Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Stable Numerical Scheme for the Magnetic Induction Equation with Hall Effect

Paolo Corti

joint work with Siddhartha Mishra

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Outline

- Formulation and motivation of the problem
- Theoretical analysis
- DG Formulation
- 1D Model
- Numerical Tests

Magnetic Reconnection

- Change in topology of the magnetic field

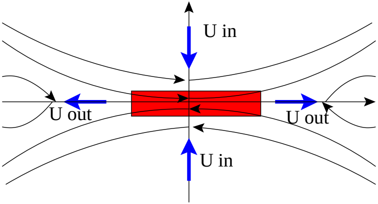

Figure: Schematic of a reconnection.

- Magnetic energy $\Rightarrow$ kinetic and thermal energy
- Dissipation

MHD Equations

The equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}) \\
\frac{\partial (\rho \mathbf{u})}{\partial t} &= -\nabla \left\{ \rho \mathbf{u}\mathbf{u}^\top + \left(p + \frac{\mathbf{B}^2}{2} \right) \mathbf{I}_{3\times 3} - \mathbf{B}\mathbf{B}^\top \right\} \\
\frac{\partial \mathcal{E}}{\partial t} &= -\nabla \left\{ \left(\mathcal{E} + p - \frac{B^2}{2} \right) \mathbf{u} + \mathbf{E} \times \mathbf{B} \right\} \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}
\end{aligned}
$$

are coupled through the equation of state

$$
\mathcal{E} = \frac{p}{\gamma - 1} + \frac{\rho \mathbf{u}^2}{2} + \frac{B^2}{2}
$$

To complete the formulation of the problem we need to state some equation for **E**

Ideal MHD

Standard model for **E**:
Ohm's Law

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}$$

Problem:
no dissipation $\Rightarrow$ "frozen" condition.

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Resistive MHD:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J}$$

not sufficient for fast reconnection.

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We need another model...
Numerical simulation and laboratory experiment $\Rightarrow$ Hall Effect

Generalized Ohm's Law

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J} + \frac{\delta_i}{L_0} \frac{\mathbf{J} \times \mathbf{B}}{\rho} - \frac{\delta_i}{L_0} \frac{\nabla \overleftrightarrow{p}}{\rho} + \left(\frac{\delta_e}{L_0} \right)^2 \frac{1}{\rho} \left[\frac{\partial \mathbf{J}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{J} \right]$$

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- Resistivity
- Hall effect
- Electron pressure
- Electron inertia

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J is the electric current given by Ampère's law

$$\mathbf{J} = \nabla \times \mathbf{B}$$

X.Qian, J.Bablás, A. Bhattacharjee, H.Yang (2009)

General Induction Equation

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- Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

- Generalized Ohm law
- Ampère's law
- $\overleftrightarrow{p}$ isotropic.

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Are combined to obtain

$$\frac{\partial}{\partial t}\left[\mathbf{B} + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\nabla \times (\nabla \times \mathbf{B})\right] = \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta \nabla \times (\nabla \times \mathbf{B})$$

$$- \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\nabla \times ((\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{B})) - \frac{\delta_i}{L_0}\frac{1}{\rho}\nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B})$$

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This equation preserve the divergence of the magnetic field

$$\frac{d}{dt}(\nabla \cdot \mathbf{B}) = 0$$

Symmetrized Equation

Using the identity

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) + \mathbf{u}(\nabla \cdot \mathbf{B}) - (\mathbf{u} \cdot \nabla)\mathbf{B}$$

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Since the magnetic field is solenoidal $\nabla \cdot \mathbf{B} = 0$ we subtract $\mathbf{u}(\nabla \cdot \mathbf{B})$ to the right side of the equation

$$\begin{aligned} \frac{\partial}{\partial t}\left[\mathbf{B} + \left(\frac{\delta_e}{L_0}\right)^2 \nabla \times (\nabla \times \mathbf{B})\right] = \\ (\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{B} - \eta \nabla \times (\nabla \times \mathbf{B}) \\ - \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho} \nabla \times ((\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{B})) - \frac{\delta_i}{L_0}\frac{1}{\rho} \nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B}) \end{aligned} \quad (1)$$

Boundary Condition

$\Omega \subset \mathbb{R}^3$ is a smooth domain
$\partial\Omega_{in} = \{x \in \partial\Omega | \mathbf{n} \cdot \mathbf{u} < 0\}$ is the inflow boundary.

