Conservation Laws in the Modeling of Moving Crowds

R.M. Colombo

Department of Mathematics, Brescia University

June 26th, 2012
Padova, HYP2012

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M. Garavello, M. Lécureux-Mercier, N. Pogodaev

Milano, Haifa, Irkutsk

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Crowd Dynamics

$$\rho = \rho(t, x) \text{ crowd density:}$$

$$[\text{people in } \Omega \text{ at time } t] = \iint_\Omega \rho(t, x)\, \mathrm{d}x.$$

$$\boxed{\partial_t \rho + \mathrm{div}_x \left(\rho\, \vec{V}(t, x, \rho)\right) = 0}$$

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$\rho(t, x) \in [0, 1]$
$\rho(t) \in \mathbf{L^1}(\mathbb{R}; [0, 1])$

$\vec{V} = v\, \vec{v}$ with $v \in \mathbb{R}$, $\|\vec{v}\| = 1$

Crowd Dynamics – $\vec{V}(t, x, \rho) = v\left(\rho(t, x)\right)\ \vec{v}(x)$

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(Kružkov Theory applies)

(Facchi: Thesis, 2008)

Crowd Dynamics – $\vec{V}(t, x, \rho) = v\big(\rho(t) * \eta\big)\ \vec{v}(t, x)$

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Theorem

If: $v, \vec v$ *sufficiently smooth*

η *smooth,* $\eta \geq 0$ *with compact support and* $\int \eta = 1$

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Then: there is a Lipschitz semigroup $(t, \rho_o) \to S_t \rho_o$

- ✓ *defined for all* $t \in [0, +\infty[$
- ✓ *differentiable w.r.t.* ρ_o
- ✓ $DS_t(\rho_o) r_o = \Sigma_t r_o$ *with* Σ *semigroup generated by*

$$\partial_t r + \operatorname{div}\Big(r\, \vec V(t,x,\rho) + \rho\, D_\rho \vec V(t,x,\rho)\, r\Big) = 0, \ (\rho = S_t \rho_o)$$

(Colombo, Herty, Mercier: ESAIM COCV, 2011)

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(also for n populations!)

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(Colombo, Lécureux–Mercier: Acta Math.Sc., 2011)

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Paths should not be prescribed *a priori*!

$$\partial_t \rho + \operatorname{div}_x \left(\rho\, v(\rho) \left(\vec{v}(x) \quad + \quad \begin{array}{c}\text{avoid}\\ \text{high}\\ \text{density}\end{array} \quad \right)\right) = 0$$

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NonLocal Route Choice

Crowd Dynamics – NonLocal Route Choice

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If: v *is smooth, non decreasing,* $v(0) = V$, $v(\rho) = 0$;

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	✓	*Lipschitz Continuous from Data and Equation*
	✓	*Viability (discomfort)*

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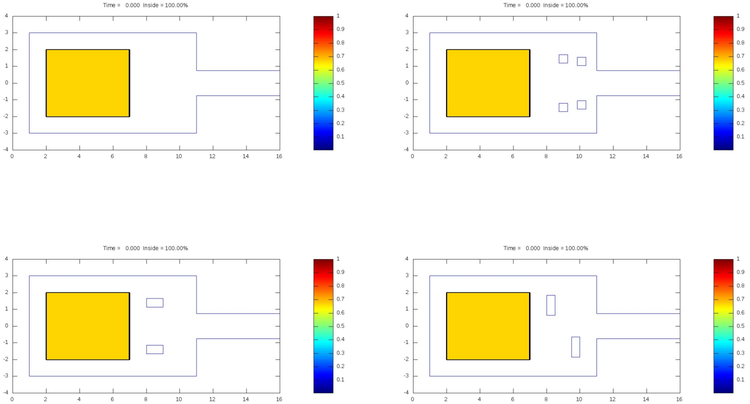
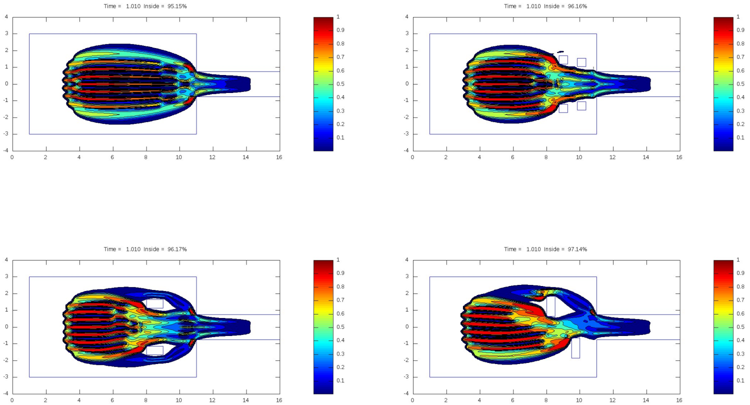
Crowd Dynamics



