Non-standard solutions of isentropic Euler with Riemann data

Camillo De Lellis

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Hyperbolic systems of conservation laws

$u : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^k$ is the unknown vector function.
$F : \mathbb{R}^k \to \mathbb{R}^{k\times m}$ is the known "flux function".

$$\begin{cases} \partial_t u + \mathrm{div}_x(F(u)) = 0 \\ u(0,\cdot) = u_0 \,. \end{cases} \tag{1}$$

The system is strictly hyperbolic if the matrix $(\partial_\ell F^{ij}(v)\xi_j)_{\ell i}$ has k distinct real eigenvalues for every $v \in \mathbb{R}^k$, $\xi \in \mathbb{S}^{k-1}$.

It is well known that solutions of (1) develop singularities (shocks) in finite time (generically!).

Problem

Develop a theory which allows to go beyond the singularities.

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Gas dynamics: a primary example

Isentropic gas dynamics in Eulerian coordinates: the unknowns of the system, which consists of $n+1$ equations, are the density ρ and the velocity v of the gas:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) = 0 \\ \partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) + \nabla[p(\rho)] = 0 \\ \rho(0,\cdot) = \rho^0 \\ v(0,\cdot) = v^0 \end{cases} \tag{2}$$

The pressure p is a function of ρ, which is determined from the constitutive thermodynamic relations of the gas in question and satisfies the assumption $p' > 0$.

A typical example is $p(\rho) = k\rho^\gamma$, with constants $k > 0$ and $\gamma > 1$,

Recall that the internal energy density ε satisfies $p(r) = r^2\varepsilon'(r)$.

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Incompressible Euler: for future reference

Incompressible Euler is a system of PDEs which is NOT a hyperbolic system of conservation laws:

$$\begin{cases} \mathrm{div}_x v = 0 \\ \partial_t v + \mathrm{div}_x (v \otimes v) + \nabla p = 0 \\ v(0, \cdot) = v^0 \end{cases} \tag{3}$$

In particular ρ is constant, p is an unknown function and the initial condition does not involve p.

Nonetheless this system will play an important role later in this talk.

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A well established theory and a lot of literature exists when $m = 1$: well-posedness holds if weak solutions are required to satisfy a suitable admissibility condition.

Much less is known for $m > 1$, aside from very interesting works on the stability of sufficiently smooth shock waves.

The space of BV functions plays a prominent role in the 1-dimensional setting, but

Theorem (Rauch 1986)

Well-posedness in BV can be expected only if the following commutator condition holds

$$DF^i \cdot DF^j = DF^j \cdot DF^i .$$

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A very simple example which develops singularities is given by

$$\begin{cases} \partial_t u + \mathrm{div}_x (f(|u|) \otimes u) = 0 \\ u(0, \cdot) = u_0 \end{cases} \tag{4}$$

which can be decoupled in a scalar conservation law and $n-1$ transport equations.

Serre: is it possible to prove well-posedness for (4) using this structure?

Theorem (Bressan 2003)

There is f Lipschitz (piecewise linear) such that (4) is ill-posed in L^∞.

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Theorem (Ambrosio-Bouchut-D 2004)

The Cauchy problem for the Keyfitz-Kranzer system is well-posed if $|u_0| \in BV_{loc} \cap L^\infty$.

The Keyfitz-Kranzer system satisfies Rauch's commutator condition, but nonetheless

Theorem (D 2005)

Generically, even if $u_0 \in BV$, the BV norm of the solution blows up instantaneously.

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Which space?

Key point: BV is a "bad space for transport phenomena" in more than one space dimension.

Problem

Is there a "better" function space?

Loosely speaking there are two options:

- Look for a larger space.
- Look for a smaller space.

We will focus on the first option.

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(a) Is it possible to recast the DiPerna-Lions theory in a more classical framework, with apriori estimates?

(b) Is there a function space which contains BV, embeds compactly in L^1 and is well behaved with respect to the transport equations?

Theorem (Crippa-D 2008)

(a) has a positive answer for the $W^{1,p}$ theory (BV still open!).

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L^∞

There does not seem to be a good candidate between L^1 (or L^∞) and BV. Is it possible to have well-posedness in L^∞?

Theorem (D-Székelyhidi 2010)

For any pressure law p there are bounded initial data (ρ_0, v_0) with $\rho_0 \geq c > 0$ with infinitely many bounded weak solutions (ρ, v) with $\rho \geq c > 0$ satisfying the "usual" entropy admissibility condition.

