Semi-Lagrangian schemes for linear and fully non-linear Hamilton-Jacobi-Bellman equations

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Outline

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Stochastic control problem $\Longrightarrow$ HJB equation

$$dX(t) = b(t, X(t), \alpha(t))\ dt + \sum_{i=1}^{M} \sigma_i(t, X(t), \alpha(t))\ dW_i(t), \quad t_0 \le t \le T,$$

$$X(t_0) = X_0 \in \mathbb{R}^N.$$

$$u(t_0, X_0) = \inf_{\alpha \in \mathfrak{A}} \mathrm{E}\left(\int_{t_0}^{T} \beta(t_0, t) f(t, X(t), \alpha(t))\ dt + \beta(t_0, T) g(X(T)) \right)$$

where

$$\beta(t_0, t) = e^{\int_{t_0}^{t} c\left(s, X(s), \alpha(s)\right)\ ds}.$$

Under appropriate assumptions: u solves HJB equation

$$-u_t(t,x) - \inf_{\alpha \in \mathcal{A}} \left\{ L[u](t,x,\alpha) + c(t,x,\alpha)u(t,x) + f(t,x,\alpha) \right\} = 0$$

$$\text{in } [0,T) \times \mathbb{R}^N, \qquad u(T,x) = g(x) \text{ in } \mathbb{R}^N,$$

$$L[u](t,x,\alpha) = \frac{1}{2}\mathrm{Tr}[\sigma(t,x,\alpha)\sigma^\top(t,x,\alpha)D^2u(t,x)] + b(t,x,\alpha)Du(t,x)$$

Stochastic control problem $\Longrightarrow$ HJB equation

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where

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Under appropriate assumptions: u solves in the viscosity sense

$$-u_t(t, x) - \inf_{\alpha \in \mathcal{A}} \left\{ L[u](t, x, \alpha) + c(t, x, \alpha) u(t, x) + f(t, x, \alpha) \right\} = 0$$

$$\text{in } [0, T) \times \mathbb{R}^N, \qquad u(T, x) = g(x) \text{ in } \mathbb{R}^N,$$

$$L[u](t, x, \alpha) = \frac{1}{2} \mathrm{Tr}[\sigma(t, x, \alpha) \sigma^\top(t, x, \alpha) D^2 u(t, x)] + b(t, x, \alpha) Du(t, x)$$

Convergence to viscosity solution via Lax type result

Barles & Souganidis 1991: The numerical approximation U obtained by the scheme $S(\Delta t, \Delta x, n, j, u_j^n, u) = 0$ converges uniformly to the viscosity solution if at least for some sequence $(\Delta t, \Delta x)$ converging to zero it is

- stable: For all $(\Delta t, \Delta x)$ there exists a solution U with a bound independent of $(\Delta t, \Delta x)$.
- consistent:

$$\lim_{\substack{\xi \to 0 \\ \Delta t, \Delta x \to 0 \\ (n\Delta t, j\Delta x) \to (t,x)}} \frac{S(\Delta t, \Delta x, n, j, \Phi_j^n + \xi, \Phi + \xi)}{\rho(\Delta t, \Delta x)} \to F\big(t, x, \Phi(t,x), \Phi_t(t,x), D\Phi(t,x), D^2\Phi(t,x)\big)$$

  for some positive function ρ, any smooth function Φ and any (x, t)

- monotone:

$$S(\Delta t, \Delta x, n, j, u_j^n, u) \le S(\Delta t, \Delta x, n, j, v_j^n, v) \text{ if } u \ge v,\ \ u_j^n = v_j^n,$$

  for any Δt, Δx, n, j, u and v

Oberman (2006), Pooley, Forsyth, Vetzal (2003): Nonmonotone methods need not converge or even converge to a wrong function

Outline

Proposed family of SL schemes

$$\text{PDE: } u_t(t,x) - \inf_{\alpha\in\mathcal{A}} \left\{ L[u](t,x,\alpha) + c(t,x,\alpha)u(t,x) + f(t,x,\alpha) \right\} = 0$$

$$\text{in } (0,T]\times\mathbb{R}^N, \qquad u(0,x) = g(x) \text{ in } \mathbb{R}^N,$$

$$L[u](t,x,\alpha) = \frac{1}{2}\mathrm{Tr}[\sigma(t,x,\alpha)\sigma^\top(t,x,\alpha)D^2u(t,x)] + b(t,x,\alpha)Du(t,x)$$

