Scalar conservation laws with moving density constraints arising in traffic flow modeling

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1. Introduction

2. The Riemann problem with moving density constraint

3. The Cauchy problem: Existence of solutions

4. Conclusions
Introduction

1. Introduction
   - Mathematical Model
   - Existing Models

2. The Riemann problem with moving density constraint
   - Riemann Problem
   - Riemann Solver

3. The Cauchy problem: Existence of solutions
   - Cauchy Problem
   - Wave-Front Tracking Method
   - Bounds on the total variation
   - Convergence of approximate solutions
   - Existence of weak solution

4. Conclusions
A slow moving large vehicle along a road reduces its capacity and generates a moving bottleneck for the cars flow. From a macroscopic point of view this can be modeled by a PDE-ODE coupled model consisting in a scalar conservation law with moving density constraint and an ODE describing the slower vehicle.¹

\[
\begin{align*}
\partial_t \rho + \partial_x f(\rho) &= 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\rho(0, x) &= \rho_0(x), & x \in \mathbb{R}, \\
\rho(t, y(t)) &\leq \alpha R, & t \in \mathbb{R}^+, \\
\dot{y}(t) &= \omega(\rho(t, y(t)+)), & t \in \mathbb{R}^+, \\
y(0) &= y_0.
\end{align*}
\]

- \( \rho = \rho(t, x) \in [0, R] \) mean traffic density.
- \( y = y(t) \in \mathbb{R} \) bus position.

\begin{itemize}
\item $v(\rho) = V(1 - \frac{\rho}{R})$ mean traffic velocity, smooth decreasing.
\item $f : [0, R] \rightarrow \mathbb{R}^+$ flux function, strictly concave $f(\rho) = \rho v(\rho)$.
\item $\omega(\rho) = \begin{cases} 
V_b & \text{if } \rho \leq \rho^* = R(1 - V_b/V), \\
v(\rho) & \text{otherwise},
\end{cases}$
\end{itemize}
with $V_b \leq V$ lower vehicle speed.
Fixing the value of the parameters:

- $\alpha \in ]0, 1[$ reduction rate of the road capacity due to the presence of the bus.
- $R = V = 1$ respectively the maximal density and the maximal velocity allowed on the road.

We obtain

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho (1 - \rho)) &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\rho(0, x) &= \rho_0(x), \quad x \in \mathbb{R}, \\
\rho(t, y(t)) &\leq \alpha, \quad t \in \mathbb{R}^+, \\
\dot{y}(t) &= \omega (\rho(t, y(t)+)), \quad t \in \mathbb{R}^+, \\
y(0) &= y_0.
\end{align*}
\]
Lattanzio, Maurizi and Piccoli Model

\[
\begin{cases}
\partial_t \rho + \partial_x f(x, y(t), \rho) = 0, \\
\rho(0, x) = \rho_0(x), \\
\dot{y}(t) = \omega(\rho(t, y(t))), \\
y(0) = y_0.
\end{cases}
\] (3)

The model \(^2\) gives a similar approach to the traffic flow problem even though with specific differences:

- Use of a cut-off function for the capacity dropping of car flows against constrained conservation laws with non-classical shocks.

  \[ f(x, y, \rho) = \rho \cdot v(\rho) \cdot \varphi(x - y(t)). \]

- Assumption that the slower vehicle has a velocity \( \omega(\rho) \) such that \( \omega(0) = V \) and \( \omega(R) = 0 \).

- ODE considered in the Filippov sense against Carathéodory approach.

Colombo and Marson Model

\[\begin{align*}
\partial_t \rho + \partial_x [\rho \cdot v(\rho)] &= 0, \\
\rho(0, x) &= \bar{\rho}(x), \\
\dot{p}(t) &= \omega(\rho(t, p)), \\
p(0) &= \bar{p}.
\end{align*}\] (4)

The model is a coupled ODE-PDE problem with:

- Assumption that \( \omega(\rho) \geq v(\rho) \).
- Weak coupling between the ODE and the PDE.
- Dependence of Filippov solutions to the ODE from the initial datum both of the ODE and of the conservation law.
- Hölder dependence on \( \bar{p} \).

