Central-Upwind Scheme for the Shallow Water System with Horizontal Temperature Gradients

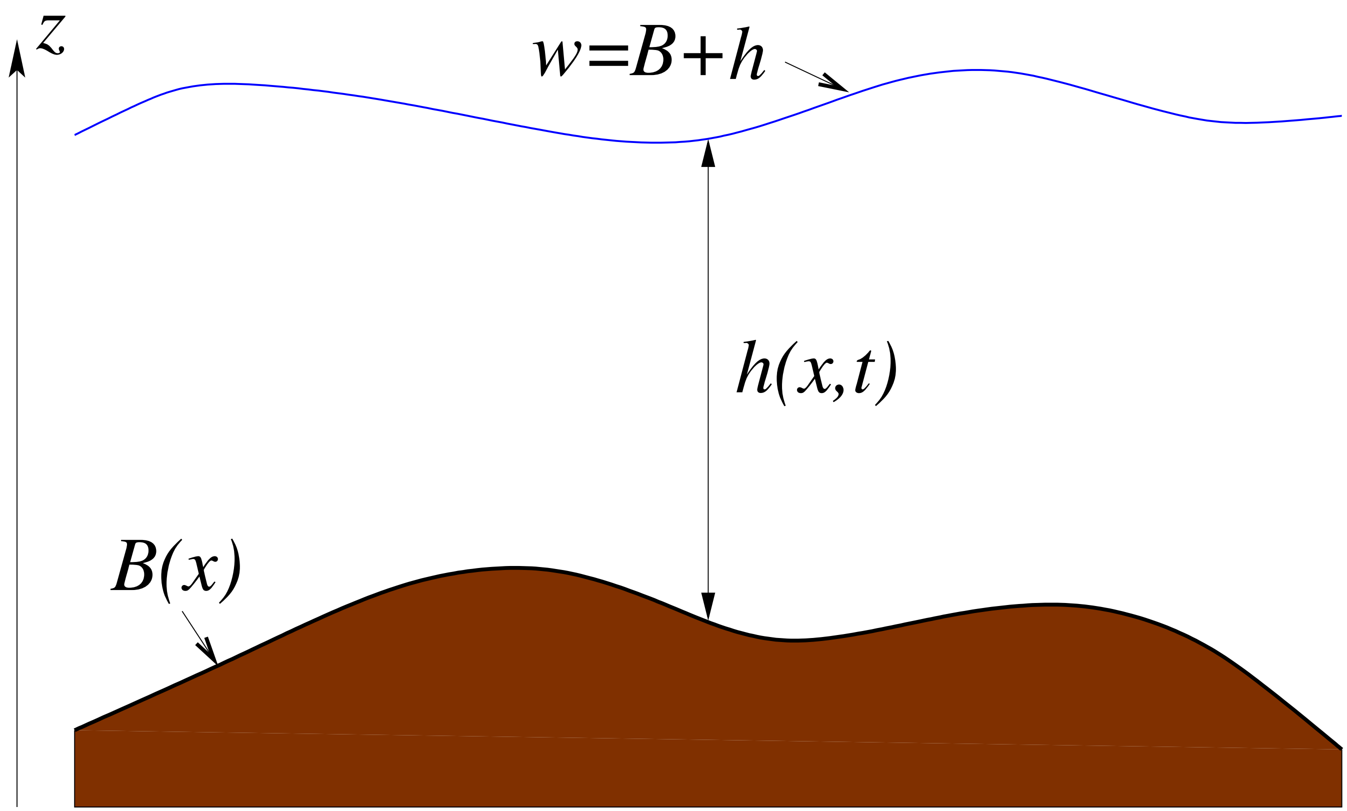
Alina Chertock

North Carolina State University
chertock@math.ncsu.edu

joint work with
Alexander Kurganov, Tulane University
Yu Liu, Tulane University

Saint-Venant System of Shallow Water

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghB_x \end{cases}$$



The 1-D Ripa System

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{g}{2}h^2\theta\right)_x = -gh\theta B_x \\ (h\theta)_t + (uh\theta)_x = 0 \end{cases}$$

- h: water height
- u: fluid velocity
- θ: potential temperature. Specifically, θ is the reduced gravity $g\Delta\Theta/\Theta_{\text{ref}}$ computed as the potential temperature difference $\Delta\Theta$ from some reference value Θ_{ref}.
- B: bottom topography
- g: gravitational constant

If $\theta \equiv \text{const}$, then the Ripa system reduces to the Saint-Venant system of shallow water equations

The Ripa System

- Introduced in [Ripa [(1993,1995), Dellar (2003)] for modeling ocean currents.

- The derivation of the system is based on considering multilayered ocean models, and vertically integrating the density, horizontal pressure gradient and velocity fields in each layer.

- The model incorporates the horizontal temperature gradients, which results in the variations in the fluid density within each layer.

The Ripa System

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{g}{2}h^2\theta\right)_x = -gh\theta B_x \\ (h\theta)_t + (uh\theta)_x = 0 \end{cases}$$

admits the energy (entropy) inequality

$$\left(\frac{hu^2}{2} + g\frac{h^2\theta}{2} + gh\theta B\right)_t + \left[u\left(\frac{hu^2}{2} + gh^2\theta + gh\theta B\right)\right]_x \leq 0$$

- The eigenvalues are $u \pm \sqrt{gh\theta}, u, 0$
- There is a nonlinear resonance when $u \pm c = 0$ (wave speeds from different families of waves coincide)
- There are no Riemann invariants for the Ripa system and therefore it is very hard to design upwind schemes since they are based on (approximate) Riemann problem solvers

Balance Law

$$\mathbf{U}_t + \mathbf{f}(\mathbf{U})_x = \mathbf{S}(\mathbf{U})$$

$$\mathbf{U} := (h, hu, h\theta)^T, \quad \mathbf{f} := (hu, hu^2 + \frac{g}{2}h^2\theta, uh\theta)^T, \quad \mathbf{S} := (0, -ghB_x, 0)^T$$

Semi-discrete central-upwind scheme:

$$\frac{d}{dt}\overline{\mathbf{U}}_j = -\frac{\mathbf{H}_{j+\frac{1}{2}} - \mathbf{H}_{j-\frac{1}{2}}}{\Delta x} + \overline{\mathbf{S}}_j,$$

$$\overline{\mathbf{U}}_j^n \approx \frac{1}{\Delta x} \int\limits_{C_j} \mathbf{U}(x, t^n)\, dx, \quad C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$$

Numerical Challenges:

- well-ballanced
- positivity preserving

Steady States

$$\begin{cases} (hu)_x = 0 \\ \left(hu^2 + \dfrac{g}{2}h^2\theta\right)_x = -gh\theta B_x \end{cases} \iff \begin{cases} (hu)_x = 0 \\ \left(\dfrac{u^2}{2} + g\theta(h+B)\right)_x = \dfrac{g}{2}h\theta_x \end{cases}$$