Semi-discretization in $(0,T)\times X_{\Delta x}$:

$$U_t(t,x) - \inf_{\alpha\in\mathcal{A}} \left\{ L_k[\mathcal{I}U](t,x,\alpha) + c(t,x,\alpha)U(t,x) + f(t,x,\alpha) \right\} = 0,$$

$$L_k[\varphi](t,x,\alpha) := \sum_{i=1}^{P} \frac{\varphi(t, x+y_{k,i}^+(t,x,\alpha)) - 2\varphi(t,x) + \varphi(t, x+y_{k,i}^-(t,x,\alpha))}{2k^2}$$

$$\text{such that } \sum_{i=1}^{P}[y_{k,i}^+ + y_{k,i}^-] = 2k^2 b + \mathcal{O}(k^4),$$

$$\sum_{i=1}^{P}[y_{k,i}^+ y_{k,i}^{+\top} + y_{k,i}^- y_{k,i}^{-\top}] = 2k^2\sigma\sigma^\top + \mathcal{O}(k^4) \implies k|\sigma| \sim \text{ stencil length}$$

$$\sum_{i=1}^{P}[\otimes_{j=1}^3 y_{k,i}^+ + \otimes_{j=1}^3 y_{k,i}^-] = \mathcal{O}(k^4), \qquad \sum_{i=1}^{P}[\otimes_{j=1}^3 y_{k,i}^+ + \otimes_{j=1}^3 y_{k,i}^-] = \mathcal{O}(k^4)$$

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$$\implies |L_k[\varphi] - L[\varphi]| \le C\big(\|D\varphi\|_\infty + \|D^2\varphi\|_\infty + \|D^3\varphi\|_\infty + \|D^4\varphi\|_\infty\big)k^2$$

Important for monotonicity: $\mathcal{I}$ monotone, i. e.

$(\mathcal{I}\varphi)(x) = \sum_j \varphi(x_j)w_j(x)$, $\;w_i(x_j) = \delta_{ij}$, and $w_j(x) \ge 0$ for all $i,j\in\mathbb{N}$

In general: Not more than second order accurate interpolation ($\mathcal{I}$ linear)

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Important for monotonicity: $\mathcal{I}$ monotone, i. e.

$$(\mathcal{I}\varphi)(x) = \sum_j \varphi(x_j)w_{\varphi,j}(x), \quad w_{\varphi,i}(x_j) = \delta_{ij}, \quad w_{\varphi,j}(x) \ge 0, \quad \sum_j w_{\varphi,j}(x) \equiv 1$$

In general: Not more than second order accurate interpolation ($\mathcal{I}$ linear)

But: Higher order monotonicity preserving interpolation is possible if φ is known to be monotone $\to$ nonlinear in φ

A unifying framework - examples

1. The approximation of Falcone (1987): $k = \sqrt{\Delta x}$, $y_k^{\pm} = k^2 b$

$$bD\varphi \approx \frac{\mathcal{I}\varphi(x + \Delta x b) - \mathcal{I}\varphi(x)}{\Delta x}$$

2. Camilli-Falcone (1995): $k = \sqrt{\Delta x}$, $y_{k,j}^{\pm} = \pm k\sigma_j + \frac{k^2}{M} b$

$$\frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top D^2\varphi] + bD\varphi$$
$$\approx \sum_{j=1}^{M} \frac{\mathcal{I}\varphi(x + \sqrt{\Delta x}\sigma_j + \frac{\Delta x}{M} b) - 2\mathcal{I}\varphi(x) + \mathcal{I}\varphi(x - \sqrt{\Delta x}\sigma_j + \frac{\Delta x}{M} b)}{2\Delta x}$$

3. New version (efficient for σ independent of α):

$$\frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top D^2\varphi] + bD\varphi$$
$$\approx \frac{\mathcal{I}\varphi(x + k^2 b) - \mathcal{I}\varphi(x)}{k^2} + \sum_{j=1}^{M} \frac{\mathcal{I}\varphi(x + k\sigma_j) - 2\mathcal{I}\varphi(x) + \mathcal{I}\varphi(x - k\sigma_j)}{2k^2}$$

Monotonicity preserving cubic Hermite interpolation

(Fritsch & Carlson 1980, Eisenstat, Jackson & Lewis 1985)