- Natural BC

$$\eta(\mathbf{B} \times \mathbf{n}) = 0 \quad \text{on } \partial\Omega \backslash \partial\Omega_{in} \tag{2}$$

- Inflow BC

$$\begin{aligned} \mathbf{B} &= 0 \quad \text{on } \partial\Omega_{in} \\ \delta_i \mathbf{J} &= 0 \quad \text{on } \partial\Omega_{in} \end{aligned} \tag{3}$$

Estimate

Theorem

For $\mathbf{u} \in C^2(\Omega)$ *and* $\mathbf{B}$ *solution of (1) satisfying (2) and (3), then this estimate holds*

$$\frac{d}{dt}\left(\|\mathbf{B}\|^2_{L^2(\Omega)} + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\|\nabla \times \mathbf{B}\|^2_{L^2(\Omega)}\right) \leq C_1 \left(\|\mathbf{B}\|^2_{L^2(\Omega)} + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\|\nabla \times \mathbf{B}\|^2_{L^2(\Omega)}\right) \quad (4)$$

with C_1 *a constant that depend on* $\mathbf{u}$ *and its derivative only.*

System of Equations with Auxiliary Variables

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbb{U}_1 \mathbf{B} + (\mathbf{u}\nabla)\mathbf{B} = -\eta \nabla \times \mathbf{J} - \alpha \nabla \times \tilde{\mathbf{E}}_1 - \beta \nabla \times \tilde{\mathbf{E}}_2$$

$$\mathbf{J} = \nabla \times \mathbf{B}$$

$$\tilde{\mathbf{E}}_1 = \mathbf{J} \times \mathbf{B}$$

$$\tilde{\mathbf{E}}_2 = (\frac{\partial \mathbf{J}}{\partial t} + (\mathbf{u}\nabla)\mathbf{J})$$

Boundary Condition:

$$\eta(\mathbf{B} \times \mathbf{n}) = 0 \quad \text{on } \partial\Omega$$

$$\mathbf{B} = G_1 \quad \text{on } \partial\Omega_{in}$$

$$\beta \mathbf{J} = G_2 \quad \text{on } \partial\Omega_{in}.$$

$\mathbb{U}_1$ depends on $\frac{\partial u_i}{\partial x_j}$, $\alpha = \frac{\delta_i}{L_0 \rho}$ and $\beta = \frac{\delta_e^2}{L_0^2 \rho}$

Define

$$(v,w)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \int_K v(x)w(x)\,dx$$

$$\langle v,w\rangle_{\text{Faces}} = \sum_{f\in\text{Faces}} \int_e v(x)w(x)\,ds$$

where $\mathcal{T}_h$ a triangulation of Ω.
Faces can be

- $\mathcal{F}_h$ set of faces in $\mathcal{T}_h$.
- $\mathcal{F}_h^{\mathcal{I}}$ set of inner faces in $\mathcal{T}_h$.
- Γ_h set of boundary faces in $\mathcal{T}_h$.
- Γ_h^+ set of outflow boundary faces in $\mathcal{T}_h$.
- Γ_h^- set of inflow faces boundary in $\mathcal{T}_h$.

To have unique valued on faces we define:

- averages $\{.\}$
- normal jumps $[\![.]\!]_N$
- tangential jumps $[\![.]\!]_T$

Find $\mathbf{B}_h, \mathbf{E}_h, \mathbf{J}_h \in \mathcal{V}_h$

$$\left(\frac{\partial \mathbf{B}_h}{\partial t} + \mathbb{U}_2 \mathbf{B}_h, \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} - \left(\mathbf{B}_h, (u\nabla)\bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} + \eta\left(\mathbf{J}_h, \nabla \times \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h}$$