\(^3\)R. M. Colombo and A. Marson. A Hölder continuous ODE related to traffic flow. 
Outline

1 Introduction
   - Mathematical Model
   - Existing Models

2 The Riemann problem with moving density constraint
   - Riemann Problem
   - Riemann Solver

3 The Cauchy problem: Existence of solutions
   - Cauchy Problem
   - Wave-Front Tracking Method
   - Bounds on the total variation
   - Convergence of approximate solutions
   - Existence of weak solution

4 Conclusions
Consider (2) with the particular choice\(^4\)

\[
y_0 = 0 \quad \text{and} \quad \rho_0(x) = \begin{cases} \rho_L & \text{if } x < 0, \\ \rho_R & \text{if } x > 0. \end{cases}
\]  

(5)

Rewriting equations in the bus reference frame i.e., setting \(X = x - V_b t\). We get

\[
\begin{cases}
\partial_t \rho + \partial_X (f(\rho) - V_b \rho) = 0, \\
\rho(0, x) = \begin{cases} \rho_L & \text{if } X < 0, \\ \rho_R & \text{if } X > 0, \end{cases}
\end{cases}
\]

(6)

under the constraint

\[
\rho(t, 0) \leq \alpha.
\]

(7)

Solving problem (6), (7) is equivalent to solving (6) under the corresponding constraint on the flux

\[
f(\rho(t, 0)) - V_b \rho(t, 0) \leq f_\alpha(\rho_\alpha) - V_b \rho_\alpha \doteq F_\alpha.
\]

The Riemann problem with moving density constraint

Riemann Problem II

Maria Laura Delle Monache (INRIA)  Scalar conservation laws with moving constraints  28, June 2012 11 / 24
The Riemann problem with moving density constraint

Riemann Solver

Definition (Riemann Solver)

The constrained Riemann solver $\mathcal{R}^\alpha$ for (2), (5) is defined as follows.

1. If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) > F_\alpha + V_b\mathcal{R}(\rho_L, \rho_R)(V_b)$, then

$$\mathcal{R}^\alpha(\rho_L, \rho_R)(x) = \begin{cases} \mathcal{R}(\rho_L, \hat{\rho}_\alpha) & \text{if } x < V_b t, \\
\mathcal{R}(\tilde{\rho}_\alpha, \rho_R) & \text{if } x \geq V_b t, \end{cases} \quad \text{and } y(t) = V_b t.$$

2. If $V_b\mathcal{R}(\rho_L, \rho_R)(V_b) \leq f(\mathcal{R}(\rho_L, \rho_R)(V_b)) \leq F_\alpha + V_b\mathcal{R}(\rho_L, \rho_R)(V_b)$, then

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \text{and } y(t) = V_b t.$$

3. If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) < V_b\mathcal{R}(\rho_L, \rho_R)(V_b)$, then

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \text{and } y(t) = v(\rho_R)t.$$

Note: when the constraint is enforced, a nonclassical shock arises, which satisfies the Rankine-Hugoniot condition but violates the Lax entropy condition.
Riemann Solver: Example

The Riemann problem with moving density constraint

Riemann Solver

Scalar conservation laws with moving constraints

Maria Laura Delle Monache (INRIA) 28, June 2012 13 / 24
Outline

1 Introduction
   - Mathematical Model
   - Existing Models

2 The Riemann problem with moving density constraint
   - Riemann Problem
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3 The Cauchy problem: Existence of solutions
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   - Existence of weak solution

4 Conclusions
A bus travels along a road modelled by

\[
\begin{cases}
\partial_t \rho + \partial_x (\rho(1 - \rho)) = 0, \\
\rho(0, x) = \rho_0(x), \\
\rho(t, y(t)) \leq \alpha.
\end{cases}
\]  

(8)

The bus influences the traffic along the road but it is also influenced by it. The bus position \( y = y(t) \) then solves

\[
\begin{cases}
\dot{y}(t) = \omega(\rho(t, y(t))), \\
y(0) = y_0.
\end{cases}
\]  

(9)

Solutions to (9) are intended in Carathéodory sense, i.e., as absolutely continuous functions which satisfy (9) for a.e. \( t \geq 0 \).
Cauchy Problem: Solution