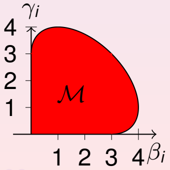
- On $[x_i, x_{i+1}]$: $(\mathcal{I}\varphi)(x) = c_0 + c_1(x-x_i) + c_2(x-x_i)^2 + c_3(x-x_i)^3$ with $(\mathcal{I}\varphi)(x_i) = \varphi_i,\ (\mathcal{I}\varphi)(x_{i+1}) = \varphi_{i+1},\ (\mathcal{I}\varphi)'(x_i) \approx \varphi'_i,\ (\mathcal{I}\varphi)'(x_{i+1}) \approx \varphi'_{i+1}$
- $d_i = \frac{\varphi_{i-2} - 8\varphi_{i-1} + 8\varphi_{i+1} - \varphi_{i+2}}{12\Delta x}, \quad \Delta_i = \frac{\varphi_{i+1} - \varphi_i}{\Delta x}, \quad \beta_i = \frac{d_i}{\Delta_i},\ \gamma_i = \frac{d_{i+1}}{\Delta_i}.$
- Use $(\mathcal{I}\varphi)'(x_i) = \beta_i \Delta_i$, $(\mathcal{I}\varphi)'(x_{i+1}) = \gamma_i \Delta_i \implies$

$$(\mathcal{I}\varphi)(x) = \varphi_i + (\varphi_{i+1} - \varphi_i) P_i(x) \text{ where}$$

$$P_i(x) = \beta_i \frac{x - x_i}{\Delta x} + (3 - \gamma_i - 2\beta_i)\left(\frac{x-x_i}{\Delta x}\right)^2 - (2 - \beta_i - \gamma_i)\left(\frac{x-x_i}{\Delta x}\right)^3.$$

$$\implies w_{\varphi,i}(x) = (1 - P_i(x))1_{[x_i, x_{i+1})}(x) + P_{i-1}(x)1_{[x_{i-1}, x_i)}(x)$$

- Modify β_i and γ_i such that $(\beta_i, \gamma_i) \in \mathcal{M}$:



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- $d_i = \frac{\varphi_{i-2} - 8\varphi_{i-1} + 8\varphi_{i+1} - \varphi_{i+2}}{12\Delta x}, \quad \Delta_i = \frac{\varphi_{i+1} - \varphi_i}{\Delta x}, \quad \beta_i = \frac{d_i}{\Delta_i},\ \gamma_i = \frac{d_{i+1}}{\Delta_i}.$
- Use $(\mathcal{I}\varphi)'(x_i) = \beta_i \Delta_i$, $(\mathcal{I}\varphi)'(x_{i+1}) = \gamma_i \Delta_i \implies$

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$$P_i(x) = \beta_i \frac{x - x_i}{\Delta x} + (3 - \gamma_i - 2\beta_i)\left(\frac{x - x_i}{\Delta x}\right)^2 - (2 - \beta_i - \gamma_i)\left(\frac{x - x_i}{\Delta x}\right)^3.$$

$$\implies w_{\varphi,i}(x) = (1 - P_i(x))1_{[x_i, x_{i+1})}(x) + P_{i-1}(x)1_{[x_{i-1}, x_i)}(x)$$

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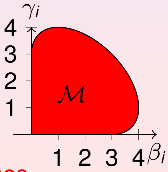


$\implies$ fourth order accurate monotonicity preserving C^0 interpolant $\implies$ New second order compact stencil schemes

Discretization in time

Semi-discretization in $(0, T) \times X_{\Delta x}$:

$$U_t(t,x) - \inf_{\alpha \in \mathcal{A}} \left\{ L_k[\mathcal{I}U](t,x,\alpha) + c(t,x,\alpha)U(t,x) + f(t,x,\alpha) \right\} = 0$$

$$L_k[\varphi](t,x,\alpha) := \sum_{i=1}^{P} \frac{\varphi\big(t, x + y^+_{k,i}(t,x,\alpha)\big) - 2\varphi(t,x) + \varphi\big(t, x + y^-_{k,i}(t,x,\alpha)\big)}{2k^2},$$

Fully discrete scheme: $\theta \in [0,1]$,

$$\frac{U_i^n - U_i^{n-1}}{\Delta t} = \inf_{\alpha \in \mathcal{A}} \left\{ L_k^\alpha[\mathcal{I}\bar{U}_\cdot^{\theta,n}]_i^{n-1+\theta} + c_i^{\alpha,n-1+\theta}\bar{U}_i^{\theta,n} + f_i^{\alpha,n-1+\theta} \right\}$$

where $\bar{U}_\cdot^{\theta,n} = (1-\theta)U_\cdot^{n-1} + \theta U_\cdot^n$, $U_i^0 = g(x_i)$ in $X_{\Delta x}$.

