Two uniqueness results for the two-dimensional continuity equation with velocity having L^1 or measure curl

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CONTINUITY EQUATION

$$\partial_t u + \operatorname{div}(bu) = 0 \qquad\qquad b(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$$

— Cauchy problem: existence, uniqueness and stability

— Further properties: compactness of solutions

— Connection with the ODE $\dot{X}(t,x) = b(t, X(t,x))$

MAIN RESULTS

— DiPerna & Lions (1989) — $b \in W^{1,p}$ with $\operatorname{div} b \in L^\infty$

— Ambrosio (2004) — $b \in BV$ with $\operatorname{div} b \in L^\infty$

— (and many others!)

ASSUMPTIONS ON THE CURL

We work in 2D and remove bounds on the *full* derivative

$$\begin{pmatrix} \partial_1 b^1 & \partial_2 b^1 \\ \partial_1 b^2 & \partial_2 b^2 \end{pmatrix} \in L^p \quad \text{or } \in \mathcal{M}$$

and require bounds just on

$$\operatorname{curl} b = -\partial_2 b^1 + \partial_1 b^2 \,.$$

Further assumptions:

$$b \in L^\infty \,, \qquad \operatorname{div} b = 0$$

(they can be somehow relaxed).

MOTIVATIONS: 2D INCOMPRESSIBLE NONVISCOUS FLUIDS

Euler equation (velocity): $\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla p \\ \operatorname{div} v = 0\,. \end{cases}$

vorticity $= \omega = \operatorname{curl} v =$ measure of local rotation of the fluid

Taking the curl of Euler (velocity):

Euler equation (vorticity): $\begin{cases} \partial_t \omega + \operatorname{div}(v\,\omega) = 0 \\ \omega = \operatorname{curl} v\,, \quad \operatorname{div} v = 0\,. \end{cases}$

Pression has been eliminated.

Still nonlinear, due to the coupling. Indeed it is nonlocal:

$$v(t,x) = K * \omega = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(t,y)\, dy\,.$$

Typical regularity of velocity $v(t,x)$:

1) Conservation kinetic energy: $v \in L_t^\infty(L^2_{x,\mathrm{loc}})$.

2) Incompressibility: $\operatorname{div} v = 0$.

3) *Formal* conservation of L^p norms of ω:

— $\operatorname{curl} v \in L_t^\infty(L_x^p)$ if $\operatorname{curl} v(0,\cdot) \in L^p$,

— $\operatorname{curl} v \in L_t^\infty(\mathcal{M}_x)$ if $\operatorname{curl} v(0,\cdot) \in \mathcal{M}$.

Known results for Euler (vorticity)

- $\omega \in L^\infty$: existence and uniqueness – Yudovich (1963)
- $\omega \in L^p$: existence – DiPerna-Majda ($\sim$1987, $p > 1$),
  Vecchi-Wu (1993, $p = 1$)
- $\omega \in H^{-1}$ and positive measure: existence – Delort (1991)
- Vortex-sheets problem: existence for ω *signed* measure in H^{-1}.

Calderon–Zygmund theory

Biot-Savart law:

$$v(t,x) = K * \omega = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(t,y)\, dy\,.$$

— If $\omega \in L^p$ with $p > 1$, then $Db \in L^p$ and so $b \in W^{1,p}$. Back to the DiPerna–Lions setting, for the *linear* equation.

— This is no more valid when $p = 1$, or when ω is a measure: we do NOT get $W^{1,1}$ or BV.

Aim of this talk: to discuss two results for the linear equation. The velocity b is given and we discuss well-posedness of

$$\partial_t u + \operatorname{div}(bu) = 0\,.$$

We assume that $\operatorname{curl} b \in L^1$, or that $\operatorname{curl} b$ is a measure.

This is different (and still far) from treating the Euler equation, but it is the "first step" (study of the "linearization").