**Definition (Weak solution)**

A couple \((\rho, y) \in C^0(\mathbb{R}^+; L^1 \cap BV(\mathbb{R})) \times W^{1,1}(\mathbb{R}^+)\) is a solution to (2) if

1. \(\rho\) is a weak solution of the conservation law, i.e. for all \(\varphi \in C^1_c(\mathbb{R}^2)\)

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) \, dx = 0 \, ; \quad (10a)
\]

2. \(y\) is a Carathéodory solution of the ODE, i.e. for a.e. \(t \in \mathbb{R}^+\)

\[
y(t) = y_0 + \int_0^t \omega(\rho(s, y(s)+)) \, ds \, ; \quad (10b)
\]

3. the constraint is satisfied, in the sense that for a.e. \(t \in \mathbb{R}^+\)

\[
\lim_{x \to y(t)\pm} (f(\rho) - \omega(\rho)\rho)(t, x) \leq F_\alpha. \quad (10c)
\]

**Note:** The above traces exist because \(\rho(t, \cdot) \in BV(\mathbb{R})\) for all \(t \in \mathbb{R}^+\).
Wave-Front Tracking Method I

- Fix $n \in \mathbb{N}$, $n > 0$ and introduce in $[0, 1]$ the mesh $\mathcal{M}_n$ by

$$\mathcal{M}_n = (2^{-n}\mathbb{N} \cap [0, 1]) \cup \{\tilde{\rho}_\alpha, \hat{\rho}_\alpha\}.$$ 

- Let $f_n$ be the piecewise linear function which coincides with $f$ on $\mathcal{M}_n$.

- Let

$$\rho^0_n = \sum_{j \in \mathbb{Z}} \rho^0_{n,j} \chi_{[x_{j-1}, x_j]}$$

with $\rho^0_{n,j} \in \mathcal{M}_n$,

such that

$$\lim_{n \to \infty} \|\rho^0_n - \rho_0\|_{L^1(\mathbb{R})} = 0,$$

and $TV(\rho^0_n) \leq TV(\rho_0)$.

- $y_0$ is given.
For small times $t > 0$, a piecewise approximate solution $(\rho^n, y_n)$ to (2) is constructed piecing together the solutions to the Riemann problems

$$\begin{align*}
\partial_t \rho + \partial_x (f^n(\rho)) &= 0, \\
\rho(0, x) &= \begin{cases} 
\rho_0 & \text{if } x < y_0, \\
\rho_1 & \text{if } x > y_0,
\end{cases} \\
\rho(t, y_n(t)) &\leq \alpha,
\end{align*}$$

and

$$\begin{align*}
\partial_t \rho + \partial_x (f^n(\rho)) &= 0, \\
\rho(0, x) &= \begin{cases} 
\rho_j & \text{if } x < x_j, \\
\rho_{j+1} & \text{if } x > x_j,
\end{cases} \\
j \neq 0,
\end{align*}$$

(11)

where $y_n$ satisfies

$$\begin{align*}
\dot{y}_n(t) &= \omega(\rho^n(t, y_n(t)+)), \\
y_n(0) &= y_0.
\end{align*}$$

(12)
Define the Glimm type functional

$$\Upsilon(t) = \Upsilon(\rho^n(t, \cdot)) = TV(\rho^n) + \gamma = \sum_j |\rho_{j+1}^n - \rho_j^n| + \gamma,$$

with

$$\gamma = \gamma(t) = \begin{cases} 0 & \text{if } \rho^n(t, y_n(t) - ) = \hat{\rho}_\alpha, \rho^n(t, y_n(t) + ) = \check{\rho}_\alpha \\ 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| & \text{otherwise.} \end{cases}$$

**Lemma (Decreasing functional)**

For any \( n \in \mathbb{N} \), the map \( t \mapsto \Upsilon(t) = \Upsilon(\rho^n(t, \cdot)) \) at any interaction either decreases by at least \( 2^{-n} \), or remains constant and the number of waves does not increase.