The system cannot be integrated, but admits several particular steady-state solutions, two of them are the following "lake at rest" ones:

1. $\theta \equiv$ constant, $\quad w := h + B \equiv$ constant, $\quad u \equiv 0$

   corresponds to flat water surface under the constant temperature

2. $B \equiv$ constant, $\quad p := \frac{g}{2}h^2\theta \equiv$ constant, $\quad u \equiv 0$

   corresponds to the contact wave across which h and θ jump while u and p remain constant

Goal: **to derive a well-balanced scheme which preserves both steady states!**

Well-Balanced Scheme

1. $\theta \equiv \text{constant}, \quad w := h + B \equiv \text{constant}, \quad u \equiv 0$

   Well-balanced scheme should exactly balance the flux and source terms so that the steady state is preserved – the same approach as in the case of the central-upwind scheme for the Saint-Venant system (Kurganov & Petrova, 2007)

2. $B \equiv \text{constant}, \quad p := \frac{g}{2}h^2\theta \equiv \text{constant}, \quad u \equiv 0$

   In a well-balanced scheme, the pressure should remain oscillation-free across the temperature jump and thus the steady state will be exactly preserved – the same approach as in the case of the interface tracking method for compressible multifluids (Chertock, Karni & Kurganov, 2008)

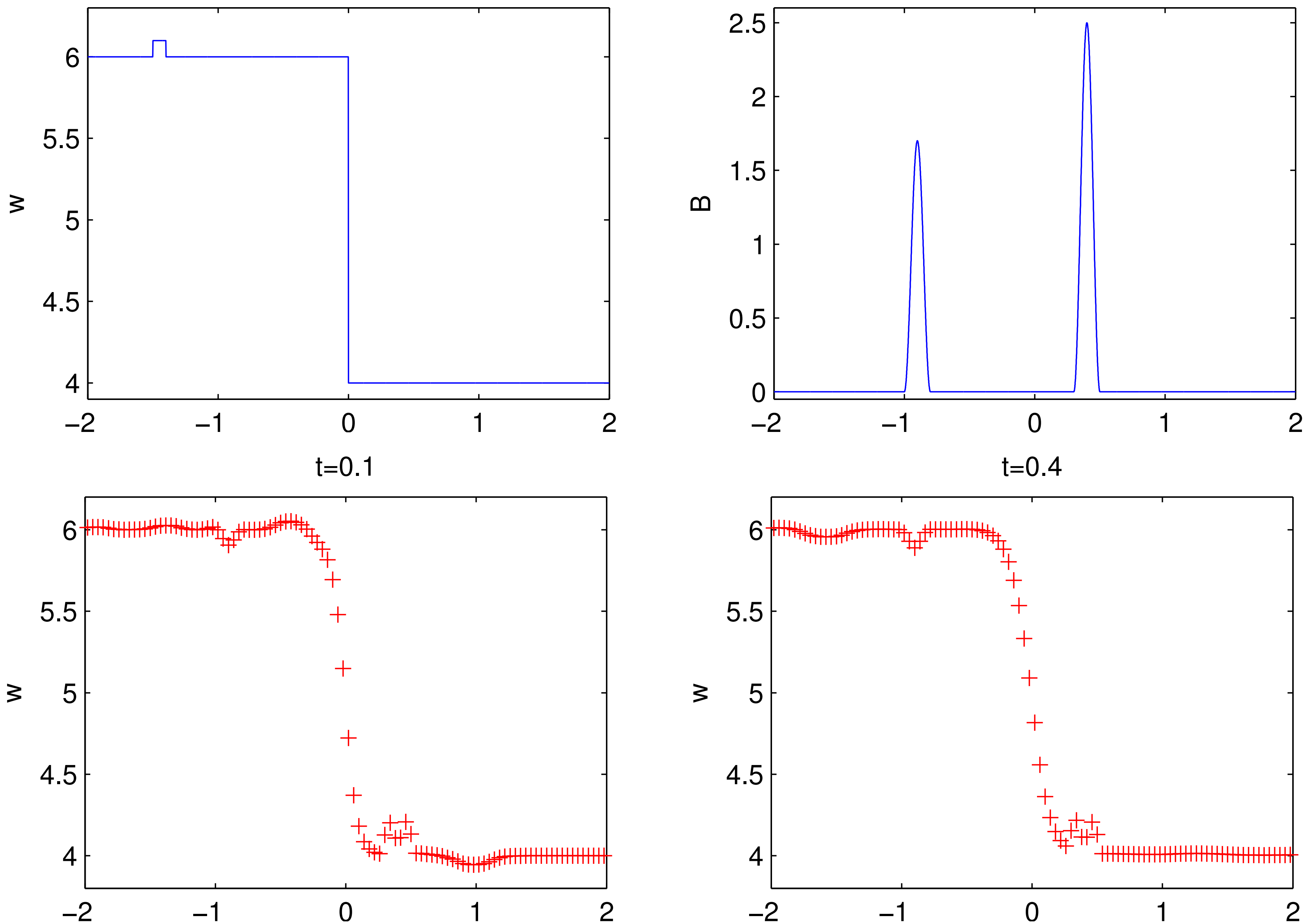
Small Perturbation of Steady-State – Numerical Example

$$B(x) = \begin{cases} 0.85(\cos(10\pi(x+0.9))+1), & -1.0 \le x \le -0.8, \\ 1.25(\cos(10\pi(x-0.4))+1), & 0.3 \le x \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$(w_s, u_s, \theta_s)^T(x) = \begin{cases} (6,0,4)^T, & x < 0 \\ (4,0,9)^T, & x > 0 \end{cases}$$

is a piecewise constant steady-state solution, which is in fact a combination of two "lake at rest" steady states of type I connected through the temperature jump, which corresponds to a steady state of type II.



