Asymptotic stabilization of the hyperelastic-rod wave equation

Giuseppe Maria Coclite

Department of Mathematics
University of Bari
Via E. Orabona 4
70125 Bari (Italy)
EMAIL: coclitegm@dm.uniba.it
URL: http://www.dm.uniba.it/Members/coclitegm/

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The Hyperelastic-Rod Wave Equation

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u\right)$$

- $t \geq 0$ time
- $x \in \mathbb{R}$ space (one-dimensional)
- $u(t,x) \in \mathbb{R}$ unknown (one-dimensional)
- $\gamma > 0$ is a given constant

Physics

Hyperelastic-Rod Waves

- Finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods.
- γ depends on the material constants and the pre-stress of the rod.

- Dai (1998 - 1998)
- Dai & Huo (2002)

Shallow Water Waves

Shallow Water Waves

$$\Updownarrow$$

$$\text{Depth of the water} \ll \text{Length of the waves}$$

- $\gamma = 1$
- unidirectional shallow water waves
- $u \equiv$ wave velocity above the bottom
- flat bottom
- Camassa-Holm equation

- Camassa & Holm (1993)
- Johnson (2002)

The Hyperelastic-Rod Wave Equation

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma\left(2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u\right)$$

is formally equivalent to the elliptic-hyperbolic system

$$\partial_t u + \gamma u \partial_x u + \partial_x P = 0, \qquad -\partial^2_{xx} P + P = \frac{3-\gamma}{2} u^2 + \frac{\gamma}{2}(\partial_x u)^2 .$$

Since

$$\frac{e^{-|x|}}{2}$$

is the Green's function of the Helmholtz operator $-\partial^2_{xx} + 1$

$$P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\left(\frac{3-\gamma}{2}u(t,y)^2 + \frac{\gamma}{2}(\partial_x u(t,y))^2\right) dy,$$
$$\partial_x P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\mathrm{sign}\,(y-x)\left(\frac{3-\gamma}{2}u(t,y)^2 + \frac{\gamma}{2}(\partial_x u(t,y))^2\right) dy.$$

Asymptotic Stabilization

Problem

Find an operator

$$f : H^1(\mathbb{R}) \longrightarrow H^{-1}(\mathbb{R})$$

such that for every initial condition $u_0 \in H^1(\mathbb{R})$ the solution of the Cauchy problem

$$\begin{cases} \partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u \right) + f[u] \\ u(0,x) = u_0(x) \end{cases}$$

decays as $t \longrightarrow \infty$, i.e.,

$$\lim_{t \longrightarrow \infty} u(t,x) = 0.$$

Physics

- Damp the waves on hyperelastic rods
- $f[u] \equiv$ external force

H^1 Regularity

- Weak solutions
  - $u(t,\cdot) \in H^1(\mathbb{R})$
  - $u(t,\cdot)$ is bounded and continuous
  - Blow-up of $\partial_x u$
- More regularity on the initial conditions gives more regularity of the solutions (there are several smooth solutions)
- Solitons and peakons are weak solutions

Literature

- Glass (2008): source compactly sopported type feedback, H^2 solutions
- Perrollaz (2010): boundary feedback, H^2 solutions

Our feedback law

$$f[u] = -\lambda(1 - \partial_{xx}^2)u, \qquad \lambda > 0$$

$$\partial_t u - \partial_{txx}^3 u + 3u\partial_x u = \gamma \left(2\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u\right) -\lambda(1 - \partial_{xx}^2)u$$

$$\Updownarrow$$

$$\begin{cases} \partial_t u + \gamma u \partial_x u + \partial_x P = -\lambda u \\ -\partial_{xx}^2 P + P = \frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \end{cases}$$