**Proof**
Lemma (Convergence of approximate solutions)

Let $\rho^n$ and $y_n$, $n \in \mathbb{N}$, be the wave front tracking approximations to (1) constructed as detailed in Section 2, and assume $TV(\rho_0) \leq C$ be bounded, $0 \leq \rho_0 \leq 1$. Then, up to a subsequence, we have the following convergences

$$
\rho^n \rightarrow \rho \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}); \quad (15a)
$$

$$
y_n(\cdot) \rightarrow y(\cdot) \quad \text{in } L^\infty([0, T]), \text{ for all } T > 0; \quad (15b)
$$

$$
\dot{y}_n(\cdot) \rightarrow \dot{y}(\cdot) \quad \text{in } L^1([0, T]), \text{ for all } T > 0; \quad (15c)
$$

for some $\rho \in C^0(\mathbb{R}^+; L^1 \cap BV(\mathbb{R}))$ and $y \in W^{1,1}(\mathbb{R}^+)$. 

Sketch of the proof:

- (15a): $TV(\rho^n(t, \cdot)) \leq \Upsilon(t) \leq \Upsilon(0) + \text{Helly's theorem}$.
- (15b): $|\dot{y}_n(t)| \leq V_b + \text{Ascoli-Arzelà theorem}$.
- (15c): $TV(\dot{y}_n; [0, T]) \leq 2 NV(\dot{y}_n; [0, T]) + \|\dot{y}_n\|_{L^\infty([0, T])} \leq 2 TV(\rho_0) + V_b$.
The Cauchy problem: Existence of solutions

Existence of weak solution

Convergence of weak solution

Theorem (Existence of solutions)

For every initial data $\rho_0 \in BV(\mathbb{R})$ such that $TV(\rho_0) \leq C$ is bounded, problem (1) admits a weak solution in the sense of Definition (Weak Solution).

Sketch of the proof:

- $\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) \, dx = 0$:
  - $\rho^n \to \rho$ in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}) \Rightarrow$ limit in the weak formulation of the conservation law.

- $\dot{y}(t) = \omega(\rho(t, y(t)+))$ for a.e. $t > 0$:
  - $\lim_{n \to \infty} \rho^n(t, y_n(t)+) = \rho^+(t) = \rho(t, y(t)+)$ for a.e. $t \in \mathbb{R}^+ \quad (15c)$.

- $\lim_{x \to y(t) \pm} (f(\rho) - \omega(\rho) \rho)(t, x) \leq F_\alpha$:
  - direct use of the convergence result already proved.

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4. Conclusions
The coupled PDE-ODE model represents a good approach to the problem of moving bottleneck, as shown by different numerical approaches (works by F. Giorgi, J. Laval, L. Leclerq, C.F. Daganzo).

We were able to develop a strong coupling between the PDE and ODE.

We proved the existence of solutions for this model.

Stability of solution is currently a work in progress.
Thank you for your attention.
**Remark 1** Definition 1 is well posed even if the classical solution \( R(\rho_L, \rho_R)(x/t) \) displays a shock at \( x = V_b t \). In fact, due to Rankine-Hugoniot equation, we have

\[
f(\rho_L) = f(\rho_R) + V_b(\rho_L - \rho_R)
\]

and hence

\[
f(\rho_L) > f_\alpha(\rho_\alpha) + V_b(\rho_L - \rho_\alpha) \iff f(\rho_R) > f_\alpha(\rho_\alpha) + V_b(\rho_R - \rho_\alpha).
\]

**Remark 2** The density constraint \( \rho(t, y(t)) \leq \alpha \) is handled by the corresponding condition on the flux

\[
f(\rho(t, y(t))) - \omega(\rho(t, y(t)))\rho(t, y(t)) \leq F_\alpha. \tag{16}
\]

The corresponding density on the reduced roadway at \( x = y(t) \) is found taking the solution to the equation

\[
f(\rho_y) + \omega(\rho_y)(\rho - \rho_y) = \rho \left( 1 - \frac{\rho}{\alpha} \right)
\]

closer to \( \rho_y \doteq \rho(t, y(t)) \).
Lemma 3: Proof I

Either two shocks collide (which means that the number of waves diminishes) or a shock and a rarefaction cancel. $\rho_l$, $\rho_r$, and the number of waves remain constant.
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\[ TV(\rho^n), \, \Upsilon \] and the number of waves remain constant.
Lemma 3: Proof II

