr} = \nabla_{\mathbf{x}_r} S_j$.

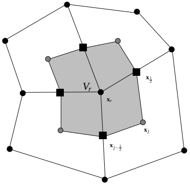
Corner or point fluxes receive increasing interest (Roe).

Corner scheme : $\begin{cases} |\Omega_j|\,\partial_t u_j(t) + \frac{1}{\varepsilon}\sum_r l_{jr}(\mathbf{v}_r, \mathbf{n}_{jr}) = 0 \\ |\Omega_j|\,\partial_t \mathbf{v}_j(t) + \frac{1}{\varepsilon}\sum_r l_{jr} u_{jr}\mathbf{n}_{jr} = -|\Omega_j|\,\frac{\sigma}{\varepsilon^2}\mathbf{v}_j \end{cases}$

Vertex-based solvers

The node $\mathbf{x}_r$ is given. The unknowns of the linear system are $(u_{jr})_j$ and $\mathbf{v}_r$.

$$\text{Lin. system : } \begin{cases} u_{jr} = u_j + (\mathbf{v}_j - \mathbf{v}_r, \mathbf{n}_{jr}), \\ \sum_j l_{jr} u_{jr} \mathbf{n}_{jr} = 0. \end{cases}$$



This linear system is non singular on many meshes.

$$\left(\sum_j l_{jr} (\mathbf{n}_{jr} \otimes \mathbf{n}_{jr} \right) \mathbf{v}_r = \sum_j l_{jr} \left(u_j \mathbf{n}_{jr} + \mathbf{n}_{jr} \otimes \mathbf{n}_{jr} \mathbf{v}_j \right)$$

Jin-Levermore and vertex-based solvers

$$\begin{cases} u_{jr} = u_j + (\mathbf{v}_j - \mathbf{v}_r, \mathbf{n}_{jr}) - \dfrac{\sigma}{\varepsilon}(\mathbf{v}_r, \mathbf{x}_r - \mathbf{x}_j), \\ \sum_j l_{jr} u_{jr} \mathbf{n}_{jr} = 0. \end{cases}$$

Good news : The linear system is still non singular on "almost" all meshes

$$\underbrace{\left(\sum_{j \in \text{cells}} l_{jr} \left(\mathbf{n}_{jr} \otimes \left(\mathbf{n}_{jr} + \frac{\sigma}{\varepsilon}(\mathbf{x}_r - \mathbf{x}_j) \right) \right) \right)}_{=A_r} \mathbf{v}_r = \sum_j l_{jr} \left(u_j \mathbf{n}_{jr} + \mathbf{n}_{jr} \otimes \right.$$

Bad news : Algebra shows that $|\Omega_j| u_j'(t) - \sum_r l_{jr} \left(\mathbf{n}_{jr}, M_r \mathbf{v}_r^{\text{classical}} \right) = 0$ where M_r is a matrix

$$M_r = A_r^{-1} \left(\sum_{j \in \text{cells}} l_{jr} l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr} \right).$$

Limit scheme for $\varepsilon \to 0^+$

It writes

$$\begin{cases} u_j'(t) - \frac{1}{|\Omega_j|} \sum_r l_{jr} \left(\mathbf{n}_{jr}, \mathbf{v}_r \right) = 0, \\ A_r \mathbf{v}_r = \sum_j l_{jr} \mathbf{n}_{jr} u_j, \end{cases}$$

with $A_r = -\left(\sum_j l_{jr} \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j) \right)$.

Theor. (very good news) : Under reasonnable technical conditions, the limit diffusion scheme is convergent for all time $T > 0$. There exists a constant $C(T) > 0$ such that

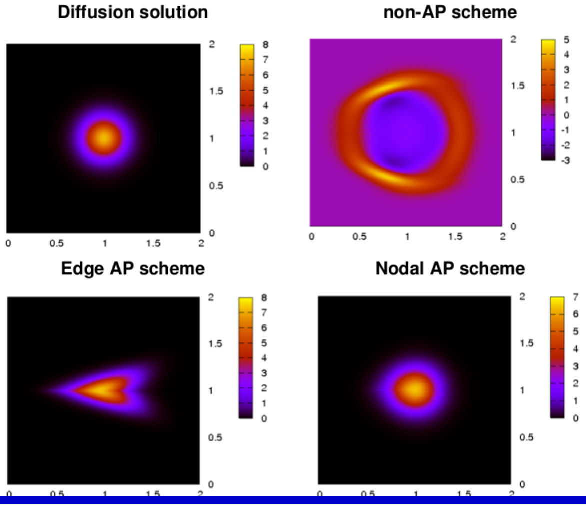
$$\|e(t)\|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2([0,t]\times\Omega)} \le C(T)h, \qquad 0 < t \le T.$$

In this sense the Jin-Levermore nodal based scheme is A.P.

The modified equation is much more complicated to undertsand than the 1D case.

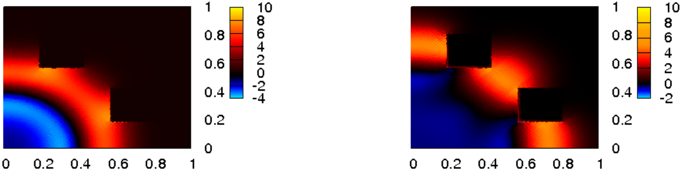
2D numerical example

Fundamental solution computed on the Kershaw mesh : $\varepsilon = 10^{-3}$.



A radiation problem

Here the opacity σ is large in the squares and small everywhere else.



The methods studies in this presentation work perfectly for non constant coeeficients.

Conclusion

- **Modified equation techniques in 1D** show a property of uniform approximation for A.P. methods.
- Adaptation in dimension $d > 1$ reveal new and difficult problems.
- To solve these problems, strong modifications of the usual Finite Volume setting are needed.
- **Nodal Finite Solver are attractive in multiD**.
  See : *Design of asymptotic preserving finite volume schemes for the hyperbolic heat equation on unstructured meshes* Numerische Math., mars 2012, with C. Buet and E. Franck
- Many open problems left.