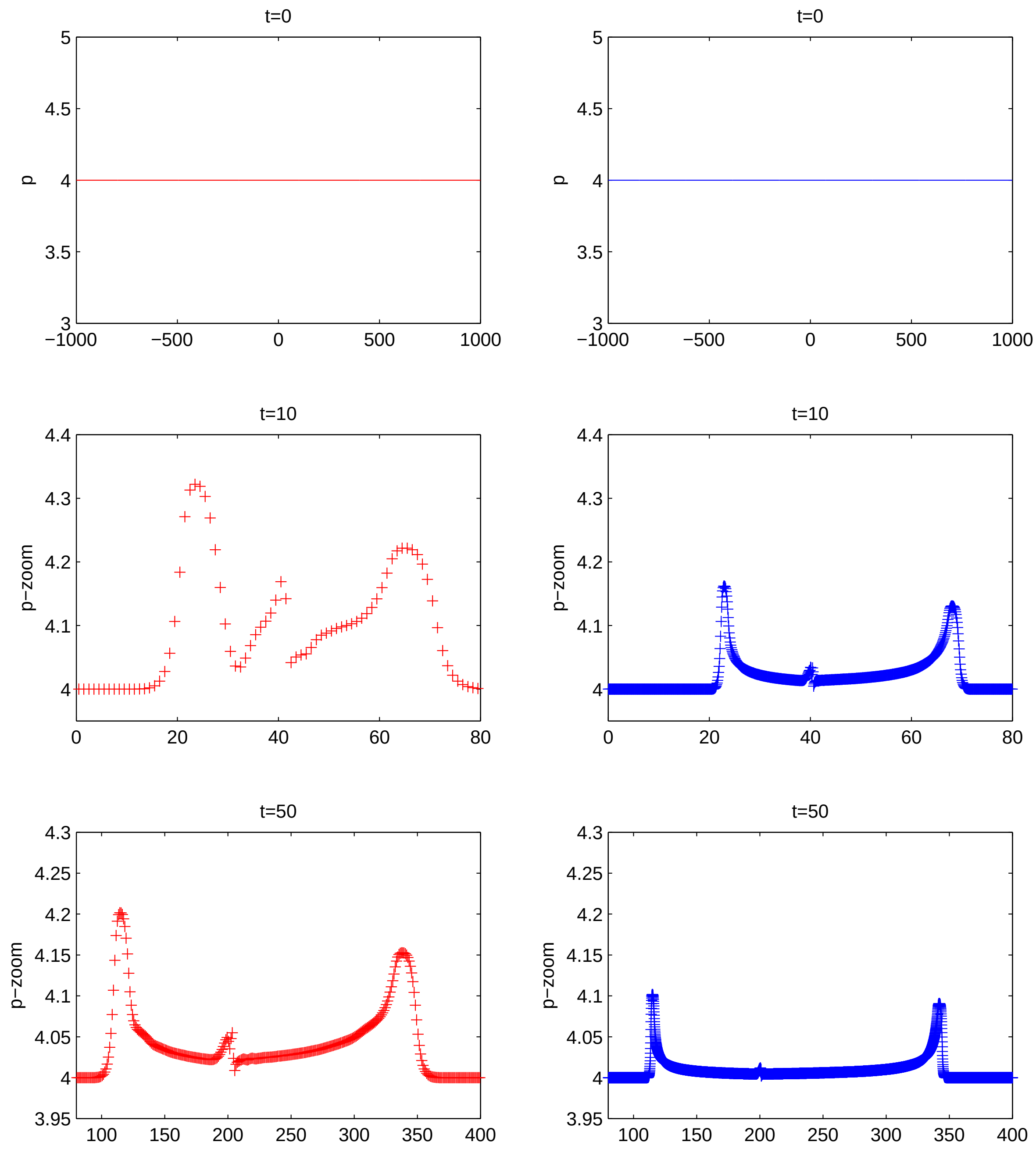
w
B
t=0.1
t=0.4

Pressure Oscillations – Numerical Example

- The initial condition is

$$(h(x,0), u(x,0), \theta(x,0)) = \begin{cases} (2\sqrt{2}, 4, 1), & x < 0 \\ (1, 4, 8), & x > 0 \end{cases}$$

- Notice that $p = 4g$ for all x, thus initially there is no pressure jump
- $g = 1$
- The bottom topography is flat



t=0
p
t=0
p
t=10
p−zoom
t=10
p−zoom
t=50
p−zoom
t=50
p−zoom

Switching to Equilibrium Variable

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{g}{2}h^2\theta\right)_x = -ghB_x \\ (h\theta)_t + (uh\theta)_x = 0 \end{cases}$$

$$\Updownarrow \qquad (h, hu, h\theta) \to (w := h + B, hu, h\theta)$$

$$\begin{cases} w_t + (hu)_x = 0 \\ (hu)_t + \left(\frac{(hu)^2}{w-B} + \frac{g}{2}\theta(w-B)^2\right)_x = -g\theta(w-B)B_x \\ (h\theta)_t + (hu\theta)_x = 0 \end{cases}$$

Semi-Discrete Central-Upwind Scheme

Central-upwind schemes were developed for multidimensional hyperbolic systems of conservation laws in 2000–2007 by Kurganov, Lin, Noelle, Petrova, Tadmor, ...

Central-upwind schemes are Godunov-type finite-volume projection-evolution methods:

- at each time level a solution is globally approximated by a piecewise polynomial function,

- which is then evolved to the new time level using the integral form of the system of hyperolic conservation/balance laws.

Semi-Discrete Central-Upwind Scheme

$$\frac{d}{dt}\overline{\mathbf{q}}_j = -\frac{\mathbf{H}_{j+\frac{1}{2}} - \mathbf{H}_{j-\frac{1}{2}}}{\Delta x} + \overline{\mathbf{S}}_j, \qquad \overline{\mathbf{q}} := (\overline{w}, \overline{hu}, \overline{h\theta})^T$$

$$\mathbf{H}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\mathbf{f}\left(\mathbf{q}^-_{j+\frac{1}{2}}, B_{j+\frac{1}{2}}\right) - a^-_{j+\frac{1}{2}}\mathbf{f}\left(\mathbf{q}^+_{j+\frac{1}{2}}, B_{j+\frac{1}{2}}\right)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\left[\mathbf{q}^+_{j+\frac{1}{2}} - \mathbf{q}^-_{j+\frac{1}{2}}\right]$$

- $\mathbf{q}^\pm_{j+\frac{1}{2}}$: right/left point values at $x_{j+\frac{1}{2}}$ of a piecewise polynomial reconstruction
- $a^\pm_{j+\frac{1}{2}}$: local right-/left-sided speeds
- $B_{j+\frac{1}{2}} = B(x_{j+\frac{1}{2}})$

Reconstruction of Equilibrium Variables

- To preserve the first steady state, we reconstruct the equilibrium variables (θ, hu, w) and obtain their point values at $x_{j+\frac{1}{2}}$:

$$\theta^-_{j+\frac{1}{2}} = \overline{\theta}_j + \frac{\Delta x}{2}(\theta_x)_j, \qquad \theta^+_{j+\frac{1}{2}} = \overline{\theta}_{j+1} - \frac{\Delta x}{2}(\theta_x)_j$$

$$(hu)^-_{j+\frac{1}{2}} = (\overline{hu})_j + \frac{\Delta x}{2}((hu)_x)_j, \qquad (hu)^+_{j+\frac{1}{2}} = (\overline{hu})_{j+1} - \frac{\Delta x}{2}((hu)_x)_j$$

$$w^-_{j+\frac{1}{2}} = \overline{w}_j + \frac{\Delta x}{2}(w_x)_j, \qquad w^+_{j+\frac{1}{2}} = \overline{w}_{j+1} - \frac{\Delta x}{2}(w_x)_j$$