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) + \nabla[p(\rho)] \;=\; 0 \\ \rho(0,\cdot) \;=\; \rho^0 \\ v(0,\cdot) \;=\; v^0 \end{cases} \tag{5}$$

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Differential inclusions and Tartar's wave analysis

The proof is based on a previous work on the incompressible Euler equations where

- Following DiPerna and Tartar we split the system of PDEs in linear equations and constitutive relations.
- Following Tartar we analyze the wave cone for the linear equation.
- We apply techniques from the theory of differential inclusions (see Cellina, Bressan, Dacorogna-Marcellini, Müller-Šverak, Kirchheim) to construct very oscillatory solutions.

With some "ad hoc" adjustments we adapt this construction to produce solutions of the compressible Euler equations.

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Is it possible to apply this framework directly to compressible Euler?

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Weak-strong uniqueness

The initial data of the last ill-posedness theorem must be sufficiently irregular because

Theorem (Dafermos-DiPerna)

As long as a Lipschitz solution exists, any bounded admissible solution must coincide with it.

In fact this theorem holds even for measure valued solutions (Brenier-D-Székelyhidi 2010).

First of all the "problem" lies in the irregularity of the velocity:

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The same ill-posedness result can occur even with pairs (ρ_0, v_0) where ρ_0 is smooth.

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Consider Riemann initial data in 2d which can be reduced to the 1d Riemann problem.

$$(\rho_0, v_0)(x) = \begin{cases} (\rho^+, v^+) & \text{if } x_1 > 0 \\ (\rho^-, v^-) & \text{if } x_1 < 0 \end{cases} \tag{7}$$

Theorem (Chiodaroli-D 2012)

There are smooth pressure laws p with $p' > 0$ for which the following holds. There are admissible L^∞ solutions of isentropic Euler with initial data (7) which depend also on x_2.

Inspired by a work of Székelyhidi which proves the same theorem for incompressible Euler with the classical shear flow initial data.

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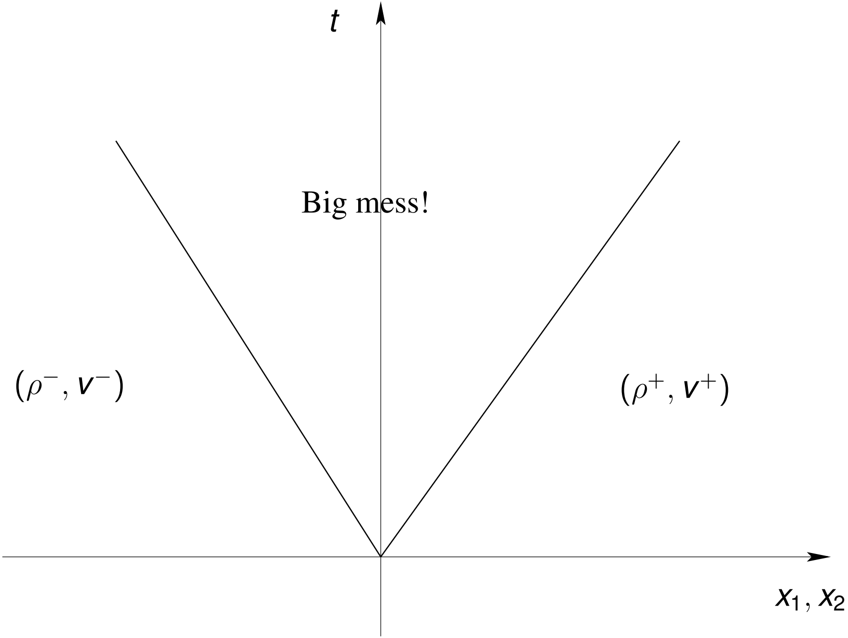
t
Big mess!
(ρ⁻, v⁻)
(ρ⁺, v⁺)
x1, x2

Important remarks

- The pressure law p is quite specific and does not satisfy $2p'(\rho) + \rho p''(\rho) > 0$, nonetheless the 1-d Riemann problem with the data of the previous theorem has a unique solution.
- The data is not "small" in L^∞;
- The solution of the Riemann problem has a contact discontinuity.
- The proof is not completely in the "compressible world", but, compared to the D-Székelyhidi result, it exploits much more several specific properties of compressible Euler.

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Theorem (Székelyhidi 2011)

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Thank you
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