$$+ \alpha\left(\tilde{\mathbf{E}}_{h,1}, \nabla \times \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} + \beta\left(\tilde{\mathbf{E}}_{h,2}, \nabla \times \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} + \sum_i \langle \widehat{\mathbf{u}B^i}_h, [\![\bar{B}^i{}_h]\!]_N \rangle_{\Gamma_h^+ \cup \mathcal{F}_h^{\mathcal{I}}}$$

$$- \eta \langle \widehat{\mathbf{J}_h}, \bar{\mathbf{B}}_h \rangle_{\mathcal{F}_h} - \alpha \langle \widehat{\tilde{\mathbf{E}}_{h,1}}, \bar{\mathbf{B}}_h \rangle_{\mathcal{F}_h} - \beta \langle \widehat{\tilde{\mathbf{E}}_{h,2}}, \bar{\mathbf{B}}_h \rangle_{\mathcal{F}_h} = -\langle (\mathbf{u} \cdot \mathbf{n})\mathbf{G}_1, \bar{\mathbf{B}}_h \rangle_{\Gamma_h^-}$$

$$\forall \bar{\mathbf{B}}_h \in \mathcal{V}_h$$

$$\left(\mathbf{J}_h, \bar{\mathbf{E}}_h\right)_{\mathcal{T}_h} - \left(\mathbf{B}_h, \nabla \times \bar{\mathbf{E}}_h\right)_{\mathcal{T}_h} + \langle \widehat{\mathbf{B}_h}, [\![\bar{\mathbf{E}}_h]\!]_T \rangle_{\mathcal{F}_h^{\mathcal{I}}} = 0$$

$$\forall \bar{\mathbf{E}}_h \in \mathcal{V}_h$$

$$\left(\tilde{\mathbf{E}}_{h,1} - \mathbf{J}_h \times \mathbf{B}_h, \bar{\mathbf{J}}_h\right)_{\mathcal{T}_h} = 0 \qquad \forall \bar{\mathbf{J}}_h \in \mathcal{V}_h$$

$$\left(\tilde{\mathbf{E}}_{h,2} + (\nabla \cdot \mathbf{u})\mathbf{J}_h - \frac{\partial \mathbf{J}_h}{\partial t}, \bar{\bar{\mathbf{J}}}_h\right)_{\mathcal{T}_h} + \left(\mathbf{B}_h, (\mathbf{u}\nabla)\bar{\bar{\mathbf{J}}}_h\right)_{\mathcal{T}_h} - \sum_i \langle \widehat{\mathbf{u}J^i_h}, [\![\bar{\bar{J}}{}^i_h]\!]_N \rangle_{\Gamma_h^+ \cup \mathcal{F}_h^{\mathcal{I}}}$$

$$= -\langle (\mathbf{u} \cdot \mathbf{n})\mathbf{G}_2, \bar{\bar{\mathbf{J}}}_h \rangle_{\Gamma_h^-} \qquad \forall \bar{\bar{\mathbf{J}}}_h \in \mathcal{V}_h$$

DG Fluxes

LDG:

$$\widehat{\mathbf{B}_h} = \{\mathbf{B}_h\} + \mathfrak{b}[\![\mathbf{B}_h]\!]_T$$

$$\widehat{\mathbf{J}_h} = \begin{cases} \{\mathbf{J}_h\} - \mathfrak{b}[\![\mathbf{J}_h]\!]_T + \mathfrak{a}^0[\![\mathbf{B}_h]\!]_T & \text{internal edges} \\ \{\mathbf{J}_h\} + \mathfrak{a}^0[\![\mathbf{B}_h]\!]_T & \text{boundary edges} \end{cases}$$

$$\widehat{\mathbf{E}_{h,i}} = \begin{cases} \{\mathbf{E}_{h,i}\} - \mathfrak{b}[\![\mathbf{E}_{h,i}]\!]_T + \mathfrak{a}^i[\![\mathbf{B}_h]\!]_T & \text{internal edges} \\ \{\mathbf{E}_{h,i}\} + \mathfrak{a}^i[\![\mathbf{B}_h]\!]_T & \text{boundary edges} \end{cases}$$

I.Perugia, D. Schötzau (2002)

Upwind:

$$\widehat{\mathbf{u}B_h^i} = \{\mathbf{u}B_h^i\} + \mathfrak{c}[\![B_h^i]\!]_N$$
$$\widehat{\mathbf{u}J_h^i} = \{\mathbf{u}J_h^i\} + \mathfrak{c}[\![J_h^i]\!]_N$$

with $\mathfrak{c} = |\mathbf{nu}|/2$.