Proposed schemes - general properties

Theorem (D./Jakobsen)

The considered scheme has a truncation error bounded by

$$\mathcal{O}\Big((1-2\theta)\Delta t + \Delta t^2 + \frac{\Delta x^r}{k^2} + k^2\Big). \qquad \begin{array}{l} r=2: \; k \sim \sqrt{\Delta x} \\ r=4: \; k \sim \Delta x \end{array}$$

Under the CFL condition

$$(1-\theta)\Delta t \left[\frac{P}{k^2} - c_i^{\alpha,n-1+\theta}\right] \leq 1 \;\; \text{and} \;\; \theta \Delta t \, c_i^{\alpha,n-1+\theta} \leq 1 \;\; \text{for all } \alpha, n, i$$

it has a unique bounded solution U, which is stable when $2\theta\Delta t \sup_\alpha |c^{\alpha,+}|_0 \leq 1$*:*

$$|U^n|_0 \leq e^{2\sup_\alpha |c^{\alpha,+}|_0 t_n}\Big[|g|_0 + t_n \sup_\alpha |f^\alpha|_0\Big].$$

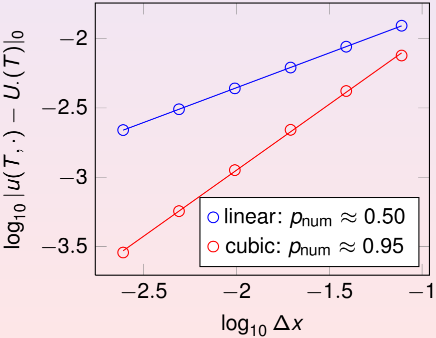
For monotone interpolation it is monotone under the above CFL condition.

Barles-Souganidis result $\Longrightarrow$ for linear interpolation, U converges uniformly to the viscosity solution u as $\Delta t, k, \frac{\Delta x^r}{k^2} \to 0$.

Test problem with non-smooth solution

$$u_t - \mathrm{Tr}\left(\begin{pmatrix} \sin^2 x_1 & \sin x_1 \sin x_2 \\ \sin x_1 \sin x_2 & \sin^2 x_2 \end{pmatrix} D^2 u\right) = f(t,x)$$

$$u(t,x) = (1+t)\sin\frac{x_2}{2}\begin{cases}\sin\frac{x_1}{2} & \text{for } -\pi < x_1 < 0, \\ \sin\frac{x_1}{4} & \text{for } 0 < x_1 < \pi.\end{cases}$$



An application from finance

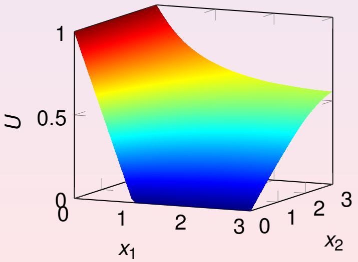
Superreplication problem of European options: Stochastic volatility model with gamma constraints (Bruder, Bokanowski, Maroso, Zidani, SIAM J. Numer. Anal. 2009)

$$\inf_{\alpha_1^2+\alpha_2^2=1} \left\{ \alpha_1^2 u_t(t,x) - \frac{1}{2} \operatorname{Tr}\left(\sigma^\alpha(t,x) \sigma^{\alpha\top}(t,x) D^2 u(t,x) \right) \right\} = 0, \quad 0 < x_1, x_2, t$$

$$u(0,x_1,x_2) = \max(0, 1-x_1), \qquad 0 \le x_1, x_2 \le 3$$

$$\sigma^\alpha(t,x) = \begin{pmatrix} \alpha_1 x_1 \sqrt{x_2} \\ \alpha_2 \eta(t,x_1,x_2) \end{pmatrix}$$

for arbitrary $\eta(t,x_1,x_2) > 0$.
Solution monotone in x_1 and x_2 direction



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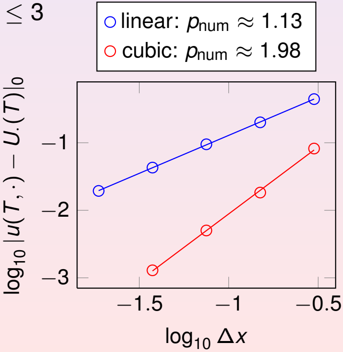
Test problem: $u(t,x) = 1 + t^2 - e^{-x_1^2 - x_2^2}$



