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Boundary treatment in cut cell finite difference methods for compressible gas dynamics in domain with moving boundaries

14th International Conference on Hyperbolic Problems: Theory, Numerics, Applications

Università di Padova, 28 June 2012

Outline

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Euler equations

The governing equations are the compressible Euler equations:

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_y = 0, \tag{1}$$

where ρ is the fluid density, u and v are the velocities, E is the total energy, and p is the pressure.

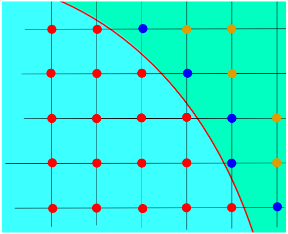
This system is closed using the equation of state (EOS), which, for ideal gases, reads:

$$E = \frac{p}{\gamma - 1} + \frac{\rho}{2}(u^2 + v^2), \quad \gamma = \text{const.} \tag{2}$$

We also introduce the notation $c := \sqrt{\gamma p/\rho}$ for the speed of sound, which will be used throughout the presentation.

Outline

Finite Difference discretization



We identifies different points:

- internal points (red) $\mathbf{X}_{jk} \in \Omega$, $(j,k) \in \mathcal{I}$. These are the points where we solve the problem, and for which we write the differential equation.
- Ghost points $\mathbf{X}_{jk}$, $(j,k) \in \mathcal{G}$.
- Inactive points

Within ghost points, we distinguish between first layer (blue) $\mathcal{L}_1$ and second layer (yellow) $\mathcal{L}_2 : \mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{G}$. The first layer of points is within one grid cell from the boundary (in either direction).
The second layer is made of points within two grid points from the boundary, which are not in the first layer.

Finite Difference discretization (cont.)

Definition of the set of ODE's

$$\frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

The Finite difference approximation is:

$$\frac{d\, u_{jk}}{dt} + \frac{\widehat{f}_{j+\frac{1}{2},k} - \widehat{f}_{j-\frac{1}{2},k}}{\Delta x} + \frac{\widehat{g}_{j,k+\frac{1}{2}} - \widehat{g}_{j,k-\frac{1}{2}}}{\Delta y}, \quad (j,k) \in \mathcal{I}.$$

These equations require the computation of the fluxes, which are at interface between internal cells, or between internal cells and first layer cells.

$$\widehat{f}_{j+\frac{1}{2},k} = \widehat{f}^{+}_{j+\frac{1}{2},k} + \widehat{f}^{-}_{j+\frac{1}{2},k} \tag{3}$$

$$\widehat{f}^{+}_{j+\frac{1}{2},k} = \widehat{F}^{+}_{j,k}\left(x_{j+\frac{1}{2}}, y_k\right), \qquad \widehat{f}^{-}_{j+\frac{1}{2},k} = \widehat{F}^{-}_{j+1,k}\left(x_{j+\frac{1}{2}}, y_k\right) \tag{4}$$

$$\widehat{g}_{j,k+\frac{1}{2}} = \widehat{g}^{+}_{j,k+\frac{1}{2}} + \widehat{g}^{-}_{j,k+\frac{1}{2}} \tag{5}$$

$$\widehat{g}^{+}_{j,k+\frac{1}{2}} = \widehat{G}^{+}_{j,k}\left(x_j, y_{k+\frac{1}{2}}\right), \qquad \widehat{g}^{-}_{j,k+\frac{1}{2}} = \widehat{G}^{-}_{j,k+1}\left(x_j, y_{k+\frac{1}{2}}\right) \tag{6}$$

The four flux functions $\widehat{F}^{\pm}$, $\widehat{G}^{\pm}$ have to be reconstructed in cells $(j,k) \in \mathcal{I} \cup \mathcal{L}_1$ from the pointwise values:

$$f_{j,k} = f(u_{j,k}), \quad g_{j,k} = g(u_{j,k}), \quad (j,k) \in \mathcal{I} \cup \mathcal{G}.$$

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Boundary treatment of Euler equations: Condition on the velocity

Let us denote by $u_n = \mathbf{u} \cdot \mathbf{n}$ and $u_\tau = \mathbf{u} \cdot \tau$ respectively the normal and tangential velocity. The condition on the normal velocity is simply:

$$u_n = 0 \text{ on } \partial\Omega \tag{7}$$

The condition on the tangential velocity is

$$\frac{\partial u_\tau}{\partial n} = u_\tau \; k \tag{8}$$

and it can be obtained in the following two manners.