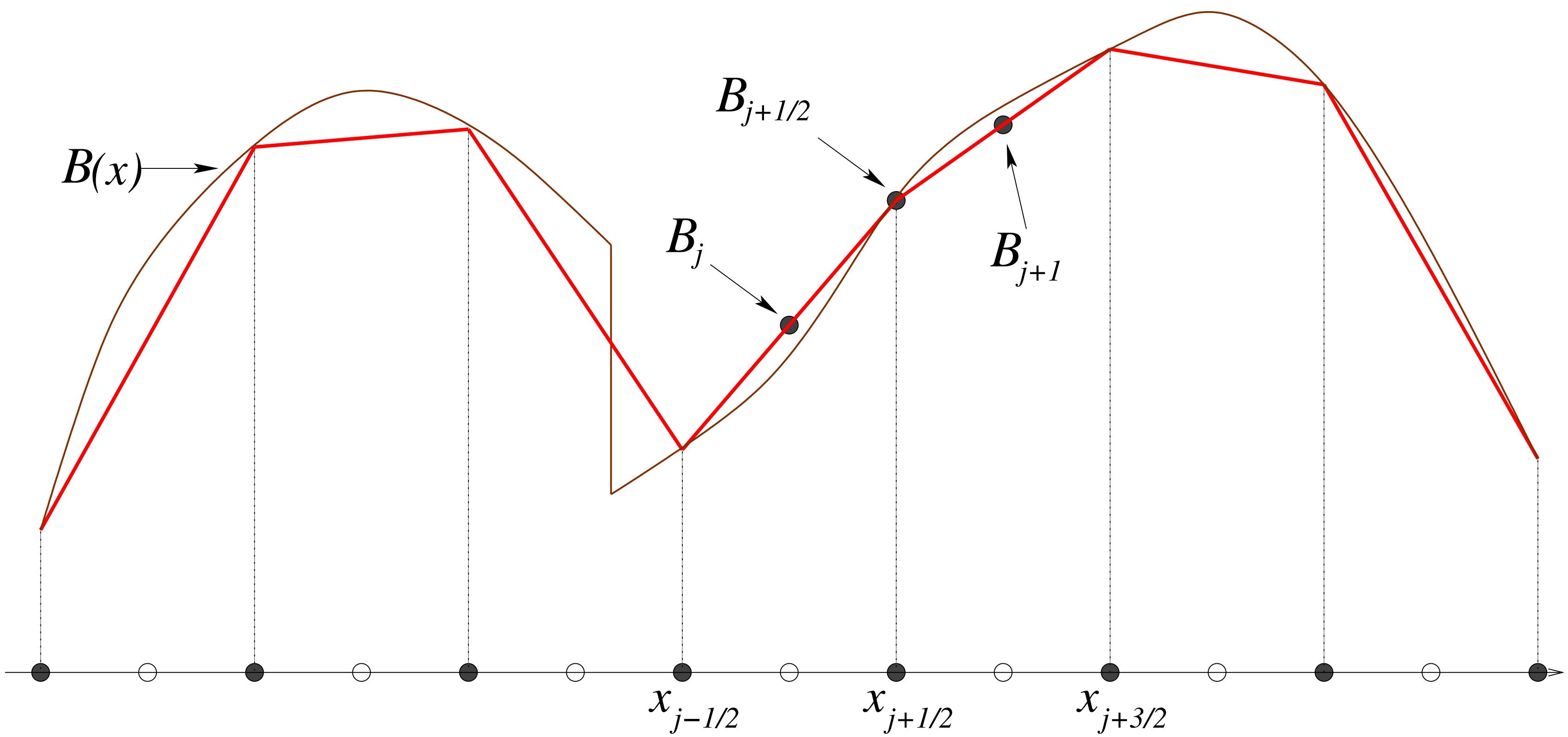
- The point values of h, u and $h\theta$ are then computed as follows:

$$h^\pm_{j+\frac{1}{2}} = w^\pm_{j+\frac{1}{2}} - B_{j+\frac{1}{2}}, \quad u^\pm_{j+\frac{1}{2}} = \frac{(hu)^\pm_{j+\frac{1}{2}}}{h^\pm_{j+\frac{1}{2}}}, \quad (h\theta)^\pm_{j+\frac{1}{2}} = h^\pm_{j+\frac{1}{2}}\theta^\pm_{j+\frac{1}{2}}$$

Preservation of Positivity

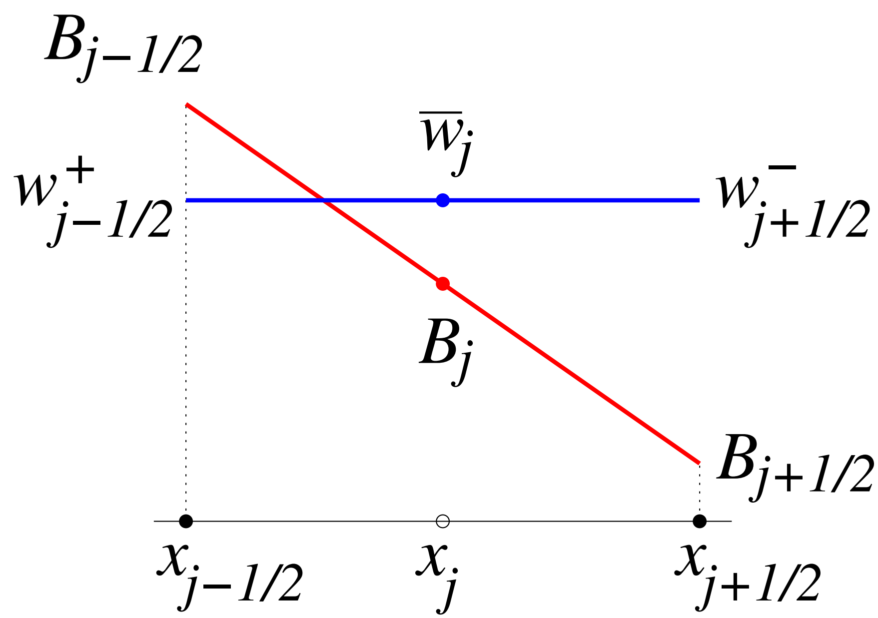
$$h^{\pm}_{j+\frac{1}{2}} = w^{\pm}_{j+\frac{1}{2}} - B_{j+\frac{1}{2}}$$