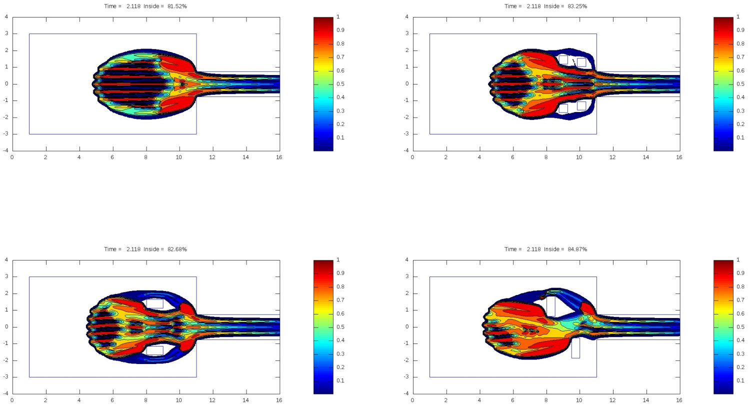
$$t = 0.000$$

Crowd Dynamics



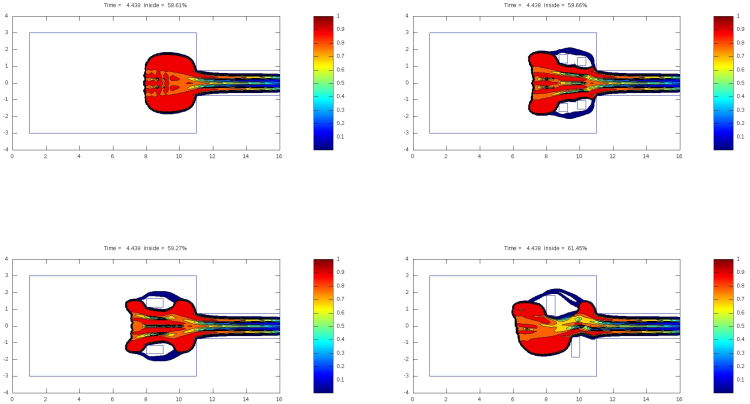
$$t = 1.010$$

Crowd Dynamics



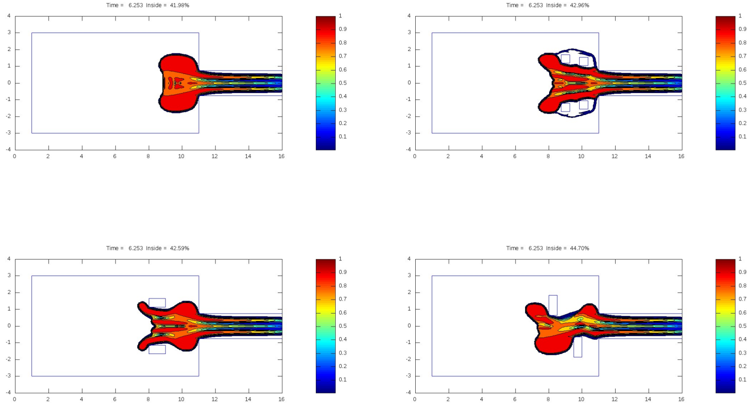
$$t = 2.118$$

Crowd Dynamics



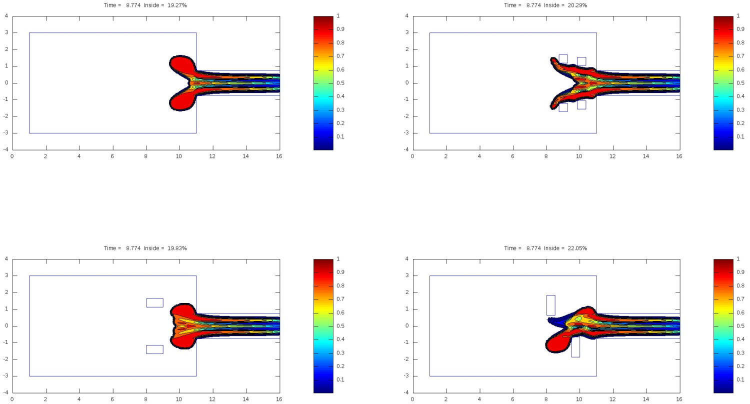
$$t = 4.438$$

Crowd Dynamics



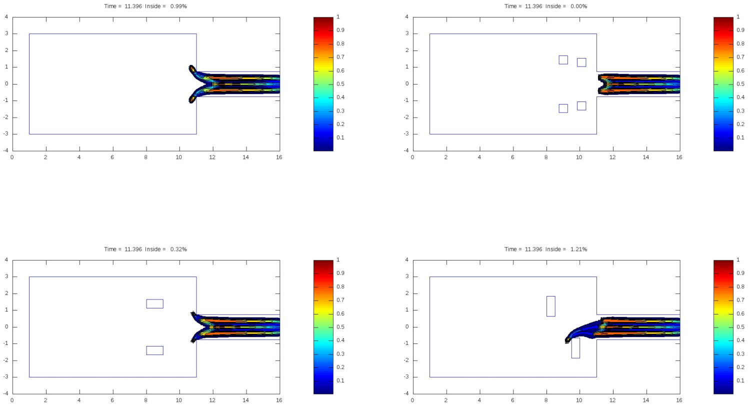
$$t = 6.253$$

Crowd Dynamics



$t = 8.774$

Crowd Dynamics – Braess Paradox



$t = 11.396$

Lanes formation – Two Groups of People Crossing

$$\partial_t \rho^1 + \mathrm{div}\left[\rho^1\, v(\rho^1)\left(\vec{v}^1(x) - \frac{\varepsilon_{11}\,\nabla(\rho^1 * \eta)}{\sqrt{1+\left\|\nabla(\rho^1 * \eta)\right\|^2}} - \frac{\varepsilon_{12}\,\nabla(\rho^2 * \eta)}{\sqrt{1+\left\|\nabla(\rho^2 * \eta)\right\|^2}}\right)\right] = 0$$

$$\partial_t \rho^2 + \mathrm{div}\left[\rho^2\, v(\rho^2)\left(\vec{v}^2(x) - \frac{\varepsilon_{21}\,\nabla(\rho^1 * \eta)}{\sqrt{1+\left\|\nabla(\rho^1 * \eta)\right\|^2}} - \frac{\varepsilon_{22}\,\nabla(\rho^2 * \eta)}{\sqrt{1+\left\|\nabla(\rho^2 * \eta)\right\|^2}}\right)\right] = 0$$

