A hyperbolic model for phase transitions in porous media

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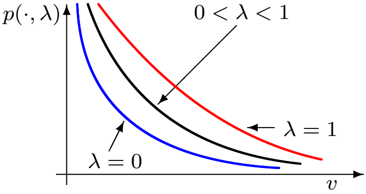
A general model for fluid phase transitions

We consider the following hyperbolic model for liquid-vapor phase transitions in fluids (H. Fan (2000)). In Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,\lambda)_x = -\nu u + \epsilon u_{xx}, \\ \lambda_t = \dfrac{\alpha}{\tau}(p - p_e)\lambda(\lambda - 1) + \beta\epsilon\lambda_{xx}\,. \end{cases}$$

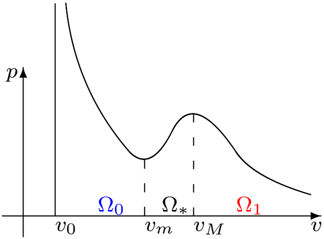


- v specific volume, u velocity, λ mass density fraction of vapor in the fluid;
- $\lambda \in [0,1]$ and $\lambda = 0$: liquid regime, $\lambda = 1$: vapor regime;
- $p = p(v,\lambda)$ is the pressure, satisfying $p_v < 0$, $p_\lambda > 0$, $p_{vv} > 0$, $p_{v\lambda} < 0$. Example:

  $$p = p(v,\lambda) = \frac{a^2(\lambda)}{v^\gamma},$$

  with $\gamma \geq 1$, $a(\lambda) > 0$, $a'(\lambda) > 0$; for instance $a(\lambda) = 1 + \lambda$; p_e a fixed equilibrium pressure;
- ϵ a viscosity coefficient, α, β positive parameters, τ a reaction time, $\nu > 0$ a parameter related to a friction force.

Physical motivations



Van der Waals fluid

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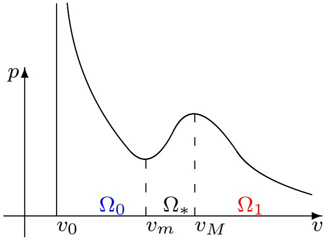

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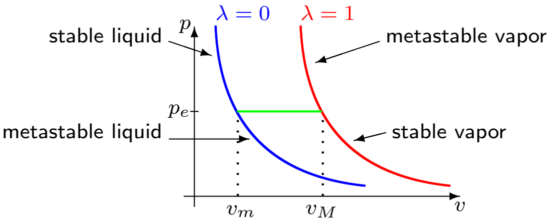

Pressure lines

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,\lambda)_x = -\nu u, \\ \lambda_t = \frac{1}{\tau}(p - p_e)\lambda(\lambda - 1), \end{cases}$$

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- Traveling waves in the porous media case ($\nu > 0$): results.
- Sketch of the proof.

The homogeneous system

Consider the *homogeneous* case[1]:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,\lambda)_x = 0, \\ \lambda_t = 0, \end{cases} \qquad (v,u,\lambda)(0,x) = (v_o,u_o,\lambda_o).$$

The system is strictly hyperbolic with two GNL and one LD eigenvalue.

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$$\mathrm{TV}\big(\log(a_o)\big) < \frac{1}{2}, \qquad \mathrm{TV}\,(\log p_o) + \frac{1}{\inf a_o}\mathrm{TV} u_o < H\big(\mathrm{TV}(\log(a_o))\big),$$

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- Proof by a wave-front tracking scheme.
- If λ_o is constant we recover the famous result by Nishida (1968).
- A similar result was obtained by Asakura-C. (Quart. Appl. Math. 2012) by using a technique of path decomposition in the wave-front tracking scheme.

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$$\begin{cases} \tilde{v}_t - \tilde{u}_x & = 0 \\ \tilde{u}_t + p(\tilde{v}, 0)_x & = 0. \end{cases}$$

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- The Riemann problem for

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,\lambda)_x = 0, \\ \left(p(v,\lambda) - p_e\right)\lambda(\lambda-1) = 0. \end{cases}$$

  can be uniquely solved for arbitrary data (C.-Fan, SIAM J: Appl. Math. 2004).