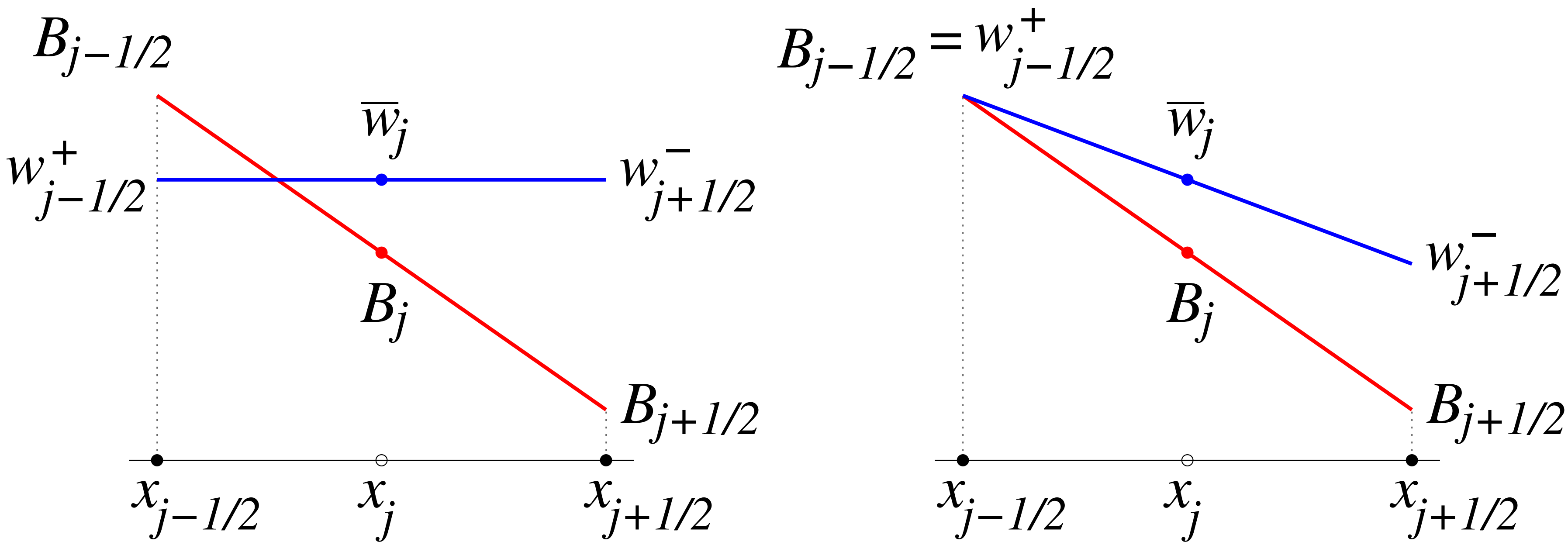
Step 1: Piecewise linear reconstruction of the bottom



Step 2: Positivity preserving reconstruction of w

$$h^{\pm}_{j+\frac{1}{2}} = w^{\pm}_{j+\frac{1}{2}} - B_{j+\frac{1}{2}}$$





if $w^-_{j+\frac{1}{2}} < B_{j+\frac{1}{2}}$ then take $w^-_{j+\frac{1}{2}} = B_{j+\frac{1}{2}},\ w^+_{j-\frac{1}{2}} = 2\overline{w}_j - B_{j+\frac{1}{2}}$

if $w^+_{j-\frac{1}{2}} < B_{j-\frac{1}{2}}$ then take $w^-_{j+\frac{1}{2}} = 2\overline{w}_j - B_{j-\frac{1}{2}},\ w^+_{j-\frac{1}{2}} = B_{j-\frac{1}{2}}$

We have proved that if an SSP ODE solver is used, then

$$\bar{h}_j^{n+1} = \alpha^-_{j-\frac{1}{2}} h^-_{j-\frac{1}{2}} + \alpha^+_{j-\frac{1}{2}} h^+_{j-\frac{1}{2}} + \alpha^-_{j+\frac{1}{2}} h^-_{j+\frac{1}{2}} + \alpha^+_{j+\frac{1}{2}} h^+_{j+\frac{1}{2}}$$

and

$$\overline{h\theta_j}^{n+1} = \beta^-_{j-\frac{1}{2}} h^-_{j-\frac{1}{2}} \theta^-_{j-\frac{1}{2}} + \beta^+_{j-\frac{1}{2}} h^+_{j-\frac{1}{2}} \theta^+_{j-\frac{1}{2}} + \beta^-_{j+\frac{1}{2}} h^-_{j+\frac{1}{2}} \theta^-_{j+\frac{1}{2}} + \beta^+_{j+\frac{1}{2}} h^+_{j+\frac{1}{2}} \theta^+_{j+\frac{1}{2}}$$

where the coefficients $\alpha^{\pm}_{j\pm\frac{1}{2}} > 0$ and $\beta^{\pm}_{j\pm\frac{1}{2}} > 0$ provided an appropriate CFL condition is satisfied.

This guarantees positivity of both h and $\theta = \dfrac{h\theta}{h}$.

Approximation of the Source Term

Substitute the "lake at rest" values $\theta^{\pm}_{j+\frac{1}{2}} \equiv \hat{\theta}, (hu)^{\pm}_{j+\frac{1}{2}} \equiv 0, w^{\pm}_{j+\frac{1}{2}} \equiv \hat{w}$ into the scheme $\Rightarrow$ the numerical fluxes $\mathbf{H}_{j+\frac{1}{2}}$ reduce to:

$$\left(H^{(1)}_{j+\frac{1}{2}}, H^{(2)}_{j+\frac{1}{2}}, H^{(3)}\right)^T = \left(0, \frac{g}{2}\hat{\theta}(\hat{w} - B_{j+\frac{1}{2}})^2, 0\right)^T$$

Thus

$$-\frac{H^{(2)}_{j+\frac{1}{2}} - H^{(2)}_{j-\frac{1}{2}}}{\Delta x} = -\frac{g\hat{\theta}}{2\Delta x}\left[(\hat{w} - B_{j+\frac{1}{2}})^2 - (\hat{w} - B_{j-\frac{1}{2}})^2\right]$$

$$= \frac{g\hat{\theta}}{2}\left(\hat{w} - B_{j+\frac{1}{2}} + \hat{w} - B_{j-\frac{1}{2}}\right)\frac{B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}}}{\Delta x}$$

The well-balanced quadrature:

$$\overline{S}^{(2)}_j = -\frac{g}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \theta(w - B)B_x\,dx$$

$$\approx -\frac{g}{2}\left[\theta^-_{j+\frac{1}{2}}\left(w^-_{j+\frac{1}{2}} - B_{j+\frac{1}{2}}\right) + \theta^+_{j-\frac{1}{2}}\left(w^+_{j-\frac{1}{2}} - B_{j-\frac{1}{2}}\right)\right]\frac{B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}}}{\Delta x}$$