After the collision, the number of discontinuities in $\rho$ diminishes and the functional $\Upsilon$ remains constant:

$$\Delta \Upsilon(\bar{t}) = \Upsilon(\bar{t}^+) - \Upsilon(\bar{t}^-) = \left| \hat{\rho}_\alpha - \rho_l \right| + 2 \left| \hat{\rho}_\alpha - \check{\rho}_\alpha \right| - \left( \left| \check{\rho}_\alpha - \rho_l \right| + \left| \hat{\rho}_\alpha - \check{\rho}_\alpha \right| \right) = 0.$$
Lemma 3: Proof II

After the collision, the number of discontinuities in $\rho^n$ diminishes and the functional $\Upsilon$ remains constant:

$$\Delta \Upsilon(\bar{t}) = \Upsilon(\bar{t}^+) - \Upsilon(\bar{t}^-)$$
$$= |\hat{\rho}_\alpha - \rho_\alpha| + 2|\hat{\rho}_\alpha - \rho_\alpha| - (|\rho_\alpha - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \rho_\alpha|)$$
$$= 0.$$
After the collision, the number of discontinuities in $\rho^n$ diminishes and the functional $\Upsilon$ remains constant:

$$\Delta \Upsilon(\bar{t}) = \Upsilon(\bar{t}^+) - \Upsilon(\bar{t}^-)$$

$$= |\rho_l - \rho_\alpha| + 2|\hat{\rho}_\alpha - \hat{\rho}_\alpha| - (|\rho_l - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \hat{\rho}_\alpha|)$$

$$= 0.$$
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$$\Delta \Upsilon(\bar{t}) = \Upsilon(\bar{t}^+) - \Upsilon(\bar{t}^-)$$
$$= |\rho_l - \rho_\alpha| + 2|\hat{\rho}_\alpha - \rho_\alpha| - (|\rho_l - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \rho_\alpha|)$$
$$= 0.$$
Lemma 3: Proof III

\[ \rho_r = \dot{\rho}_\alpha \]

New waves are created at \( \bar{t} \) and the total variation is given by:

\[ TV(\bar{t}^-) = |\hat{\rho}_\alpha - \rho_l| \leq 2^{-n}; \]

\[ TV(\bar{t}^+) = |\hat{\rho}_\alpha - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \dot{\rho}_\alpha| \leq 2 |\hat{\rho}_\alpha - \hat{\rho}_\alpha|, \]

\[ \Delta \Upsilon(\bar{t}^-) = \Upsilon(\bar{t}^+) - \Upsilon(\bar{t}^-) \leq -2^{-n}. \]
Lemma 3: Proof III

New waves are created at $\bar{t}$ and the total variation is given by:

- $TV(\bar{t}^-) = |\hat{\rho}_\alpha - \rho_l| \leq 2^{-n}$;
- $TV(\bar{t}^+) = |\hat{\rho}_\alpha - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \rho_l| \leq 2|\hat{\rho}_\alpha - \hat{\rho}_\alpha|$, \n
\[
\Delta \gamma(\bar{t}) = \gamma(\bar{t}^+) - \gamma(\bar{t}^-) \leq -2^{-n}
\]
Lemma 3: Proof III

New waves are created at $\bar{t}$ and the total variation is given by:

- $\text{TV}(\bar{t}^-) = |\hat{\rho}_\alpha - \rho_l| \leq 2^{-n};$
- $\text{TV}(\bar{t}^+) = |\hat{\rho}_\alpha - \check{\rho}_\alpha| + |\check{\rho}_\alpha - \rho_l| \leq 2|\hat{\rho}_\alpha - \check{\rho}_\alpha|,

$$\Delta \Upsilon(\bar{t}) = \Upsilon(\bar{t}^+) - \Upsilon(\bar{t}^-) \leq -2^{-n}$$
Lemma 3: Proof III

New waves are created at $\bar{t}$ and the total variation is given by:

- $TV(\bar{t}^-) = |\hat{\rho}_\alpha - \rho_l| \leq 2^{-n}$;
- $TV(\bar{t}^+) = |\hat{\rho}_\alpha - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \rho_l| \leq 2|\hat{\rho}_\alpha - \hat{\rho}_\alpha| $,

$$\Delta \gamma(\bar{t}) = \gamma(\bar{t}^+) - \gamma(\bar{t}^-) \leq -2^{-n}$$

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- $TV(\bar{t}^-) = |\hat{\rho}_\alpha - \rho_l| \leq 2^{-n}$;
- $TV(\bar{t}^+) = |\hat{\rho}_\alpha - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \rho_l| \leq 2|\hat{\rho}_\alpha - \hat{\rho}_\alpha| $,

$$\Delta \gamma(\bar{t}) = \gamma(\bar{t}^+) - \gamma(\bar{t}^-) \leq -2^{-n}$$
Convergence of approximate solutions II

Proof:
From 3 we have \( TV(\rho^n(t, \cdot)) \leq \Upsilon(t) \leq \Upsilon(0) \). Using Helly’s Theorem we ensure the existence of a subsequence converging to some function \( \rho \in C^0(\mathbb{R}^+; L^1 \cap BV(\mathbb{R})) \), proving (15a).

Since \( |\dot{y}_n(t)| \leq V_b \), the sequence \( \{y_n\} \) is uniformly bounded and equicontinuous on any compact interval \([0, T]\). By Ascoli-Arzelà Theorem, there exists a subsequence converging uniformly, giving (15b).

We can estimate the speed variation at interactions times \( \bar{t} \) by the size of the interacting front:

\[
|\dot{y}_n(\bar{t}+) - \dot{y}_n(\bar{t}-)| = |\omega(\rho_l) - \omega(\rho_r)| \leq |\rho_l - \rho_r|.
\]

\( \dot{y}_n \) increases only at interactions with rarefaction fronts, which must be originated at \( t = 0 \). Therefore,

\[
TV(\dot{y}_n; [0, T]) \leq 2 N V(\dot{y}_n; [0, T]) + \|\dot{y}_n\|_{L^\infty([0,7])} \leq 2 TV(\rho_0) + V_b
\]
is uniformly bounded, proving (15c).
Convergence of weak solution I

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) \, dx = 0 ;
\]  \hspace{1cm} (17)

**Proof:**
Since \(\rho^n\) converge strongly to \(\rho\) in \(L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})\), it is straightforward to pass to the limit in the weak formulation of the conservation law, proving that the limit function \(\rho\) satisfies (17).
Convergence of weak solution I

\[ \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left( \rho \partial_t \varphi + f(\rho) \partial_x \varphi \right) dx \, dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) \, dx = 0 ; \quad (17) \]

**Proof:**
Since \( \rho^n \) converge strongly to \( \rho \) in \( L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \), it is straightforward to pass to the limit in the weak formulation of the conservation law, proving that the limit function \( \rho \) satisfies (17).

\[ \dot{y}(t) = \omega(\rho(t, y(t)+)) \quad \text{for a. e.} \ t > 0. \quad (18) \]

**Proof:**
We prove that
\[
\lim_{n \to \infty} \rho^n(t, y_n(t)+) = \rho^+(t) = \rho(t, y(t)+) \quad \text{for a. e.} \ t \in \mathbb{R}^+. \quad (19)
\]

By pointwise convergence a. e. of \( \rho^n \) to \( \rho \), \( \exists \) a sequence \( z_n \geq y_n(t) \) s.t. \( z_n \to y(t) \) and \( \rho^n(t, z_n) \to \rho^+(t) \).
For a.e. \( t > 0 \), the point \((t, y(t))\) is for \( \rho(t, \cdot) \) either a continuity point, or it belongs to a discontinuity curve (either a classical or a non-classical shock).

Fix \( \epsilon^* > 0 \) and assume \( TV(\rho(t, \cdot); [y(t) - \delta, y(t) + \delta[) \leq \epsilon^* \), for some \( \delta > 0 \).

Then by weak convergence of measure \(^{6}\) \( TV(\rho^n(t, \cdot); [y(t) - \delta, y(t) + \delta[) \leq 2\epsilon^* \) for \( n \) large enough, and

\[
|\rho^n(t, y_n(t)^+) - \rho^+(t)| \leq |\rho^n(t, y_n(t)^+) - \rho^n(t, z_n)| + |\rho^n(t, z_n) - \rho^+(t)| \leq 3\epsilon^*
\]

for \( n \) large enough.