Formal Energy Estimate

$$\begin{cases} \partial_t u + \gamma u \partial_x u + \partial_x P = -\lambda u \\ -\partial_{xx}^2 P + P = \frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \end{cases}$$

$$\Downarrow$$

$$\partial_t \left(\frac{u^2 + (\partial_x u)^2}{2} \right) + \partial_x \left(\frac{\gamma}{2} u (\partial_x u)^2 - \frac{1-\gamma}{2} u^3 + uP \right) = -\lambda \left(u^2 + (\partial_x u)^2 \right)$$

The total energy

$$E(t) := \|u(t,\cdot)\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} \left(u(t,x)^2 + (\partial_x u(t,x))^2 \right) dx$$

satisfies the following ordinary differential equation

$$\frac{d}{dt} E(t) = -2\lambda E(t)$$

and therefore

$$E(t) = E(0) e^{-2\lambda t}, \qquad t \geq 0.$$

Definition (Weak Solutions)

A function $u : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a weak solution of the Cauchy problem

$$\begin{cases} \partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u \partial^2_{xx} u + u \partial^3_{xxx} u \right) u - \lambda (1 - \partial^2_{xx}) u \\ u(0,x) = u_0(x) \end{cases}$$

if

- u is continuous;
- $u(t,\cdot) \in H^1(\mathbb{R})$ at every $t \in [0,\infty)$;
- the map $t \longrightarrow u(t,\cdot)$ is Lipschitz continuous from $[0,\infty)$ into $L^2(\mathbb{R})$ and satisfies the initial condition together with the following equality between functions in $L^2(\mathbb{R})$:

$$\frac{d}{dt} u = -\gamma u \partial_x u - \partial_x P - \lambda u, \qquad \text{for a.e. } t \in [0,\infty).$$

Let $\gamma,\,\lambda > 0$ be fixed. There exists a semigroup

$$S : [0,\infty) \times H^1(\mathbb{R}) \longrightarrow H^1(\mathbb{R})$$

such that the following properties hold.

- For every $u_0 \in H^1(\mathbb{R})$, $u(t,x) = S_t(u_0)(x)$ is a weak solution of

$$\begin{cases} \partial_t u - \partial_{txx}^3 u + 3u\partial_x u = \gamma\left(2\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u\right) u - \lambda(1-\partial_{xx}^2)u, \\ u(0,x) = u_0(x). \end{cases}$$

- $E(t) \le E(0)e^{-2\lambda t},\ t \ge 0$.
- For every $u_0 \in H^1(\mathbb{R})$ there exists a constant $C > 0$ such that

$$\partial_x S_t(u_0)(x) \le C\left(1 + \frac{1}{t}\right), \qquad t > 0.$$

- For every $\{u_{0,n}\}_n \subset H^1(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R})$

$$u_{0,n} \longrightarrow u_0 \text{ in } H^1(\mathbb{R}) \implies S(u_{0,n}) \longrightarrow S(u_0) \text{ in } L^\infty_{loc}((0,\infty)\times\mathbb{R}).$$

Remark

- *Semigroup of solutions*
  - *no uniqueness of weak solutions*
  - *solitons interaction*
  - *conservative and dissipative solutions*
- *Energy exponential decay*
- *Oleinik type estimate*
  - $\partial_x u$ *is bounded from above*
  - $\partial_x u$ *may go to* $-\infty$
- S *is not continuous as a map with values in* H^1.

Strategy

- We introduce a new set of variable that solves a semilinear system of ordinary differential equations.
- Short time existence of solutions for the semilinear system.
- An energy estimate gives the existence of global in time solutions for the semilinear system.
- Continuous dependence with respect to the initial conditions for the semilinear system.
- We come back to the original variables and prove our result.

Bressan & Constantin (Anal. Appl. - 2007)

New variables

Let u be a smooth solution of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u\right) u - \lambda(1 - \partial^2_{xx})u \\ u(0, x) = u_0(x). \end{cases}$$

Therefore $(u,\ P)$ is a smooth solution of

$$\begin{cases} \partial_t u + \gamma u \partial_x u + \partial_x P = -\lambda u \\ -\partial^2_{xx} P + P = \frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} \left(\partial_x u\right)^2 \\ u(0, x) = u_0(x). \end{cases}$$

Energy variable $\xi \in \mathbb{R}$.