Flow in a porous medium

The flow is hosted in a medium that induces a friction force (extraction of petroleum from underground oil sand, modeling of polymer electrolyte fuel cells):

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- An important issue is to establish whether Darcy's law

$$p_x = -\nu u$$

holds as $t \to +\infty$, see Pan (2006). We do not address to such a problem.

Traveling waves and the case $c = 0$

For $U = (v, u, \lambda)$ we look for solutions of the form

$$U(\xi) = U\left(\frac{x - ct}{\tau}\right).$$

Then, for $A = \nu\tau$,

$$\begin{cases} -cv' - u' = 0\,, \\ -cu' + p' = -Au\,, \\ -c\lambda' = (p - p_e)\lambda(\lambda - 1)\,, \end{cases} \qquad \text{with} \qquad \begin{cases} (v, u, \lambda)(\pm\infty) = (v_\pm, u_\pm, \lambda_\pm)\,, \\ (v', u', \lambda')(\pm\infty) = 0\,, \end{cases}$$

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As a consequence, $c(v_+ - v_-) = 0$. Then

$$\text{either} \quad c = 0 \quad \text{or} \quad v_- = v_+.$$

If $c = 0$, then $u \equiv u_\pm = 0$, $p \equiv p_\pm = \bar{p}$ and $(\bar{p} - p_e)\lambda(\lambda - 1) \equiv 0$.

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Lemma

For every fixed value $\bar{p}$, any pair of functions $v(\xi)$, $\lambda(\xi)$ satisfying

$$p\big(v(\xi), \lambda(\xi)\big) \equiv \bar{p} \quad \textit{and} \quad (\bar{p} - p_e)\lambda(\xi)\big(\lambda(\xi) - 1\big) \equiv 0$$

provides a traveling-wave solution $(v, u, \lambda) = (v, 0, \lambda)(\xi)$ with speed $c = 0$.

Traveling waves, the case $c > 0$

Then we consider the case

$$c > 0, \quad \text{hence,} \quad v_- = v_+ =: \bar{v}.$$

We can rewrite the traveling wave system as

$$\begin{cases} (c^2 + p_v)v' = Ac(v - \bar{v}) + \frac{1}{c} p_\lambda (p - p_e)\lambda(\lambda - 1), \\ \lambda' = -\frac{1}{c}(p - p_e)\lambda(\lambda - 1), \end{cases} \qquad (v, \lambda)(\pm\infty) = (\bar{v}, \lambda_\pm).$$

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We define

$$\begin{aligned} s(v, \lambda) &:= c^2 + p_v, \\ g(v, \lambda) &:= Ac(v - \bar{v}) + \frac{1}{c}p_\lambda(p - p_e)\lambda(\lambda - 1), \end{aligned}$$

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The equilibrium points are $(\bar{v}, 0)$, $(\bar{v}, 1)$ and the points $(\bar{v}, \bar{\lambda})$ satisfying

$$p(\bar{v}, \bar{\lambda}) = p_e. \tag{1}$$

For each $\bar{v}$ there is at most one $\bar{\lambda}$ satisfying (1), because $p_\lambda \neq 0$.

The curves $\mathcal{S}$, $\mathcal{G}$, $\mathcal{P}$

Recall

$$\begin{cases} sv' = g, \\ \lambda' = -\frac{1}{c}(p - p_e)\lambda(\lambda - 1). \end{cases}$$

We introduce

$$\mathcal{S}\colon s(v,\lambda) = 0, \qquad \mathcal{G}\colon g(v,\lambda) = 0, \qquad \mathcal{P}\colon p(v,\lambda) = p_e.$$

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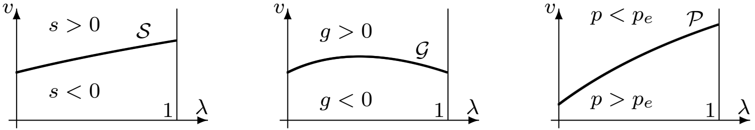
$$\mathcal{S}\colon s(v,\lambda) = 0, \qquad \mathcal{G}\colon g(v,\lambda) = 0, \qquad \mathcal{P}\colon p(v,\lambda) = p_e.$$

Notice that the system is degenerate along the curve $\mathcal{S}$. The curves $\mathcal{S}$ and $\mathcal{P}$ define increasing functions of λ.