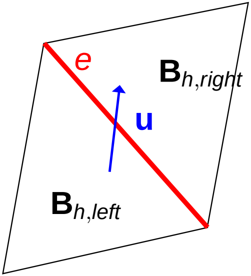

Figure: In the case the flux $\widehat{\mathbf{u}B_h^i}$ will be $\mathbf{u}B_{h,left}^i$.

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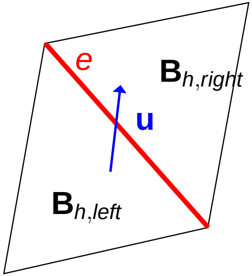

Figure: In the case the flux $\widehat{\mathbf{u}B_h^i}$ will be $\mathbf{u}B_{h,left}^i$.

With this Fluxes $\Rightarrow$ Discrete Energy estimate.

1D Model

$$\partial_t b + u\partial_x b - \eta\partial_{xx} b - \beta(\partial_{xxt} b + \partial_x(u\partial_{xx} b)) = 0 \text{ in } (0,1).$$

With boundary conditions

$$\begin{aligned} b(0,t) = b(1,t) &= 0, \\ \beta\partial_x b(0,t) = g_l(t) &\qquad \text{when } v(0,t) > 0, \\ \beta\partial_x b(1,t) = g_r(t) &\qquad \text{when } v(1,t) < 0. \end{aligned}$$

$\Rightarrow$ Energy Estimate.

Auxiliary variables

$$\partial_t b + u\partial_x b - \eta\partial_x j - \beta\partial_x e = 0$$
$$j = \partial_x b$$
$$e = \partial_t j + u\partial_x j$$

- Build DG Formulation (as for the Vector Equation)
- Use Upwind fluxes for Advection Part.
- Use LDG fluxes for the diffusive part.
- Matrix Formulation (Semi-discrete).
- Eliminate Auxiliary Variables.

$$(\mathbb{M} - \beta\mathbb{W}(\mathfrak{b}))\mathbf{b}_t = -(\eta\mathfrak{a}^0 + \beta\mathfrak{a}^1)\mathbb{Q}\mathbf{b} + \eta\mathbb{W}(\mathfrak{b})\mathbf{b} + \mathbb{Z}(\mathfrak{c})\mathbf{b} - \tilde{\mathbb{Z}}(\mathfrak{c})\mathbf{b}$$

- Mass Matrix
- "Laplace Parts"
- "Advection Parts"

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Implicit-Explicit Time Discretization:

$$(\mathbb{M} - \beta\mathbb{W})\frac{\mathbf{b}^{n+1} - \mathbf{b}^n}{\Delta t} = -(\eta\mathfrak{a}^0 + \beta\mathfrak{a}^1)\mathbb{Q}\mathbf{b}^{n+1} + \eta\mathbb{W}\mathbf{b}^{n+1} + \mathbb{Z}\mathbf{b}^n - \tilde{\mathbb{Z}}\mathbf{b}^n$$

Choosing: $\mathfrak{a}^1 = \frac{\mathfrak{a}^0}{\Delta t}$

$$(\mathbb{M} - \underbrace{(\beta + \Delta t \eta)}_{:=\gamma}(\mathbb{W} - \mathfrak{a}^0 \mathbb{Q}))\mathbf{b}^{n+1} = (\mathbb{M} - \beta \mathbb{W} + \Delta t(\mathbb{Z} - \tilde{\mathbb{Z}}))\mathbf{b}^n$$

$$\mathbb{A}_{DG}(\gamma) := \mathbb{M} - \gamma(\mathbb{W} - \mathfrak{a}^0 \mathbb{Q})$$

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$$\mathbb{A}_{DG}(\gamma) := \mathbb{M} - \gamma(\mathbb{W} - \mathfrak{a}^0\mathbb{Q})$$

- Solve $\mathbb{A}_{DG}(\gamma)\mathbf{x} = \mathbf{l}$
- $\mathbb{A}_{DG}(\gamma)$ is the DG discretization of $u - \gamma\partial_{xx}u$.
- Use Conform Discretization ($\mathbb{A}_{CG}(\gamma)$) $\Rightarrow$ Auxiliary Space Preconditioner.