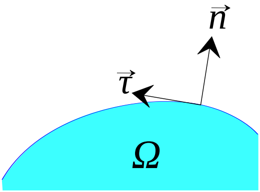


Figure: Locally convex boundary $k < 0$.

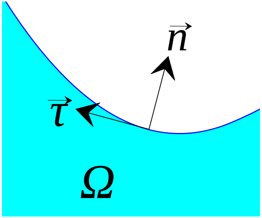
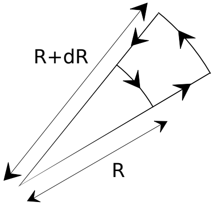


Figure: Locally concave boundary $k > 0$.

- By imposing that the vorticity is zero. This means imposing that

$$\oint_\Gamma \mathbf{u} \cdot d\mathbf{l} = 0$$

for each closed circuit Γ.



Supposing that the u_τ is constants along the arcs, we obtain:

$$\oint_\Gamma \mathbf{u} \cdot d\mathbf{l} = (R + \Delta R)\, u_\tau|_{R+\Delta R} - R\, u_\tau|_R = 0. \tag{9}$$

For Taylor we have:

$$u_\tau|_{R+\Delta R} = u_\tau|_R - \frac{\partial u_\tau}{\partial n}\Delta R + \mathcal{O}(\Delta R^2).$$

Plugging it into (9) and neglecting $\mathcal{O}(\Delta R^2)$ terms, we obtain:

$$\frac{\partial u_\tau}{\partial n} = \frac{1}{R} u_\tau|_R.$$

- By imposing that the normal derivative of the total enthalpy is zero on the boundary. Let us recall that the enthalpy is $h = \frac{1}{2}u^2 + e + \frac{p}{\rho}$, where $e = e(\rho, p) = \frac{p}{(\gamma - 1)\rho}$ is the internal energy. The condition reads:

$$0 = \frac{\partial h}{\partial n} = \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial n} + \left(\frac{1}{\gamma - 1} + 1\right)\left(\frac{\partial p}{\partial n}\frac{1}{\rho} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial n}\right).$$

Using the boundary conditions on density and pressure (which will be explained later) and the fact that $\mathbf{u} = u_\tau \tau$ on the boundary, we obtain:

$$0 = \frac{\partial u_\tau}{\partial n} u_\tau + \frac{\gamma}{\gamma - 1}\frac{\partial p}{\partial n}\left(\frac{1}{\rho} - \frac{p}{c_s^2 \rho^2}\right) = \frac{\partial u_\tau}{\partial n} u_\tau - \frac{\gamma}{\gamma - 1} k\, u_\tau^2 \left(1 - \frac{p}{c_s^2 \rho}\right)$$

where c_s^2 is the square of the speed sound. For a polytropic gas we have $c_s^2 = \gamma p / \rho$. Therefore:

$$0 = \frac{\partial u_\tau}{\partial n} u_\tau - k\, u_\tau^2 \Longrightarrow \frac{\partial u_\tau}{\partial n} = k\, u_\tau.$$

Condition on the pressure

The equation of motion for a fluid particle (balance of momentum) for Euler equations reads:

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p = 0 \tag{10}$$

where $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ denotes the Lagrangian derivative.
Along the boundary of the domain, the velocity vector can be defined as follows:

$$\mathbf{u} = u_\tau \tau$$

It is therefore:

$$\frac{D\mathbf{u}}{Dt} = \frac{Du_\tau}{Dt}\tau + u_\tau \frac{D\tau}{Dt} = a_\tau \tau + u_\tau^2 \, k \, \mathbf{n} \tag{11}$$

where k denotes the curvature. The sign of k is negative for locally convex regions, and positive for locally concave regions, and a_τ denotes the tangential acceleration of the fluid. By projecting Eq. (10) on the normal direction, and making use of (11), one obtain the boundary condition on the pressure:

$$\frac{\partial p}{\partial n} = -\rho \, u_\tau^2 \, k.$$

Condition on the density

Finally, the condition on the density is given by the requirement that the boundary is adiabatic:

$$\frac{\partial p}{\partial n} = c_s^2 \frac{\partial \rho}{\partial n}.$$

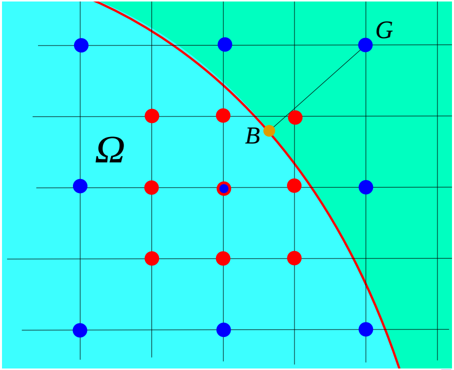
Outline

Discretization of the boundary conditions

We write a linear equation for each unknown of the system, i.e. for each $v_x(G)$, $v_y(G)$, $p(G)$, $\rho(G)$, where $G \in \mathcal{G}$ is a ghost point. In details: let G be a ghost point. We compute the projection point B on the interface:

$$B \equiv (x_B, y_B) = G - \phi(G)\mathbf{n}_G = G - \phi(G)\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\bigg|_G .$$

Let us define two 3×3 stencils: $St_G^{(I)}$ (blue) and $St_G^{(II)}$ (red).



Let us recall the boundary conditions we want to use in order to extrapolate **u**, p, ρ in ghost points:

$$
\begin{aligned}
u_n &= 0 && (12)\\
\frac{\partial u_\tau}{\partial n} - u_\tau\, k &= 0 && (13)\\
\frac{\partial p}{\partial n} &= -\rho\, k\, u^2 && (14)\\
\frac{\partial \rho}{\partial n} &= \frac{1}{c_s^2}\,\frac{\partial p}{\partial n} && (15)
\end{aligned}
$$

Let us denote by $\mathcal{L}[\omega; St]$ the biquadratic interpolant of ω in the stencil St. The four linear equations for the ghost point G are:

$$
\begin{aligned}
\mathcal{L}[u_n; St_G^{(I)}](B) &= 0 && (16)\\
\frac{\partial \mathcal{L}[u_\tau; St_G^{(I)}](B)}{\partial n} - \mathcal{L}[u_\tau; St_G^{(I)}](B)\, k &= 0 && (17)\\
\frac{\partial \mathcal{L}[p; St_G^{(I)}](B)}{\partial n} &= -\rho\, k\, \left(\mathcal{L}[u_\tau; St_G^{(II)}](B)\right)^2 && (18)\\
\frac{\partial \mathcal{L}[\rho; St_G^{(I)}](B)}{\partial n} &= \frac{1}{c_s^2}\,\frac{\partial \mathcal{L}[p; St_G^{(II)}](B)}{\partial n} && (19)
\end{aligned}
$$

The linear system is solved by an iterative scheme. In particular, we transform the system of the boundary conditions into a time dependent problem with a fictitious time τ:

$$\frac{\partial u_n}{\partial \tau} + u_n = 0 \tag{20}$$

$$\frac{\partial u_\tau}{\partial \tau} + \mu_1 \frac{\partial u_\tau}{\partial n} = \mu_1 u_\tau \, k \tag{21}$$

$$\frac{\partial p}{\partial p} + \mu_2 \frac{\partial p}{\partial n} = -\mu_2 \rho \, k \, u^2 \tag{22}$$

$$\frac{\partial \rho}{\partial \rho} + \mu_3 \frac{\partial \rho}{\partial n} = \mu_3 \frac{1}{c_s^2} \frac{\partial p}{\partial n} \tag{23}$$

where μ_i, $i = 1, 2, 3$ are suitable constants.