Interface Tracking Method

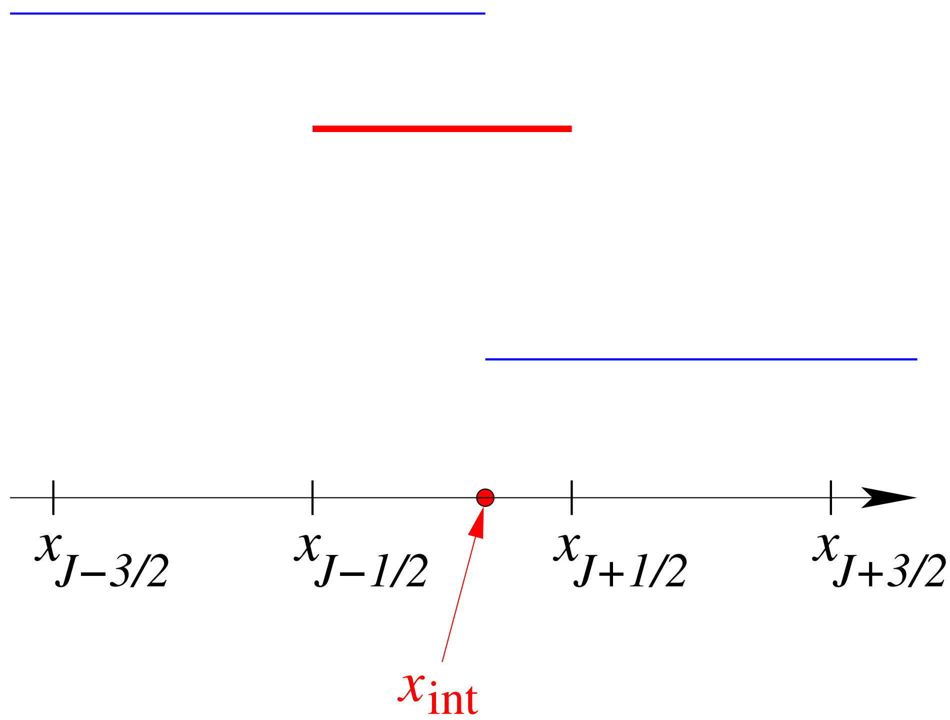
The Ripa system is a "toy" model for cold/warm currents in the ocean. In the oceanographic scales, the currents have sharp boundaries across which the temperature jumps, while in the rest of the ocean, the temperature varies smoothly.

To track the temperature jumps, we use the level set approach:

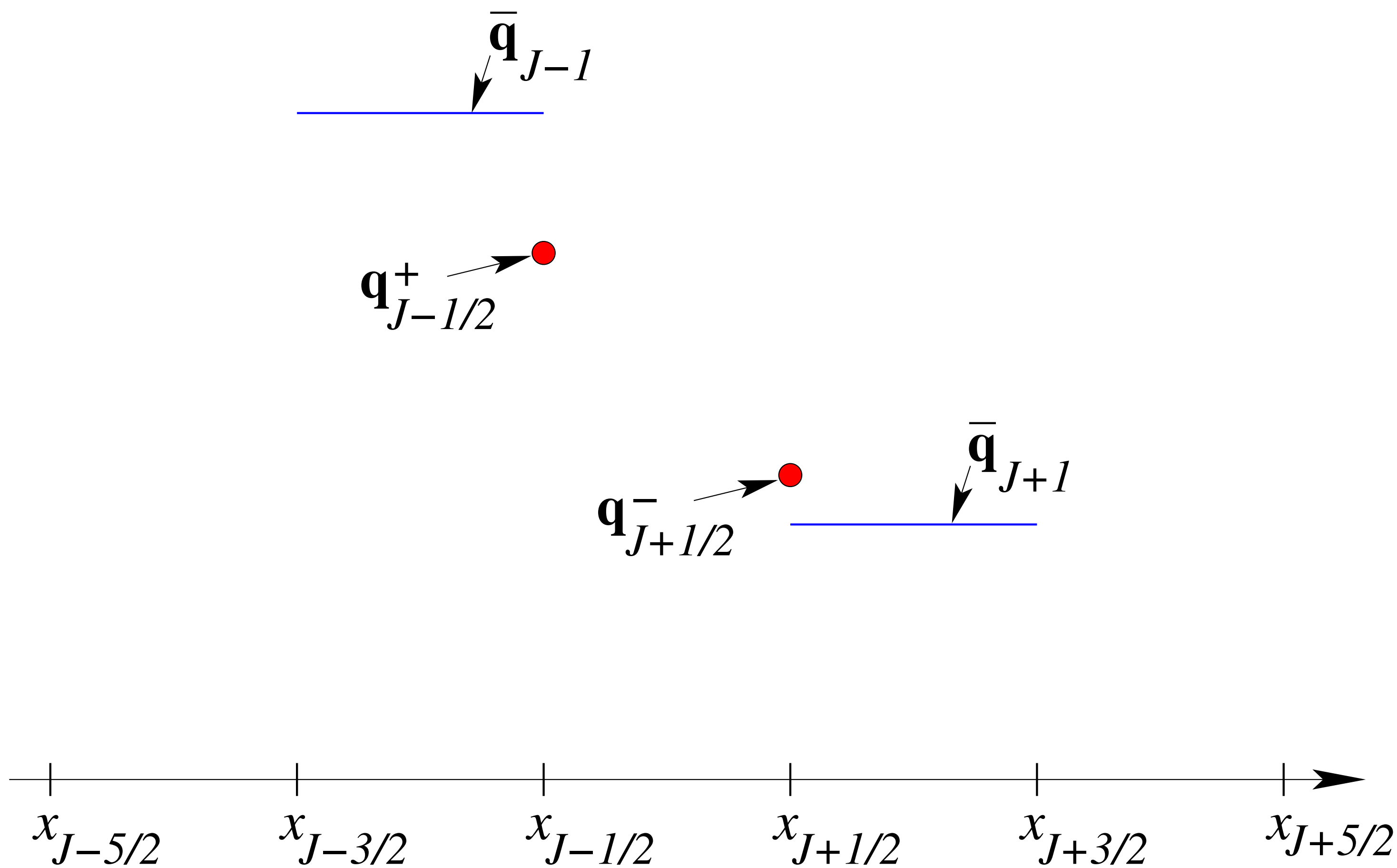
$$\begin{cases} w_t + (hu)_x = 0 \\ (hu)_t + \left(\dfrac{(hu)^2}{w-B} + \dfrac{g}{2}\theta(w-B)^2 \right)_x = -g\theta(w-B)B_x \\ (h\theta)_t + (hu\theta)_x = 0 \\ (h\phi)_t + (hu\phi)_x = 0 \end{cases}$$