If \( \rho(t, \cdot) \) has a discontinuity of strength greater than \( \epsilon^* \) at \( y(t) \), then

\[
|\rho^n(t, y_n(t)^+) - \rho^n(t, y_n(t)^-)| \geq \epsilon^*/2 \text{ for } n \text{ sufficiently large.}
\]

---

We set $\rho^{n,+} = \rho^n(t, y_n(t)+)$ and we show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n$ large enough there holds

$$|\rho^n(s, x) - \rho^{n,+}| < \varepsilon \quad \text{for } |s - t| \leq \delta, |x - y(t)| \leq \delta, x > y_n(s). \quad (20)$$

In fact, if (20) does not hold, we could find $\varepsilon > 0$ and sequences $t_n \to t$, $\delta_n \to 0$ s. t. $\text{TV} (\rho^n(t_n, \cdot); ]y_n(t_n), y_n(t_n) + \delta_n[) \geq \varepsilon$.

By strict concavity of the flux function $f$, there should be a uniformly positive amount of interactions in an arbitrarily small neighborhood of $(t, y(t))$, giving a contradiction. Therefore (20) holds and we get

$$|\rho^n(t, y_n(t)+) - \rho^+(t)| \leq |\rho^n(t, y_n(t)+) - \rho^n(t, z_n)| + |\rho^n(t, z_n) - \rho^+(t)| \leq 2\varepsilon$$

for $n$ large enough, thus proving (19).

Combining (15c) and (19) we get (18).
Convergence of weak solution IV

\[
\lim_{x \to y(t) \pm} (f(\rho) - \omega(\rho)\rho)(t, x) \leq F_\alpha.
\]  

(21)

Proof:

Introduce the sets

\[
\Omega^\pm = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}: x \leq y(t)\}
\]

and

\[
\Omega^\pm_n = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}: x \leq y_n(t)\},
\]

Consider a test function \( \varphi \in C^1_c(\mathbb{R}^+ \times \mathbb{R}) \), \( \varphi \geq 0 \), s. t. \( \text{supp}(\varphi) \cap \{(t, y(t)): t > 0\} \neq 0 \) and \( \text{supp}(\varphi) \cap \{(t, y_n(t)): t > 0\} \neq 0 \).
Then by conservation on $\Omega_n^+$ we have

$$\iint \chi_{\Omega_n^+} (\rho^n \partial_t \varphi + f^n(\rho^n) \partial_x \varphi) \, dx \, dt$$

$$= \int_0^{+\infty} \int_{y_n(t)}^{+\infty} (\rho^n \partial_t \varphi + f^n(\rho^n) \partial_x \varphi) \, dx \, dt$$

$$= \int_0^{+\infty} \left( f^n(\rho^n(t, y_n(t)+)) - \dot{y}_n(t) \rho^n(t, y_n(t)+) \right) \varphi(t, y_n(t)) \, dt$$

$$= \int_0^{+\infty} \left( f^n(\rho^n(t, y_n(t)+)) - \omega(\rho^n(t, y_n(t)+)) \rho^n(t, y_n(t)+) \right) \varphi(t, y_n(t)) \, dt$$

$$\leq \int_0^T F_\alpha \varphi(t, y_n(t)) \, dt,$$

(22)
The same can be done for the limit solutions $\rho$ and $y(t)$ that, by conservation on $\Omega$, satisfy

$$\int\int \chi_{\Omega^+} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt = \int_0^{+\infty} \int_{y(t)}^{+\infty} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt$$

$$= \int_0^{+\infty} (f(\rho(t, y(t)+)) - \dot{y}(t)\rho(t, y(t)+)) \varphi(t, y(t)) \, dt$$

$$= \int_0^{+\infty} (f(\rho(t, y(t)+)) - \omega(\rho(t, y(t)+))\rho(t, y(t)+)) \varphi(t, y(t)) \, dt \quad (23)$$

By (15a) and (15b) we can pass to the limit in (22) and (23), which gives

$$\int_0^{+\infty} (f(\rho(t, y(t)+)) - \omega(\rho(t, y(t)+))\rho(t, y(t)+) - F_\alpha) \varphi(t, y(t)) \, dt \leq 0.$$ 

Since the above inequality holds for every test function $\varphi \geq 0$, we have proved (21).