Proposed schemes - convergence result

Theorem (D./Jakobsen)

Under the above assumptions,

$$|u - U| \leq C(|1 - 2\theta|\Delta t^{1/4} + \Delta t^{1/3} + k^{1/2} + \frac{\Delta x}{k^2}).$$

$k \sim \Delta x^{2/5}, \; \Delta t \sim k^2 \sim \Delta x^{4/5} \implies |u - U| = \mathcal{O}(\Delta x^{1/5})$

- applies to both linear and monotonicity preserving cubic interpolation
- no convergence for $k \sim \sqrt{\Delta x}$ ($r = 2$) and $k \sim \Delta x$ ($r = 4$)
- holds also for unstructured grids

Ideas of proof:

1. $|U - V| \leq C\frac{\Delta x}{k^2}$
2. $|u - V|$: extension of ideas used in the stationary case by Camilli/Falcone, Barles/Jakobsen

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Conclusion

- class of semi-Lagrangian schemes for Hamilton-Jacobi-Bellman equations
- unifying framework for several known first order schemes, as well as new versions, including new second order compact stencil schemes for essentially monotone solutions
- for linear interpolation: stability and convergence in the general case, also for HJB-Isaacs equations
- both linear and cubic interpolation: convergence with rate $\Delta x^{\frac{1}{5}}$

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Thank you very much for your attention!

Outline

4 Hamilton-Jacobi-Bellman equations and viscosity solutions

Stochastic control problem - example (Merton 1969)

Investment and consumption problem:

$$dR(t) = rR(t)\,dt$$
$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$$

$$\begin{aligned} dX(t) &= r\big(1-\pi(t)\big)X(t)\,dt + \pi(t)X(t)\big(\mu\,dt + \sigma\,dW(t)\big) - C(t)\,dt \\ &= \Big[\big(r + (\mu - r)\pi(t)\big)X(t) - C(t)\Big]\,dt + \pi(t)X(t)\sigma\,dW(t) \end{aligned}$$

$$X(t_0) = X_0.$$

$$u(t_0, X_0) = \sup_{(\pi, C)\in\mathfrak{A}} \mathrm{E}\left(\int_{t_0}^{T} e^{-\rho(t-t_0)} f(C(t))\,dt + e^{-\rho(T-t_0)} g(X(T))\right)$$

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$$dR(t) = rR(t)\,dt$$
$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$$

$$\begin{aligned} dX(t) &= r\big(1-\pi(t)\big)X(t)\,dt + \pi(t)X(t)\big(\mu\,dt + \sigma\,dW(t)\big) - C(t)\,dt \\ &= \big[\big(r + (\mu - r)\pi(t)\big)X(t) - C(t)\big]\,dt + \pi(t)X(t)\sigma\,dW(t) \\ X(t_0) &= X_0. \end{aligned}$$

$$u(t_0, X_0) = \sup_{(\pi,C)\in\mathfrak{A}} \mathrm{E}\left(\int_{t_0}^{T} e^{-\rho(t-t_0)} f\big(C(t)\big)\,dt + e^{-\rho(T-t_0)} g\big(X(T)\big)\right)$$

Stochastic control problem $\Longrightarrow$ HJB equation

$$dX(t) = b\big(t, X(t), \alpha(t)\big)\ dt + \sum_{i=1}^{M} \sigma_i\big(t, X(t), \alpha(t)\big)\ dW_i(t), \quad t_0 \le t \le T,$$

$$X(t_0) = X_0 \in \mathbb{R}^N.$$

$$u(t_0, X_0) = \inf_{\alpha \in \mathfrak{A}} \mathrm{E}\left(\int_{t_0}^{T} \beta(t_0, t) f\big(t, X(t), \alpha(t)\big)\ dt + \beta(t_0, T) g\big(X(T)\big) \right)$$

where

$$\beta(t_0, t) = e^{\int_{t_0}^{t} c\big(s, X(s), \alpha(s)\big)\ ds}.$$

Under appropriate assumptions: u solves HJB equation

$$-u_t(t,x) - \inf_{\alpha \in \mathcal{A}} \Big\{ L[u](t,x,\alpha) + c(t,x,\alpha) u(t,x) + f(t,x,\alpha) \Big\} = 0$$

$$\text{in } [0, T) \times \mathbb{R}^N, \qquad u(T,x) = g(x) \text{ in } \mathbb{R}^N,$$

$$L[u](t,x,\alpha) = \frac{1}{2} \mathrm{Tr}[\sigma(t,x,\alpha)\sigma^{\top}(t,x,\alpha) D^2 u(t,x)] + b(t,x,\alpha) Du(t,x)$$