The map

$$y \in \mathbb{R} \longmapsto \int_0^y \left(1 + (\partial_x u_0)^2\right) dx$$

is continuous, increasing, and goes to ∞ and $-\infty$ as $y \longrightarrow \infty$ and $y \longrightarrow -\infty$, respectively. So we can define implicitly the function $y_0 = y_0(\xi)$ by the relation

$$\int_0^{y_0(\xi)} \left(1 + (\partial_x u_0)^2\right) dx = \xi, \qquad \xi \in \mathbb{R}.$$

- ξ plays the role of a Lagrangian variable (it is constant along characteristics)

Characteristic curve $t \longmapsto y(t,\xi)$

$$\partial_t y(t,\xi) = \gamma u(t, y(t,\xi)), \qquad y(0,\xi) = y_0(\xi).$$

Notation

$$u(t,\xi) := u(t, y(t,\xi)), \qquad P(t,\xi) := P(t, y(t,\xi)).$$

New variables $v = v(t,\xi)$ **and** $q = q(t,\xi)$

$$v := 2\arctan(\partial_x u), \qquad q := (1 + (\partial_x u)^2)\partial_\xi y.$$

- v is bounded
- $q \geq 0$

The semilinear system for $u = u(t,\xi)$, $v = v(t,\xi)$, $q = q(t,\xi)$

$$
\begin{cases}
\partial_t u = -\partial_x P - \lambda u \\
\partial_t v = \left(\frac{3-\gamma}{2} u^2 - P\right)(1+\cos(v)) - \gamma \sin^2\left(\frac{v}{2}\right) - \lambda \sin(v) \\
\partial_t q = \left(\frac{3-\gamma}{2} u^2 - P + \frac{\gamma}{2}\right)\sin(v) q - 2\lambda \sin^2\left(\frac{v}{2}\right) q \\
u(0,\xi) = u_0(y_0(\xi)) \\
v(0,\xi) = 2\arctan(\partial_x u_0(y_0(\xi))) \\
q(0,\xi) = 1
\end{cases}
$$

The nonlocal term $P = P(t,\xi)$

$$
\begin{aligned}
P(t,\xi) =& \frac{1}{2}\int_{\mathbb{R}} e^{-\left|\int_{\xi}^{\xi'} \cos^2\left(\frac{v(t,s)}{2}\right) q(t,s)ds\right|} \times \\
&\times \left(\frac{3-\gamma}{2} u(t,\xi')^2 \cos^2\left(\frac{v(t,\xi')}{2}\right) + \frac{\gamma}{2}\sin^2\left(\frac{v(t,\xi')}{2}\right)\right) \times \\
&\times q(t,\xi')d\xi',
\end{aligned}
$$

$$
\begin{aligned}
\partial_x P(t,\xi) =& \frac{1}{2}\int_{\mathbb{R}} e^{-\left|\int_{\xi}^{\xi'} \cos^2\left(\frac{v(t,s)}{2}\right) q(t,s)ds\right|} \times \\
&\times \operatorname{sign}(\xi - \xi') \times \\
&\times \left(\frac{3-\gamma}{2} u(t,\xi')^2 \cos^2\left(\frac{v(t,\xi')}{2}\right) + \frac{\gamma}{2}\sin^2\left(\frac{v(t,\xi')}{2}\right)\right) \times \\
&\times q(t,\xi')d\xi'.
\end{aligned}
$$

In order to obtain global dissipative solutions, a modification of the system for u, v, q is needed. Assume that, along a given characteristic $t \longmapsto y(t,\xi)$, the wave breaks at a first time $t = \tau(\xi)$. Arguing as for the Burgers equation and reminding that $\partial_x u$ satisfies an Oleinik type inequality, the wave break means $\partial_x u(t,\xi) \longrightarrow -\infty$, as $t \longrightarrow \tau(\xi)^-$. For all $t \geq \tau(\xi)$ we then set $v(t,\xi) \equiv -\pi$ and remove the values of $u(t,\xi)$, $v(t,\xi)$, $q(t,\xi)$ from the computation of P and $\partial_x P$.