Fast Approximated Solution: Auxiliary Space

joint work with Ralf Hiptmair

Solve $\mathbb{A}_{DG}(\gamma)\mathbf{x} = \mathbf{l}$:

$\mathbf{x} \leftarrow \mathbf{x_0}$
for $i \leq N_{\text{iter}}$ **do**
$\mathbf{x} \leftarrow \text{Smoother}(\mathbf{x}, \mathbb{A}_{DG}, \mathbf{l})$
$\mathbf{r} \leftarrow \mathbf{l} - \mathbb{A}_{DG}\mathbf{x}$
$\rho \leftarrow \mathbb{P}^\top \mathbf{r}$
$\kappa \leftarrow \mathbb{A}_{CG}\kappa = \rho$
$\mathbf{x} \leftarrow \mathbf{x} + \mathbb{P}\kappa$
$\mathbf{x} \leftarrow \text{Smoother}(\mathbf{x}, \mathbb{A}_{DG}, \mathbf{l})$
end for

Auxiliary $\mathbb{P}$ Prolongation operator from Conform to Discontinuous Space.

Numerical Tests

- Advection: $\beta = \eta = 0$

$$b_0(x) = \begin{cases} (1-(4x^2-1))^4 & 0 \le x < 1/2 \\ 0 & 1/2 \le x \le 1 \end{cases}$$

$$u(x) = c \Rightarrow b(x,t) = b_0(x-ct),$$
$$u(x) = cx \Rightarrow b(x,t) = b_0(xe^{-ct}).$$

- Heat Equation: $\beta = u = 0$.

$$b(x,t) = e^{-\pi^2\eta t}\sin(\pi x) + \frac{1}{2}e^{-4\pi^2\eta t}\sin(2\pi x)$$

- Advection Diffusion: $\beta = 0$ and $u = c$

$$b(x,t) = e^{\frac{c}{2\eta}x}\left(e^{-\lambda_1 t}\sin(\pi x) + \frac{e^{-\lambda_1 t}}{2}\sin(2\pi x) + \frac{e^{-\lambda_1 t}}{4}\sin(3\pi x)\right).$$

with $\lambda_k = \frac{c^2+4\pi^2k^2\eta^2}{4\eta}$

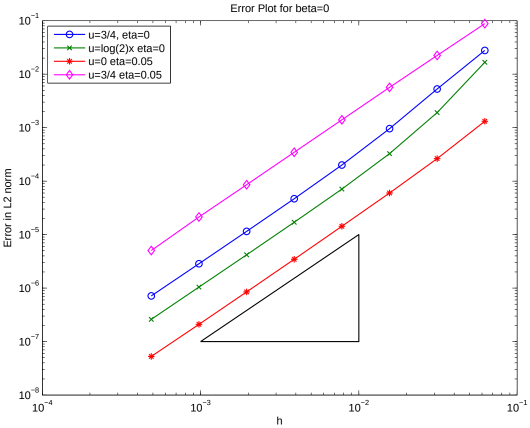


Figure: Convergence plot for test problems, at time $T = 1$. The reference triangle has a slope of 2, i.e. convergence order of 2.

Forced Solution:
Solving

$$\partial_t b + u\partial_x b = \eta\partial_{xx} b + \beta(\partial_{xxt} b + \partial_x(u\partial_{xx} b)) + f$$

Choose $f(x, t, \beta)$ so that

$$b(x, t) = e^{\frac{c}{2\eta}x}(e^{-\lambda_1 t}\sin(\pi x) + \frac{e^{-\lambda_1 t}}{2}\sin(2\pi x) + \frac{e^{-\lambda_1 t}}{4}\sin(3\pi x)).$$

with $\lambda_k = \frac{c^2+4\pi^2k^2\eta^2}{4\eta}$ is solution.