Stochastic control problem $\Longrightarrow$ HJB equation

$$dX(t) = b\big(t, X(t), \alpha(t)\big)\ dt + \sum_{i=1}^{M} \sigma_i\big(t, X(t), \alpha(t)\big)\ dW_i(t), \quad t_0 \le t \le T,$$

$$X(t_0) = X_0 \in \mathbb{R}^N.$$

$$u(t_0, X_0) = \inf_{\alpha \in \mathfrak{A}} \mathrm{E}\left(\int_{t_0}^{T} \beta(t_0, t) f\big(t, X(t), \alpha(t)\big)\ dt + \beta(t_0, T) g\big(X(T)\big) \right)$$

where

$$\beta(t_0, t) = e^{\int_{t_0}^{t} c\left(s, X(s), \alpha(s)\right)\ ds}.$$

Under appropriate assumptions: u solves HJB equation

$$-u_t(t, x) - \inf_{\alpha \in \mathcal{A}} \Big\{ L[u](t, x, \alpha) + c(t, x, \alpha) u(t, x) + f(t, x, \alpha) \Big\} = 0$$

$$\text{in } [0, T) \times \mathbb{R}^N, \qquad u(T, x) = g(x) \text{ in } \mathbb{R}^N,$$

$$L[u](t, x, \alpha) = \frac{1}{2} \mathrm{Tr}[\sigma(t, x, \alpha) \sigma^{\top}(t, x, \alpha) D^2 u(t, x)] + b(t, x, \alpha) Du(t, x)$$

Stochastic control problem $\Longrightarrow$ HJB equation

$$dX(t) = b\big(t, X(t), \alpha(t)\big)\, dt + \sum_{i=1}^{M} \sigma_i\big(t, X(t), \alpha(t)\big)\, dW_i(t), \quad t_0 \le t \le T,$$

$$X(t_0) = X_0 \in \mathbb{R}^N.$$

$$u(t_0, X_0) = \inf_{\alpha \in \mathfrak{A}} \mathrm{E}\left(\int_{t_0}^{T} \beta(t_0, t) f\big(t, X(t), \alpha(t)\big)\, dt + \beta(t_0, T) g\big(X(T)\big) \right)$$

where

$$\beta(t_0, t) = e^{\int_{t_0}^{t} c\left(s, X(s), \alpha(s)\right)\, ds}.$$

Under appropriate assumptions: u solves in the viscosity sense

$$-u_t(t, x) - \inf_{\alpha \in \mathcal{A}} \Big\{ L[u](t, x, \alpha) + c(t, x, \alpha) u(t, x) + f(t, x, \alpha) \Big\} = 0$$

$$\text{in } [0, T) \times \mathbb{R}^N, \qquad u(T, x) = g(x) \text{ in } \mathbb{R}^N,$$

$$L[u](t, x, \alpha) = \frac{1}{2} \mathrm{Tr}[\sigma(t, x, \alpha) \sigma^\top(t, x, \alpha) D^2 u(t, x)] + b(t, x, \alpha) D u(t, x)$$

$\mathcal{A} \subset \mathbb{R}^k$
Control processes $\mathfrak{A}_0$: set of all progressively measurable processes $\nu = \{\nu(t),\ t \geq 0\}$ valued in $\mathcal{A}$.
Admissible control processes $\mathfrak{A}$: subset of all $\nu \in \mathfrak{A}_0$ for which

$$\mathrm{E} \int_{t_0}^{T} \left(|b(t, x, \nu(t))| + |\sigma(t, x, \nu(t))|^2 \right) \, dt < \infty \quad \text{for } x \in \mathbb{R}^n$$

(guarantees existence of a controlled process for each given initial condition and control).
Cost functional: $\|c\|_\infty < \infty$,

$$|f(t, x, \alpha)| + |g(x)| \leq C(1 + \|x\|^2)$$

for some constant C independent of (t, α)