– Compactness for the linear PDE $\Rightarrow$ Existence for nonlinear Euler.

– Uniqueness for Euler is apparently unrelated from this approach.

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$b \in L^\infty$, autonomous, $\operatorname{div} b = 0$

$\operatorname{curl} b = \omega \in \mathcal{M}$ with no sign restrictions

COLLABORATIONS WITH:

G. ALBERTI AND S. BIANCHINI

RESULTS:

— Uniqueness and Renormalization for the PDE;

— Uniqueness for the ODE regular Lagrangian flow.

STRATEGY OF PROOF.

The 2D flow can be decomposed into 1D flows on

$$\left\{ x \ : \ \Psi(x) = c \right\} \qquad c \in \mathbb{R} \,.$$

These are stationary level sets.

Uniqueness on level sets is characterized via the weak Sard property: it forbids concentrations of the solution in a square-root-like fashion.

The weak Sard property (informally) amounts to requiring restrictions on the critical set of Ψ. It is implied by:

Ψ has the Lusin property with functions of class C^2.

Lusin property C^2: for every $\varepsilon > 0$ exists $\widetilde{\Psi} \in C^2$ such that $\widetilde{\Psi} = \Psi$ at every point except a set of measure less than ε.

$$\begin{aligned} \Psi \in C^2 \quad &\Longrightarrow \quad \text{Sard: critical values are negligible} \\ \text{Lusin property } C^2 \quad &\Longrightarrow \quad \text{weak Sard property,} \\ &\qquad\ \ \text{a suitable measure-theoretical weakening} \\ &\qquad\ \ \text{of “critical values are negligible”} \end{aligned}$$

$$\underset{\substack{\uparrow \\ \text{push forward}}}{\Psi_\#} \Big(\mathscr{L}^2 \underset{\substack{\uparrow \\ \text{restriction}}}{\llcorner} \big(\{\nabla\Psi = 0\} \setminus D\big)\Big) \underset{\substack{\uparrow \\ \text{singular}}}{\perp} \mathscr{L}^1$$

Main result: for $b = K * \omega$ with $\omega \in \mathcal{M}$ there holds:

b is L^p-differentiable at almost all point, for $1 \leq p < 2$.

This means existence of a Taylor expansion at order 1, with a rest "small in average" for h small:

$$b(x+h) = P^1_x(h) + R^1_x(h)\,, \qquad \left[\fint_{B_\rho} |R^1_x(h)|^p \, dh \right]^{1/p} = o(\rho)$$

This implies that Ψ (such that $b = \nabla^\perp \Psi$) has the same property of order 2, and the L^p version of the Whitney extension theorem gives the Lusin property C^2.

Advantages:

– works for a measure vorticity, with no sign assumptions.

Possible extensions:

– some time dependence can be considered: a scalar such that

$$b(t,x) = a(t,x)\nabla^{\perp}\Psi(x)\,;$$

– for such b, it is enough to have bounded divergence.

Disadvantages:

– not clear how to deal with “effective” time dependence;

– difficult to relax $b \in L^{\infty}$: we want topological properties of $\{\Psi = c\}$.

b **possibly unbounded,**

time dependent, $\operatorname{div} b = 0$

$\operatorname{curl} b = \omega \in L^1$

COLLABORATION WITH:

F. BOUCHUT

RESULTS:

Quantitative estimates for the regular Lagrangian flow:

— Existence, Uniqueness and Stability;

— Compactness.

Well-posedness for *Lagrangian* solutions of the continuity equation.

Advantages:

– time-dependent vector fields, not necessarily in L^∞;

– divergence bounds are relaxed;

– it works in any space dimension;

– other singular kernels than Biot-Savart;

– quantitative compactness and stability rates.

Disadvantages:

– it seems difficult to reach the case of a measure curl.