The dissipative semilinear system for
$u = u(t,\xi)$, $v = v(t,\xi)$, $q = q(t,\xi)$

$$
\begin{cases}
\partial_t u = -\partial_x P - \lambda u \\
\partial_t v = \begin{cases} \left(\frac{3-\gamma}{2}u^2 - P\right)(1+\cos(v)) - \gamma \sin^2\left(\frac{v}{2}\right) - \lambda \sin(v) & \text{if } v > -\pi \\ 0 & \text{if } v \leq -\pi \end{cases} \\
\partial_t q = \begin{cases} \left(\frac{3-\gamma}{2}u^2 - P + \frac{\gamma}{2}\right)\sin(v)q - 2\lambda \sin^2\left(\frac{v}{2}\right) q & \text{if } v > -\pi \\ 0 & \text{if } v \leq -\pi \end{cases} \\
u(0,\xi) = u_0(y_0(\xi)) \\
v(0,\xi) = 2\arctan(\partial_x u_0(y_0(\xi))) \\
q(0,\xi) = 1
\end{cases}
$$

The dissipative nonlocal term $P = P(t,\xi)$

$$
\begin{aligned}
P(t,\xi) =& \frac{1}{2}\int_{\{v(\xi')>-\pi\}} e^{-\left|\int_{\{\xi'\le\xi,\, v(\xi')>-\pi\}} \cos^2\left(\frac{v(t,s)}{2}\right) q(t,s)ds\right|} \times \\
&\times \left(\frac{3-\gamma}{2} u(t,\xi')^2 \cos^2\left(\frac{v(t,\xi')}{2}\right) + \frac{\gamma}{2}\sin^2\left(\frac{v(t,\xi')}{2}\right)\right) \times \\
&\times q(t,\xi')d\xi',
\end{aligned}
$$

$$
\begin{aligned}
\partial_x P(t,\xi) =& \frac{1}{2}\int_{\{v(\xi')>-\pi\}} e^{-\left|\int_{\{\xi'\le\xi,\, v(\xi')>-\pi\}} \cos^2\left(\frac{v(t,s)}{2}\right) q(t,s)ds\right|} \times \\
&\times \operatorname{sign}(\xi-\xi') \times \\
&\times \left(\frac{3-\gamma}{2} u(t,\xi')^2 \cos^2\left(\frac{v(t,\xi')}{2}\right) + \frac{\gamma}{2}\sin^2\left(\frac{v(t,\xi')}{2}\right)\right) \times \\
&\times q(t,\xi')d\xi'.
\end{aligned}
$$

Global existence and uniqueness for the semilinear system

Local existence and uniqueness

- General theorem on directionally continuous ordinary differential equations in functional spaces
- Bressan & Shen - 2006

Global existence

- Energy estimate
- Global bound on the total energy

$$E(t) = \|u(t,\cdot)\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \left(u(t,x)^2 + (\partial_x u(t,x))^2\right) dx$$
$$= \int_{\{v(t,\xi)>-\pi\}} \left(u^2(t,\xi)\cos^2\left(\frac{v(t,\xi)}{2}\right) + \sin^2\left(\frac{v(t,\xi)}{2}\right)\right) q(t,\xi)d\xi$$

Stability for the semilinear system

Claim

Let $\{u_{0,n}\}_n \subset H^1(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R})$. If

$$u_{0,n} \longrightarrow u_0 \qquad \text{in } H^1(\mathbb{R}),$$

then

$$u_n \longrightarrow u \qquad \text{in } L^\infty((0,T)\times\mathbb{R}) \text{ for every } T > 0,$$

where u_n and u are the solutions of the semilinear dissipative system in correspondence of $u_{0,n}$ and u_0, respectively.