The main result

Theorem

Consider the traveling-wave system and denote $p_\pm = p(\bar{v}, \lambda_\pm)$ *and* $s_\pm = s(\bar{v}, \lambda_\pm)$*. Then, the end states* $(\bar{v}, \lambda_\pm)$ *must satisfy one of the following necessary conditions:*

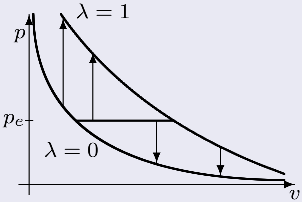
(i) $p_\pm > p_e$ *and* $\lambda_- = 0$, $\lambda_+ = 1$;

(ii) $p_\pm < p_e$ *and* $\lambda_- = 1$, $\lambda_+ = 0$;

(iii) $p_+ > p_e = p_-$ *and* $\lambda_- < 1$, $\lambda_+ = 1$;

(iv) $p_+ < p_e = p_-$ *and* $\lambda_- > 0$, $\lambda_+ = 0$;

(v) $p_- = p_+ = p_e$ *and* $\lambda_- = \lambda_+$.



The main result

Theorem

Consider the traveling-wave system and denote $p_{\pm} = p(\bar{v}, \lambda_{\pm})$ *and* $s_{\pm} = s(\bar{v}, \lambda_{\pm})$*. Then, the end states* $(\bar{v}, \lambda_{\pm})$ *must satisfy one of the following necessary conditions:*

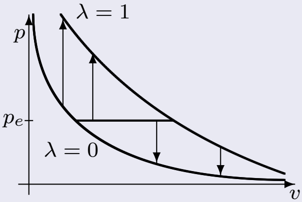
(i) $p_{\pm} > p_e$ *and* $\lambda_- = 0,\ \lambda_+ = 1$;

(ii) $p_{\pm} < p_e$ *and* $\lambda_- = 1,\ \lambda_+ = 0$;

(iii) $p_+ > p_e = p_-$ *and* $\lambda_- < 1,\ \lambda_+ = 1$;

(iv) $p_+ < p_e = p_-$ *and* $\lambda_- > 0,\ \lambda_+ = 0$;

(v) $p_- = p_+ = p_e$ *and* $\lambda_- = \lambda_+$.



Moreover, assume

$$s(\bar{v}, \lambda_{\pm}) \neq 0.$$

In case (i), traveling waves exist if $\mathcal{S}$ and $\mathcal{G}$ intersect at most once;
in case (iii) the same result holds if $\mathcal{S}$ and $\mathcal{G}$ intersect at most once in $\mathbb{R} \times (\lambda_-, 1]$.
In case (ii), traveling waves exist if $\mathcal{S}$ and $\mathcal{G}$ do not intersect;
in case (iv) the same result holds if $\mathcal{S}$ and $\mathcal{G}$ do not intersect in $\mathbb{R} \times [0, \lambda_-)$.
In case (v) the only solution is the trivial constant solution.
For the given speed c, the traveling wave solutions are unique up to a shift in ξ.[a]

[a] C.-Fan, Appl. Anal. 2012

Proof of Case *(i)*, $s_\pm < 0$ and $s_\pm > 0$

Consider Case *(i)*: $p_\pm > p_e$, $\lambda_- = 0$, $\lambda_+ = 1$.
There are three cases, according to the possible mutual positions of $\mathcal{S}$ and $\mathcal{G}$.

Proof of Case *(i)*, $s_\pm < 0$ and $s_\pm > 0$

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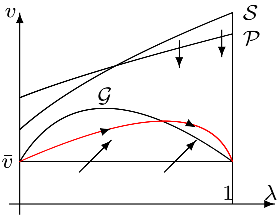
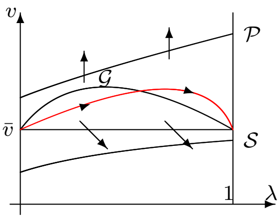
There are three cases, according to the possible mutual positions of $\mathcal{S}$ and $\mathcal{G}$.