- $x_{\mathsf{int}} = x_{\mathsf{int}}(t)$: position of the interface at time t

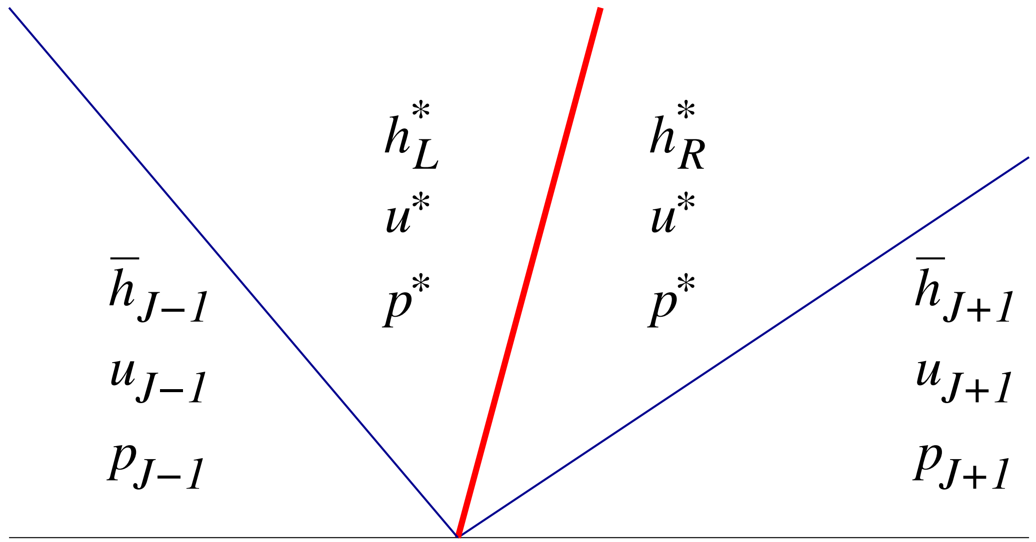
- Assume that it is located in cell J, i.e., $x_{J-\frac{1}{2}} \le x_{\mathsf{int}} \le x_{J+\frac{1}{2}}$



- Cell J is a "mixed" cell: the information (cell averages) there is NOT reliable!



- The point values $\mathbf{q}^{+}_{J-\frac{1}{2}}$ and $\mathbf{q}^{-}_{J+\frac{1}{2}}$ are to be obtained by interpolating between $\overline{\mathbf{q}}_{J-1}$ and $\overline{\mathbf{q}}_{J+1}$ in the phase space (via an approximate solution of the Riemann problem for the linearized system)



- We obtain point values $\mathbf{q}^-_{J+\frac{1}{2}}$ and $\mathbf{q}^+_{J+\frac{1}{2}}$ by connecting reliable neighboring cell averages $\overline{\mathbf{q}}_{J-1}$ and $\overline{\mathbf{q}}_{J+1}$ in the phase space:

if $h_L^* > 0,\ p_L^* > 0,\ u^* - c_L^* < 0$

then

$$\mathbf{q}^+_{J-\frac{1}{2}} = \mathbf{q}^*_L,$$

otherwise,

$$\mathbf{q}^+_{J-\frac{1}{2}} = \overline{\mathbf{q}}_{J-1},$$

if $h_R^* > 0,\ p_R^* > 0,\ u^* + c_R^* > 0$

then

$$\mathbf{q}^-_{J+\frac{1}{2}} = \mathbf{q}^*_R,$$

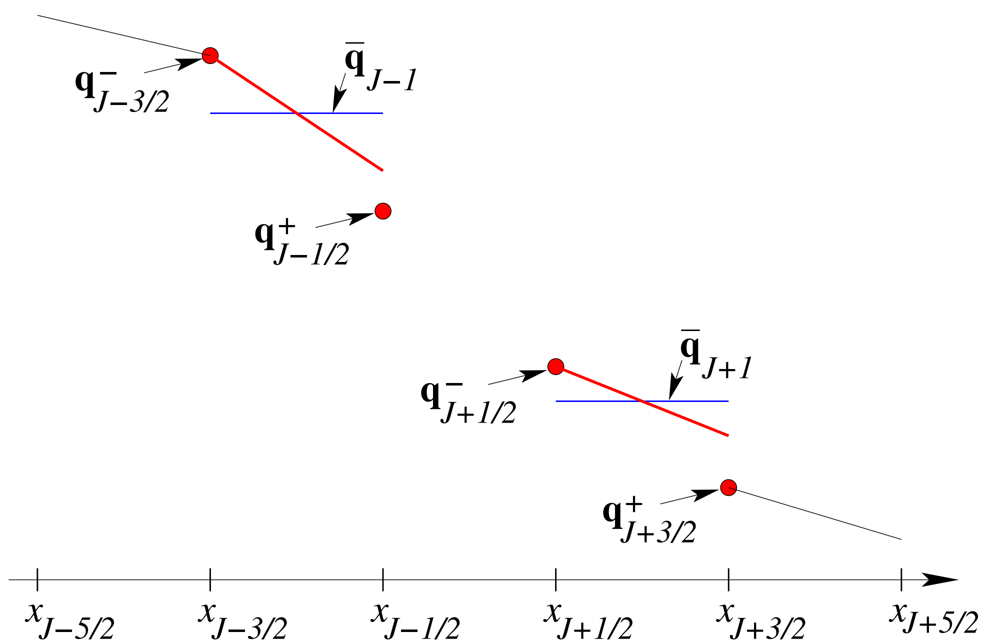
otherwise

$$\mathbf{q}^-_{J+\frac{1}{2}} = \overline{\mathbf{q}}_{J+1}.$$

- The point values $\mathbf{q}^+_{J-\frac{1}{2}}$ and $\mathbf{q}^-_{J+\frac{1}{2}}$ are used to compute the numerical derivatives in the neighboring cells:

$$(\mathbf{q}_x)_{J-1} = \text{minmod}\left(\frac{\overline{\mathbf{q}}_{J-1} - \mathbf{q}^-_{J-\frac{3}{2}}}{\Delta x/2}, \frac{\mathbf{q}^+_{J-\frac{1}{2}} - \overline{\mathbf{q}}_{J-1}}{\Delta x/2}\right)$$

$$(\mathbf{q}_x)_{J+1} = \text{minmod}\left(\frac{\overline{\mathbf{q}}_{J+1} - \mathbf{q}^-_{J+\frac{1}{2}}}{\Delta x/2}, \frac{\mathbf{q}^+_{J+\frac{3}{2}} - \overline{\mathbf{q}}_{J+1}}{\Delta x/2}\right)$$



- If $x_{\text{int}}(t+\Delta t)$ remains in cell J, we proceed to the next time step.