Viscosity solutions (Crandall & Lions 1983)

$$F\Big(t,x,u(t,x),u_t(t,x),Du(t,x),D^2u(t,x)\Big) = 0 \text{ on } (0,T)\times\Omega \quad (\star)$$

fulfilling the degenerate ellipticity condition: ($A, B \in S^N$)

$$F(t,x,r,s,p,A) \le F(t,x,r,s,p,B) \text{ whenever } A \ge B$$
$$\forall (t,x,r,s,p) \in [0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N.$$

Example (HJB equation)

$$F(t,x,r,s,p,A) = -s - \inf_{\alpha\in\mathcal{A}} \Big\{ \frac{1}{2}\mathrm{Tr}[\sigma(t,x,\alpha)\sigma^\top(t,x,\alpha)A] + b(t,x,\alpha)p + c(t,x,\alpha)r + f(t,x,\alpha)\Big\}$$

Viscosity solutions (Crandall & Lions 1983)

$$F\big(t, x, u(t,x), u_t(t,x), Du(t,x), D^2u(t,x)\big) = 0 \text{ on } (0,T)\times\Omega \quad (\star)$$

fulfilling the degenerate ellipticity condition: $(A, B \in S^N)$

$$F(t,x,r,s,p,A) \leq F(t,x,r,s,p,B) \text{ whenever } A \geq B$$
$$\forall (t,x,r,s,p) \in [0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N.$$

Then $v \in C^{1,2}\big((0,T)\times\Omega\big)$ is a classical supersolution of $(\star)$ iff

for all $(t_0, x_0, \varphi) \in (0,T) \times \Omega \times C^{1,2}\big((0,T)\times\Omega\big)$ such that

(t_0, x_0) is a minimizer of $v - \varphi$ on (t_0, Ω)

$$F\big(t_0, x_0, v(t_0,x_0), \varphi_t(t_0,x_0), D\varphi(t_0,x_0), D^2\varphi(t_0,x_0)\big) \geq 0.$$

Viscosity solutions (Crandall & Lions 1983)

$$F\big(t, x, u(t,x), u_t(t,x), Du(t,x), D^2u(t,x)\big) = 0 \text{ on } (0,T)\times\Omega \quad (\star)$$

fulfilling the degenerate ellipticity condition: $(A, B \in S^N)$

$$F(t,x,r,s,p,A) \leq F(t,x,r,s,p,B) \text{ whenever } A \geq B$$
$$\forall (t,x,r,s,p) \in [0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N.$$

Then $v \in C^{0,0}\big((0,T)\times\Omega\big)$ is a classical $\begin{matrix}\text{supersolution}\\ \text{subsolution}\end{matrix}$ of $(\star)$ iff for all $(t_0, x_0, \varphi) \in (0,T) \times \Omega \times C^{1,2}\big((0,T)\times\Omega\big)$ such that (t_0, x_0) is a $\begin{matrix}\text{minimizer}\\ \text{maximizer}\end{matrix}$ of $v - \varphi$ on (t_0, Ω)

$$F\big(t_0, x_0, v(t_0,x_0), \varphi_t(t_0,x_0), D\varphi(t_0,x_0), D^2\varphi(t_0,x_0)\big) \begin{matrix}\geq\\ \leq\end{matrix} 0.$$

Viscosity solutions (Crandall & Lions 1983)

$$F\big(t,x,u(t,x),u_t(t,x),Du(t,x),D^2u(t,x)\big)=0 \text{ on } (0,T)\times\Omega \quad (\star)$$

fulfilling the degenerate ellipticity condition: $(A,B\in S^N)$

$$F(t,x,r,s,p,A)\le F(t,x,r,s,p,B) \text{ whenever } A\ge B$$
$$\forall (t,x,r,s,p)\in[0,T]\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N.$$

Then $v\in C^{0,0}\big((0,T)\times\Omega\big)$ is a viscosity $\begin{matrix}\text{supersolution}\\ \text{subsolution}\end{matrix}$ of $(\star)$ iff for all $(t_0,x_0,\varphi)\in(0,T)\times\Omega\times C^{1,2}\big((0,T)\times\Omega\big)$ such that (t_0,x_0) is a $\begin{matrix}\text{minimizer}\\ \text{maximizer}\end{matrix}$ of $v-\varphi$ on (t_0,Ω)

$$F\big(t_0,x_0,v(t_0,x_0),\varphi_t(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)\big) \begin{matrix}\ge\\ \le\end{matrix} 0.$$