Treatment of moving boundaries

Let us assume that the normal boundary velocity is given by a function $V(\mathbf{x}, t)$. Such a function can be easily computed, for example, in the case in which the distance function is known analytically, as is the case of the motion of a rigid body: if $\phi(\mathbf{x}, t)$ is a signed distance function, then one has

$$\frac{\partial \phi}{\partial t} + V |\nabla \phi| = 0 \Rightarrow V(\mathbf{x}, t) = -\frac{\partial \phi}{\partial t}.$$

In the case of a moving boundary, the boundary conditions on $\partial\Omega$ for the velocity (i.e. (7) and (8)) become:

$$u_n = V \qquad (24)$$

$$\frac{\partial u_\tau}{\partial n} = u_\tau \, k. \qquad (25)$$

The condition for the pressure is obtained as follows. The velocity of the fluid on the boundary is given by:

$$\mathbf{u} = u_\tau \tau + V\mathbf{n}. \tag{26}$$

From the equation of motion, one has:

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p = 0.$$

Projecting this relation along the normal to the line, one has:

$$-\frac{\partial p}{\partial n} = \rho \frac{D\mathbf{u}}{Dt} \cdot \mathbf{n}.$$

Differentiating (26) along the trajectory, taking the scalar product with $\mathbf{n}$, considering that $\tau \cdot \mathbf{n} = 0$ and $\dfrac{D\mathbf{n}}{Dt} \cdot \mathbf{n} = 0$, one has:

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} = u_\tau \mathbf{n} \cdot \frac{D\tau}{Dt} + \frac{DV}{Dt}$$

Furthermore,

$$\frac{D\tau}{Dt} = \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau = \frac{\partial \tau}{\partial t} + u_\tau \frac{\partial \tau}{\partial \tau} = \frac{\partial \tau}{\partial t} + u_\tau \, k \, \mathbf{n}$$

which gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} = \mathbf{n} \cdot \frac{\partial \tau}{\partial t} u_\tau + u_\tau^2 \, k + \frac{DV}{Dt}. \tag{27}$$

Outline

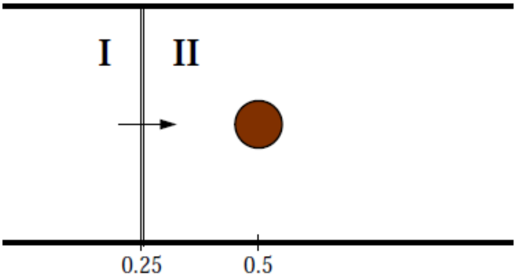
Numerical tests: Moving Shock- Steady Ball

All the numerical tests are taken from:

- Chertock, A., Kurganov, A., A simple Eulerian finite-volume method for compressible fluids in domains with moving boundaries, *Commun. Math. Sci.*, Volume no. **6** (2008), 531-556.

We consider a flow generated by a right moving vertical shock, initially positioned at $x = 0.25$:

$$(\rho(x,y,0), u(x,y,0), v(x,y,0), p(x,y,0)) = \begin{cases} (4/3, 35/99, 0, 1.5), & x < 0.25, \\ (1, 0, 0, 1), & x \geq 0.25. \end{cases}$$