- $p_\pm > p_e$, $\lambda_- = 0$, $\lambda_+ = 1$ and $s_\pm < 0$ (subsonic end states).
  Both end points $(\lambda_\pm, \bar{v}) \in \mathcal{G}$ lie below $\mathcal{S}$. By the assumption of this theorem, the curves $\mathcal{S}$ and $\mathcal{G}$ cannot intersect. The curve $\mathcal{S}$ lies above the curve $\mathcal{G}$.
- $p_\pm > p_e$, $\lambda_- = 0$, $\lambda_+ = 1$ and $s_\pm > 0$ (supersonic end states).
  Both end points $(\lambda_\pm, \bar{v}) \in \mathcal{G}$ lie above $\mathcal{S}$. The curve $\mathcal{S}$ lies below the line $v = \bar{v}$. The curves $\mathcal{S}$ and $\mathcal{G}$ cannot intersect.



$s_\pm < 0$



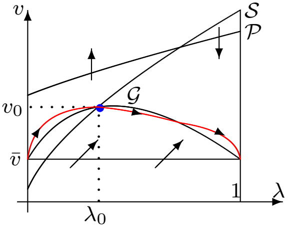
$s_\pm > 0$

Proof of Case *(i)*, $s_- > 0 > s_+$

- $p_\pm > p_e$, $\lambda_- = 0$, $\lambda_+ = 1$ and $s_- > 0 > s_+$ (left supersonic, right subsonic end states).
  In this case $\mathcal{S}$ and $\mathcal{G}$ must intersect at some point (λ_0, v_0), $\lambda_0 \in (0,1)$. By assumption there is only one such point; we have $v_0 > \bar{v}$, $p(v_0, \lambda_0) > p_e$, $g_v(v_0, \lambda_0) > 0$. Consider the following problem:

$$\begin{cases} sv' = g, \\ \lambda' = -\frac{1}{c}(p - p_e)\lambda(\lambda - 1), \\ (v, \lambda)(0) = (v_0, \lambda_0). \end{cases}$$

The first equation is singular: $s(v_0, \lambda_0) = g(v_0, \lambda_0) = 0$.



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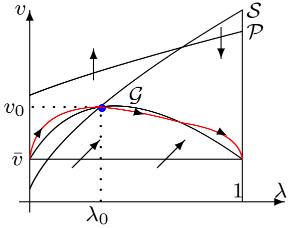

Lemma

Consider the point $(v_0,\lambda_0) \in \mathcal{S} \cap \mathcal{G}$. Then the system has exactly two C^1 solutions for $\xi \in (-\delta,\delta)$, for some constant $\delta > 0$. As ξ increases across 0:
one solution crosses (v_0,λ_0) from $s > 0$ into $s < 0$,
the other solution crosses (v_0,λ_0) from $s < 0$ into $s > 0$.

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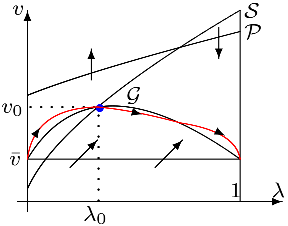

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one solution crosses (v_0,λ_0) from $s > 0$ into $s < 0$,
the other solution crosses (v_0,λ_0) from $s < 0$ into $s > 0$.

Let $\big(v(\xi),\lambda(\xi)\big)$ be the C^1 solution crossing (v_0,λ_0) from $s > 0$ to $s < 0$ as ξ increases across 0. Then $\big(v(\xi),\lambda(\xi)\big)$ is a solution to the traveling wave system.

Sketch of the proof of the lemma

Consider the transformation $\xi \mapsto \zeta$ defined by $\frac{d\xi}{d\zeta} = s(v(\xi), \lambda(\xi))$ and $\xi(\pm\infty) = 0$. Denote $\tilde{v}(\zeta) = v(\xi)$, $\tilde{\lambda}(\zeta) = \lambda(\xi)$. The system becomes

$$\begin{cases} \dfrac{d\xi}{d\zeta} = s(\tilde{v}, \tilde{\lambda}), \\ \dfrac{d\tilde{v}}{d\zeta} = g(\tilde{v}, \tilde{\lambda}), \\ \dfrac{d\tilde{\lambda}}{d\zeta} = -\dfrac{s(\tilde{v}, \tilde{\lambda})}{c}(p(\tilde{v}, \tilde{\lambda}) - p_e)\tilde{\lambda}(\tilde{\lambda} - 1), \end{cases} \qquad (\xi, \tilde{v}, \tilde{\lambda})(\pm\infty) = (0, v_0, \lambda_0).$$