Let (u, v, q) and $(\widetilde{u}, \widetilde{v}, \widetilde{q})$ be any two solutions of the semilinear dissipative system and $T > 0$.
For every

$$\xi \in \{\xi \in \mathbb{R}; v(T, \xi) = -\pi\} \cup \{\xi \in \mathbb{R}; \widetilde{v}(T, \xi) = -\pi\}.$$

Let $\tau(\xi)$ be the first time at which one of the two solutions reaches the value $-\pi$, namely

$$\tau(\xi) = \inf\{t \in [0, T]; \min\{v(t, \xi), \widetilde{v}(t, \xi)\} = -\pi\}.$$

Since the map $\tau(\cdot)$ is measurable, we can construct a measure-preserving, measurable map $\alpha \longmapsto \xi(\alpha)$ from $[0, \alpha^*]$ onto Λ such that

$$\alpha \le \alpha' \Longleftrightarrow \tau(\xi(\alpha')) \le \tau(\xi(\alpha)).$$

The inverse mapping $\xi \longmapsto \alpha(\xi)$ from Λ into $[0, \alpha^*]$ is still measure-preserving.

Distance functional

$$
\begin{aligned}
j(t) =& J((u(t,\cdot), v(t,\cdot), q(t,\cdot)), (\widetilde{u}(t,\cdot), \widetilde{v}(t,\cdot), \widetilde{q}(t,\cdot))) \\
=& \|u(t,\cdot) - \widetilde{u}(t,\cdot)\|_{L^\infty(\mathbb{R})} \\
& + \|v(t,\cdot) - \widetilde{v}(t,\cdot)\|_{L^2(\mathbb{R})} + \|q(t,\cdot) - \widetilde{q}(t,\cdot)\|_{L^2(\mathbb{R})} \\
& + K_0 \int_0^{\alpha^*} e^{K\alpha} |v(t,\xi(\alpha)) - \widetilde{v}(t,\xi(\alpha))| d\alpha,
\end{aligned}
$$

where K and K_0 are two positive constants.

Choosing suitably K and K_0

$$\frac{d}{dt}j(t) \leq Mj(t),$$

for some constant $M > 0$. Therefore

$$j(t) \leq e^{Mt}j(0), \qquad 0 \leq t \leq T,$$

and in particular

$$\|u(t,\cdot) - \widetilde{u}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq ce^{Mt}\|u(0,\cdot) - \widetilde{u}(0,\cdot)\|_{H^1(\mathbb{R})}.$$

Global Dissipative Solutions in the Original Variables $u = u(t,x)$, $P = P(t,x)$

Let (u, v, q) be the solution of the semilinear system.
Define

$$y(t,\xi) = y_0(\xi) + \int_0^t u(\tau,\xi)d\tau.$$

For each fixed ξ, the function $t \longmapsto y(t,\xi)$ solves

$$\partial_t y(t,\xi) = u(t,\xi), \qquad y(0,\xi) = y_0(\xi).$$

We set

$$u(t,x) = u(t,\xi) \quad \text{if } y(t,\xi) = x.$$

Clearly

$$u(0,x) = u_0(x) \quad x \in \mathbb{R}.$$

The facts

- the energy estimate on $\|u(t,\cdot)\|_{H^1(\mathbb{R})}$
- the image of the singular set where $v = -\pi$ has measure zero (in the x-variable), i.e.,

$$\text{meas}(\{y(t,\xi); v(t,\xi) = -\pi\}) = 0$$

give that

- u is continuous
- $t \longmapsto u(t,\cdot) \in L^2(\mathbb{R})$ is Lipschitz continuous
- $\frac{d}{dt}u = -\gamma u \partial_x u - \partial_x P - \lambda u.$

Therefore

u is a weak solution of the hyperelastic rod wave equation.