$$\vec{v}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \delta \qquad \eta(x,y) = \left[1-(2x)^2\right]^3 \left[1-(2y)^2\right]^3 \chi_{[-0.5,0.5]^2}(x,y)$$

$$\vec{v}^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \delta \qquad v(\rho) = 4\,(1-\rho) \qquad \begin{array}{lllll} \varepsilon_{11} & = & 0.3 & \varepsilon_{12} & = & 0.7 \\ \varepsilon_{21} & = & 0.7 & \varepsilon_{22} & = & 0.3 \end{array}$$

Lanes formation – Two Groups of People Crossing

Film

Cluster formation – Part of an Audience Leaves a Room

$$\begin{cases} \partial_t \rho^1 + \operatorname{div}\left(\rho^1\, v(\rho^1)\left(\vec{v}^1(x) - \varepsilon \frac{\nabla(\rho^2 * \eta)}{\sqrt{1+\left\|\nabla(\rho^2*\eta)\right\|^2}}\right)\right) = 0 \\ \partial_t \rho^2 + \operatorname{div}\left(\rho^2\, v(\rho^2)\left(\vec{v}^2(x) - \varepsilon \frac{\nabla(\rho^1 * \eta)}{\sqrt{1+\left\|\nabla(\rho^1*\eta)\right\|^2}}\right)\right) = 0 \end{cases}$$