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Denote $B := \frac{1}{c}\big(p(v_0, \lambda_0) - p_e\big)\lambda_0(\lambda_0 - 1) < 0$. The linearized system at (v_0, λ_0) is

$$\begin{cases} \dfrac{d\xi}{d\zeta} = s_v V + s_\lambda \Lambda, \\ \dfrac{dV}{d\zeta} = g_v V + g_\lambda \Lambda, \\ \dfrac{d\Lambda}{d\zeta} = -B(s_v V + s_\lambda \Lambda). \end{cases}$$

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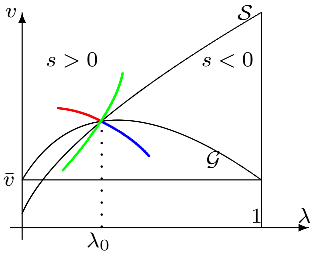
The eigenvalues are $\nu_- < 0 < \nu_+$ and $\nu_3 = 0$. Thus:

- the 2×2 system has one stable and one unstable manifold;
- moreover, there are stable (or unstable) trajectories on both sides of the curve $\mathcal{S}$.

Sketch of the proof of the lemma (cont'd)

- Consider the *stable* trajectory $(\xi, \tilde{v}, \tilde{\lambda})(\zeta)$, which is defined in a neighborhood of $\zeta = +\infty$; hence $(\xi, \tilde{v}, \tilde{\lambda})(+\infty) = (0, v_0, \lambda_0)$. We get
$$\xi = \int_{+\infty}^{\zeta} s\big(\tilde{v}(\eta), \tilde{\lambda}(\eta)\big) d\eta.$$
On the $s > 0$ side, the function $\xi(\zeta)$ strictly increases towards $\xi = 0$ as $\zeta \to +\infty$. Thus we construct $(v, \lambda)(\xi) := (\tilde{v}, \tilde{\lambda})\big(\zeta(\xi)\big)$ which satisfies the 2×2 system for $\xi \in (-\delta_1, 0]$.



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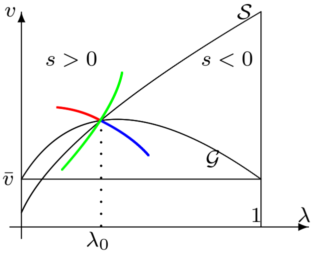

- Similarly, we use the *stable* trajectory on the $s < 0$ side to construct $(v, \lambda)(\xi)$ that satisfies the 2×2 system for $\xi \in [0, \delta_2)$.

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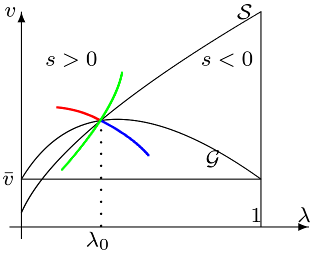

- Similarly, we use the *stable* trajectory on the $s < 0$ side to construct $(v, \lambda)(\xi)$ that satisfies the 2×2 system for $\xi \in [0, \delta_2)$.
- We paste the two "half"-solutions and prove that we obtain a full solution of class C^1 at $\xi = 0$.
  Across $\xi = 0$, the trajectory through (v_0, λ_0) passes from the $s > 0$ region into the $s < 0$ region.

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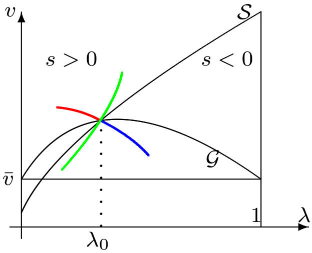

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Across $\xi = 0$, the trajectory through (v_0, λ_0) passes from the $s > 0$ region into the $s < 0$ region.
- Similarly, we use the *unstable* trajectory of to construct a solution of that passes from the $s < 0$ region into the $s > 0$ region as ξ increases across $\xi = 0$.

Thank you!