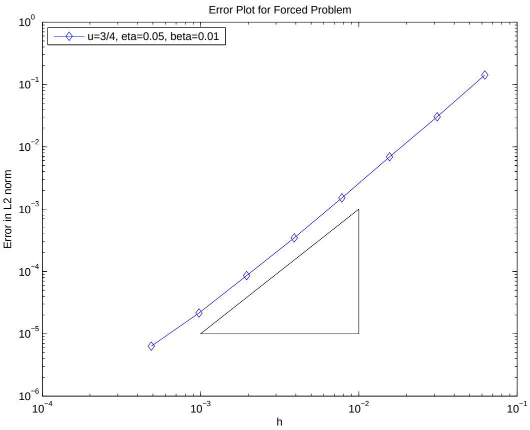
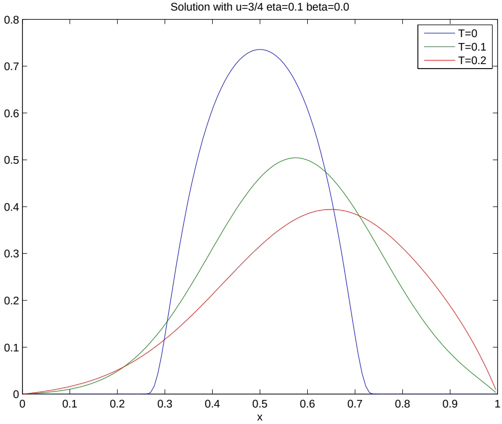


Figure: Convergence plot for solution of Forced Problem at $T = 1$. The reference triangle has a slope of 2, i.e. convergence order of 2.

Initial Data

$$b_0(x) = \begin{cases} 2e^{\frac{-1}{(1-(4x-2)^2)}} & 1/4 < x < 3/4 \\ 0 & \text{elsewhere.} \end{cases}$$



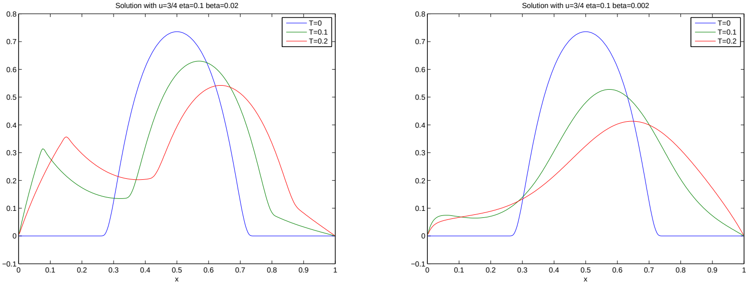


Figure: Solution for $u = 3/4$, $\eta = 0.1$. On the left β is 0.02 on the right 0.002.

Preconditoner

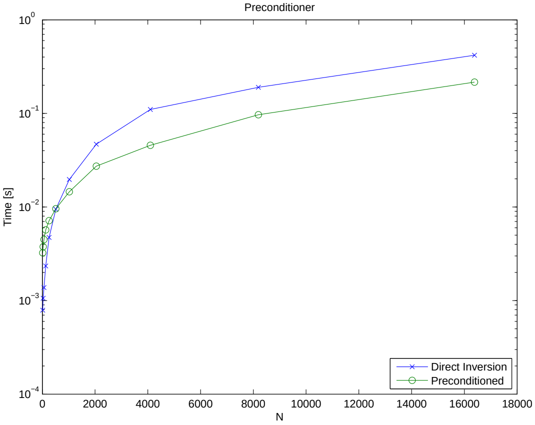


Figure: Time to obtain the solution for $u = 3/4$, $\eta = 0.1$, $\beta = 0.02$ and $T = 0.2$ with and without preconditioner.

Conclusion

- The solution of the induction equation with Hall term possesses an energy estimate .
- We can build a DG discretization which satisfies a similar estimate.
- $\Rightarrow$ stability granted for exact-time evolution of the discrete system.
- We presented a1D Model based on the full equations.
    - Space-time Discretization was presented.
    - Preconditioner for time evolution.
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Thank You!