TECHNIQUE OF PROOF:

As in Crippa-De Lellis, a functional measuring the non uniqueness:

$$\Phi_\delta(t) = \int_{\mathbb{R}^2} \log\left(1 + \frac{|X_1(t,x) - X_2(t,x)|}{\delta}\right)\, dx\,,$$

where X_1 and X_2 are flows of b.

If $X_1 \neq X_2$ then $|X_1 - X_2| \geq \gamma > 0$ on a set A of meas. $\alpha > 0$ and so

$$\Phi_\delta(t) \geq \int_A \log\left(1 + \frac{\gamma}{\delta}\right)\, dx \geq \alpha \log\left(1 + \frac{\gamma}{\delta}\right)\,.$$

A condition for uniqueness is then

$$\frac{\Phi_\delta(t)}{\log\left(\dfrac{1}{\delta}\right)} \to 0 \qquad \text{as } \delta \downarrow 0.$$

Basic case in which this holds:

$$|\Phi_\delta(t)| \leq C \qquad \text{uniformly in } \delta. \tag{$\star$}$$

Differentiating in time

$$\begin{aligned}
\Phi'_\delta(t) &\leq \int_{\mathbb{R}^2} \frac{|b(X_1) - b(X_2)|}{\delta + |X_1 - X_2|}\, dx \\
&\leq \int_{\mathbb{R}^2} \min\left\{ \frac{2\|b\|_\infty}{\delta} \; ; \; \frac{|b(X_1) - b(X_2)|}{|X_1 - X_2|} \right\} dx \,,
\end{aligned}$$

and when $b \in W^{1,p}$, $p > 1$, the maximal function estimate

$$\frac{|b(x) - b(y)|}{|x - y|} \leq C\big(MDb(x) + MDb(y)\big)$$

allows to conclude ($\star$). This was contained in Crippa-De Lellis 2008.

The key point was the strong estimate $\|Mf\|_{L^p} \le C\|f\|_{L^p}$, $p > 1$.

Obstruction for $\operatorname{curl} b = \omega \in \mathcal{M}$: only weak estimates (both for the maximal function and for the singular integral $Db = \mathscr{K} * \omega$).

1) New *smooth* maximal function

$$M_\rho f(x) = \sup_{r>0} \left| \int_{\mathbb{R}^2} \rho_r(x-y) f(y)\, dy \right|$$

so that there are cancellations in the composition $M_\rho\big(\mathscr{K} * \omega\big)$.

2) This composition enjoys the weak estimate

$$\mathscr{L}^2\Big(\big\{x \,:\, |M_\rho\big(\mathscr{K} * \omega\big)(x)| > \lambda\big\}\Big) \le C\frac{\|\omega\|_{\mathcal{M}}}{\lambda}\,.$$

3) Going back to the estimate for Φ'_δ we have

$$\Phi'_\delta(t) \le \int_{\mathbb{R}^2} \min\left\{ \frac{2\|b\|_\infty}{\delta}\ ;\ 2M_\rho\big(\mathscr{K} * \omega\big) \right\} dx\,.$$

4) Interpolating between

$$\|f\|_\infty \quad \text{and} \quad \|f\|_{M^1} = \sup_{\lambda>0} \Big\{ \lambda \mathscr{L}^2\big(\{|f| > \lambda\}\big) \Big\}$$

gives

$$\Phi'_\delta(t) \leq C\|\omega\|_{\mathcal{M}} \left[1 + \log \left(\frac{C}{\delta\|\omega\|_{\mathcal{M}}} \right) \right],$$

exactly the critical rate for uniqueness.

5) Only now we need $\omega \in L^1$: we gain smallness since we can write $\omega = \omega^1 + \omega^2$, with ω^1 small in L^1 and $\omega^2 \in L^2$.

Similar arguments: quantitative stability and compactness.

Consequence: a new existence of *Lagrangian* solutions to 2D Euler with L^1 vorticity.

Open question: can we treat $\omega \in \mathcal{M}$ with this approach?