The Semigroup

Given an initial data $u_0 \in H^1(\mathbb{R})$, we denote by $u(t,x) = S_t(u_0)(x)$ the corresponding global solution of the hyperelastic-rod wave equation. We have to prove

$$S_t(S_\tau(u_0)) = S_{t+\tau}(u_0), \qquad t,\, \tau > 0.$$

Let (u, v, q) be the solution of the problem in the auxiliary variables. Call $\widetilde{u} = S_\tau(u_0)$. We choose ξ_0 such that

$$y(\tau, \xi_0) = 0$$

and consider the new energy variable $\sigma = \sigma(\xi)$ as a solution of

$$\frac{d}{d\xi}\sigma(\xi) = \begin{cases} q(\tau,\xi) & \text{if } v(\tau,\xi) > -\pi, \\ 0 & \text{if } v(\tau,\xi) = -\pi, \end{cases} \qquad \sigma(\xi_0) = 0.$$

We have

$$\int_0^{y(\tau,\xi)} \left(1 + (\partial_x u(\tau,x))^2\right) dx = \sigma(\xi).$$

The map $\xi = \xi(\sigma)$

$$\xi(\overline{\sigma}) = \sup\{s; \sigma(s) \le \overline{\sigma}\}$$

provides almost everywhere an inverse of $\sigma(\cdot)$.
Define

$$\begin{aligned}\widetilde{u}(t,\sigma) =& u(\tau + t, \xi(\sigma)),\\ \widetilde{v}(t,\sigma) =& v(\tau + t, \xi(\sigma)),\\ \widetilde{q}(t,\sigma) =& \frac{q(\tau + t, \xi(\sigma))}{q(\tau, \xi(\sigma))}.\end{aligned}$$

Since $(\widetilde{u}, \widetilde{v}, \widetilde{q})$ solves the same equations of (u, v, q).

S is a semigroup.

The decay as $t \longrightarrow \infty$

Given an initial data $u_0 \in H^1(\mathbb{R})$, let $u(t,x)$ be the corresponding global solution of the hyperelastic-rod wave equation and (u,v,q) be the solution of the problem in the auxiliary variables. Consider

$$E(t) = \|u(t,\cdot)\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \left(u(t,x)^2 + (\partial_x u(t,x))^2\right) dx$$
$$= \int_{\{v(t,\xi)>-\pi\}} \left(u^2(t,\xi)\cos^2\left(\frac{v(t,\xi)}{2}\right) + \sin^2\left(\frac{v(t,\xi)}{2}\right)\right) q(t,\xi)d\xi.$$

We have

$$\frac{d}{dt}E \leq -2\lambda E(t).$$

Therefore

$$E(t) \leq e^{-2\lambda t}E(0), \qquad t \geq 0.$$

The Oleinik type estimate

Since

$$\partial_t v\Big|_{v=\pi} = \left(\frac{3-\gamma}{2}u^2 - P\right)(1+\cos(v)) - \gamma\sin^2\left(\frac{v}{2}\right) - \lambda\sin(v)\Big|_{v=\pi} = -\gamma,$$

we can choose $\delta > 0$ so that

$$\partial_t v(t,\xi) \le -\frac{\gamma}{2}, \qquad v \in [\pi-\delta, \pi).$$

As a consequence

$$v(t,\xi) < \min\left\{\pi-\delta, \pi - \frac{t\gamma}{2}\right\}, \qquad v \in [\pi-\delta, \pi).$$

Hence,

$$\partial_x u \le C\left(1 + \frac{1}{t}\right)$$

follows from the identity

$$\partial_x u = \frac{\sin(v)}{1+\cos(v)}.$$