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$$\vec{v}^2 = \delta \qquad v(\rho) = 4\,(1-\rho) \qquad \varepsilon = 0.3$$

Cluster formation – Part of an Audience Leaves a Room

Film

Crowd Dynamics

Measure Valued NonLocal Crowd Models

Crippa, Lécureux–Mercier: NODEA, 2011

Cristiani, Piccoli, Tosin: Multisc. Mod. & Simul., 2011

Piccoli, Tosin: ARMA, 2010

Constrained Desired Velocity, Gradient Flow

Maury, Roudneff–Chupin, Santambrogio: M3AS, 2010

1D Approximate “geodesic”

Di Francesco, Markowich, Pietschmann, Wolfram: JDE, 2011

El–Khatib, Goatin, Rosini: ZAMP, to appear

With Acceleration Equation

Coscia, Canavesio: M3AS, 2008

Reviews

Bellomo, Dogbé: M3AS, 2008

Individuals vs. Collective

Shepherd Dogs

$$\begin{cases} \text{h.c.l. for sheep density } \rho \\ \text{o.d.e. for dogs' positions } p_1,\ p_2 \\ \text{Initial sheep distribution} \\ \text{Initial dogs' position} \end{cases}$$

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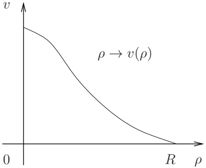

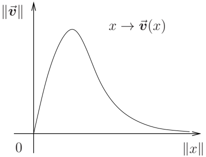

$$\begin{cases} \partial_t \rho + \mathrm{div}_x \left(\rho\, v(\rho) \left(\dfrac{\beta\, x}{1 + \|x\|^2} + \gamma \displaystyle\sum_{d=1}^{2} \vec{v}(x - p_d) \right) \right) = 0 \\ \dot{p}_d = V_d \dfrac{\left(\nabla(\rho * \eta)\right)^{\perp}}{\sqrt{1 + \left\| \left(\nabla(\rho * \eta)\right)^{\perp} \right\|^2}} \qquad d = 1, 2 \\ \rho(0, x) = \chi_{B(0,1)}(x) \\ p(0) = p_o \end{cases}$$

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Film

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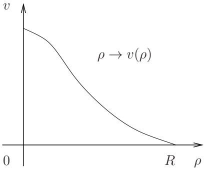

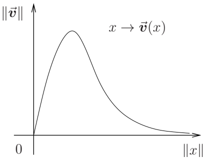

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Shepherd Dog – without p.d.e.s

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A Sheep:

Shepherd Dog – without p.d.e.s

A Sheep: $x = x(t)$

Shepherd Dog – without p.d.e.s

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A Wandering Sheep:

Shepherd Dog – without p.d.e.s

A Sheep:	$x = x(t)$
A Wandering Sheep:	$\dot{x} \in B(0, c)$

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A Shepherd Dog:

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Sheep & Dog:

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Sheep & Dog: $\dot{x} \in v\left(\xi(t), x\right) + B(0, c)$

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Differential Inclusions

(Aubin, Cellina: Differential Inclusions, 1984)

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Differential Inclusions

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Confinement Problems

(Bressan, Zhang: Set Valued and Var. An., 2012)
(Colombo, Pogodaev: SIAM J. Appl. Dyn. Syst., 2012)

Shepherd Dogs – Differential Inclusions

Film

Successful

Shepherd Dogs – Differential Inclusions

Film

NOT Successful

Shepherd Dogs – Differential Inclusions

Film

Movement

Differential Inclusions – Confinement Problems

$$\begin{cases} \dot{K} = f(K, \xi) \\ K(0) = K_o \end{cases}$$

QuasiDifferential Equation

(Colombo & Guerra: DCDS, 2009)

The state K is a compact set

Regularity of K? (from t, K_o, f, $\ldots$)

Measure/perimeter of K?

Controllability?

(Colombo & Pogodaev: Preprint, 2012)

Thank You!