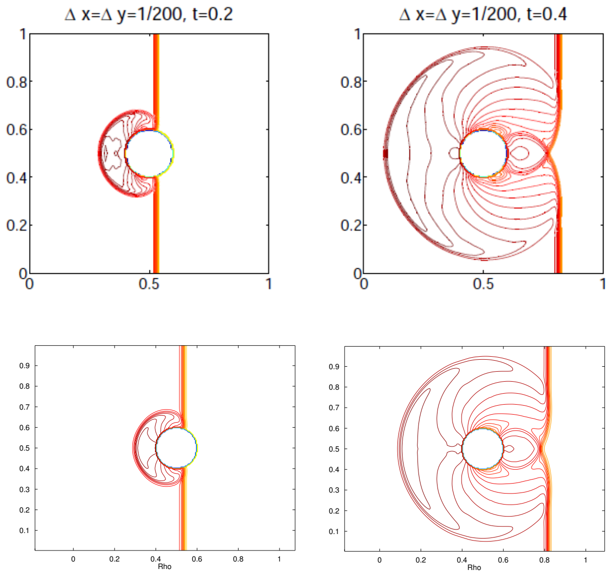
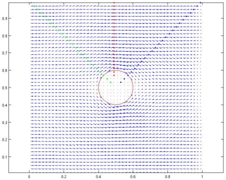
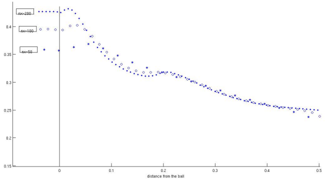
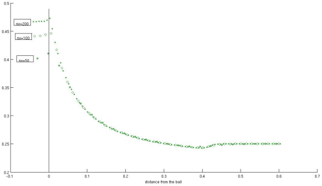


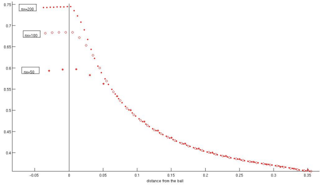
Figure: Top: Finite-Volume; bottom: finite-difference.

Validation of the BC for u_τ: $\dfrac{\partial u_\tau}{\partial n} = 0$

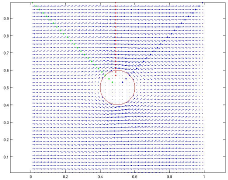


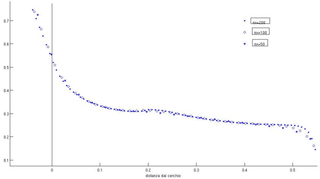


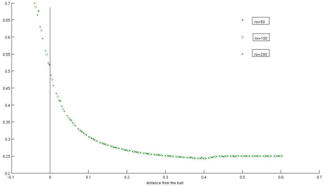


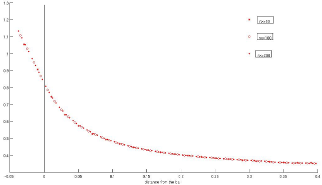


Validation of the BC for u_τ: $\dfrac{\partial u_\tau}{\partial n} = u_\tau\, k$









No Shock - Slow-Moving Ball

In this example, the ball moves periodically up and down with a small amplitude ($A = 0, B = 0.01, \omega = 10\pi$):

$$x_c(0) = 0, \quad \dot{x}_c(t) = A\,\omega\, \cos(\omega\, t),$$

$$y_c(0) = 0, \quad \dot{y}_c(t) = B\,\omega\, \cos(\omega\, t),$$

and it is placed in an initially steady flow with

$$\rho(x, y, 0) \equiv p(x, y, 0) \equiv 1, \;\; u(x, y, 0) \equiv v(x, y, 0) \equiv 0.$$

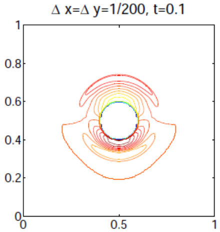


Figure: Finite-volume.

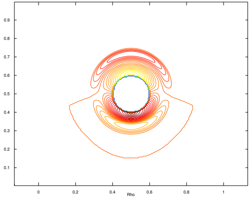


Figure: Finite-difference.

No Shock - Vertically Moving Ball

We now consider the same setting as in the previous example, but with 10 times larger and faster ball oscillations ($A = 0, B = 0.1$).
Top: finite-volume; Bottom: finite-difference.