  Otherwise

`if` $x_{\text{int}}(t+\Delta t) \in I_{J+1}$ `then`

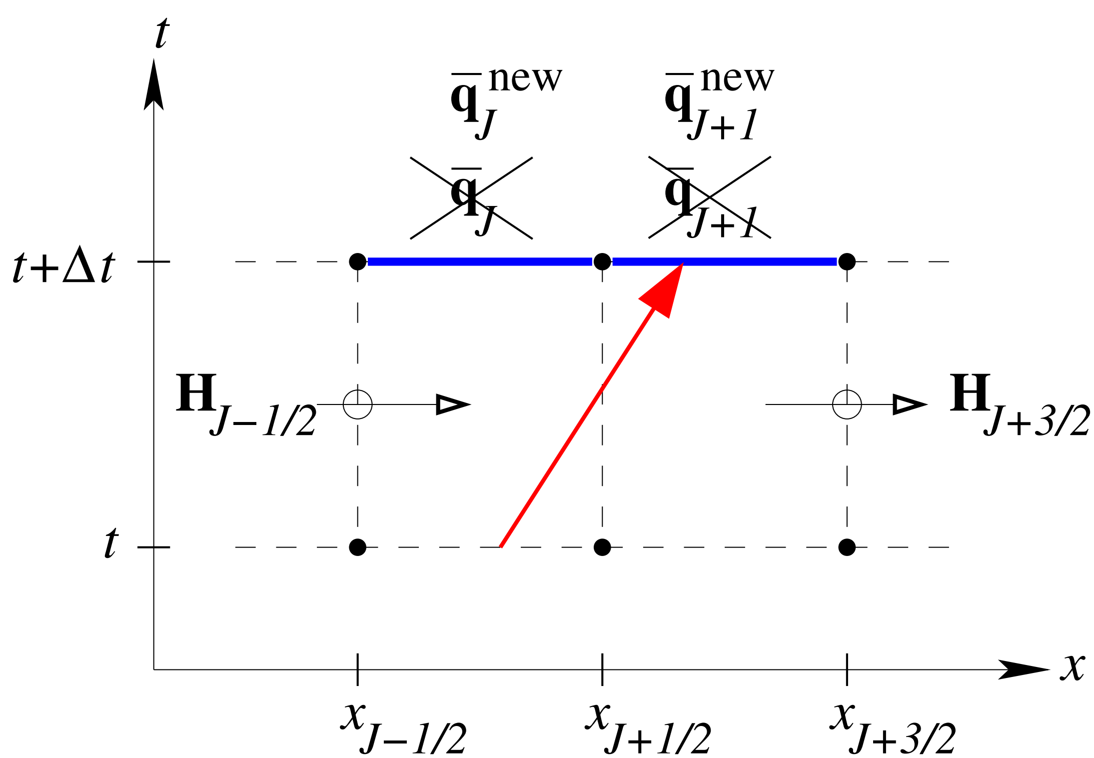
$$\overline{\mathbf{q}}_J^{\text{new}} = \mathbf{q}_L^*$$

$$\overline{\mathbf{q}}_{J+1}^{\text{new}} = \overline{\mathbf{q}}_J + \overline{\mathbf{q}}_{J+1} - \overline{\mathbf{q}}_L^*$$

`if` $x_{\text{int}}(t+\Delta t) \in I_{J-1}$ `then`

$$\overline{\mathbf{q}}_J^{\text{new}} = \mathbf{q}_R^*$$

$$\overline{\mathbf{q}}_{J-1}^{\text{new}} = \overline{\mathbf{q}}_{J-1} + \overline{\mathbf{q}}_J - \overline{\mathbf{q}}_R^*$$



Approximate Riemann Problem Solver

Step 1: Switching to primitive variables

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \dfrac{g}{2}h^2\theta\right)_x = -gh\theta B_x \\ (h\theta)_t + (uh\theta)_x = 0 \end{cases}$$

$$\Updownarrow \qquad (h, hu, h\theta) \to (h, u, p := \frac{g}{2}h^2\theta, B)$$

$$\begin{cases} h_t + h_x u + h u_x = 0 \\ u_t + u u_x + \frac{1}{h} p_x + g\theta B_x = 0 \\ p_t + u p_x + 2 p u_x = 0 \\ B_t = 0 \end{cases} \qquad (1)$$

Step 2: Linearization

$$\begin{pmatrix} h \\ u \\ p \\ B \end{pmatrix}_t + \begin{pmatrix} \hat{u} & \hat{h} & 0 & 0 \\ 0 & \hat{u} & \frac{1}{\hat{h}} & g\theta \\ 0 & 2\hat{p} & \hat{u} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ u \\ p \\ B \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which is obtained by linearizing the Jacobian matrix of (1) about

$$\hat{h} = \frac{1}{2}(h_L + h_R), \quad \hat{u} = \frac{1}{2}(u_L + u_R), \quad \hat{p} = \frac{1}{2}(p_L + p_R), \quad \hat{\theta} = \frac{1}{2}(\theta_L + \theta_R)$$

- *has the same eigenvalues as the conservative system*
- *is not diagnolizable*
- *the system is not strictly hyperbolic*

Step 3: Operator splitting

$$\mathbf{U} := (h, u, p, B)^T, \quad \mathcal{M} := \begin{pmatrix} 0 & \hat{h} & 0 & 0 \\ 0 & 0 & \frac{1}{\hat{h}} & g\hat{\theta} \\ 0 & 2\hat{p} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L} := \begin{pmatrix} \hat{u} & 0 & 0 & 0 \\ 0 & \hat{u} & 0 & 0 \\ 0 & 0 & \hat{u} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} \mathbf{U}_t + \mathcal{M}\mathbf{U}_x + \mathcal{L}\mathbf{U}_x = \mathbf{0} \\ \mathbf{U}(x,0) = \begin{cases} \mathbf{U}_L := (h_L, u_L, p_L, B_L)^T, & x < 0 \\ \mathbf{U}_R := (h_R, u_R, p_R, B_R)^T, & x > 0 \end{cases} \end{cases} \tag{2}$$

We split the system (2) into the two simpler systems:

$$\begin{cases} \mathbf{U}_t + \mathcal{M}\mathbf{U}_x = \mathbf{0} & \text{with the solution operator } S_{\mathcal{M}}(t - \cdot) \\ \mathbf{U}_t + \mathcal{L}\mathbf{U}_x = \mathbf{0} & \text{with the solution operator } S_{\mathcal{L}}(t - \cdot) \end{cases}$$

The splitting approach is then based on

$$\mathbf{U}(t + \Delta t) \approx S_{\mathcal{L}}(\Delta t) S_{\mathcal{M}}(\Delta t) \mathbf{U}(t)$$

$$\mathbf{U}(t + \Delta t) \approx S_{\mathcal{L}}(\Delta t) S_{\mathcal{M}}(\Delta t) \mathbf{U}(t)$$

is exact iff

$$\mathcal{M}\mathcal{L} - \mathcal{L}\mathcal{M} = 0$$

Here,

$$\mathcal{M}\mathcal{L} - \mathcal{L}\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g\hat{\theta}\hat{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

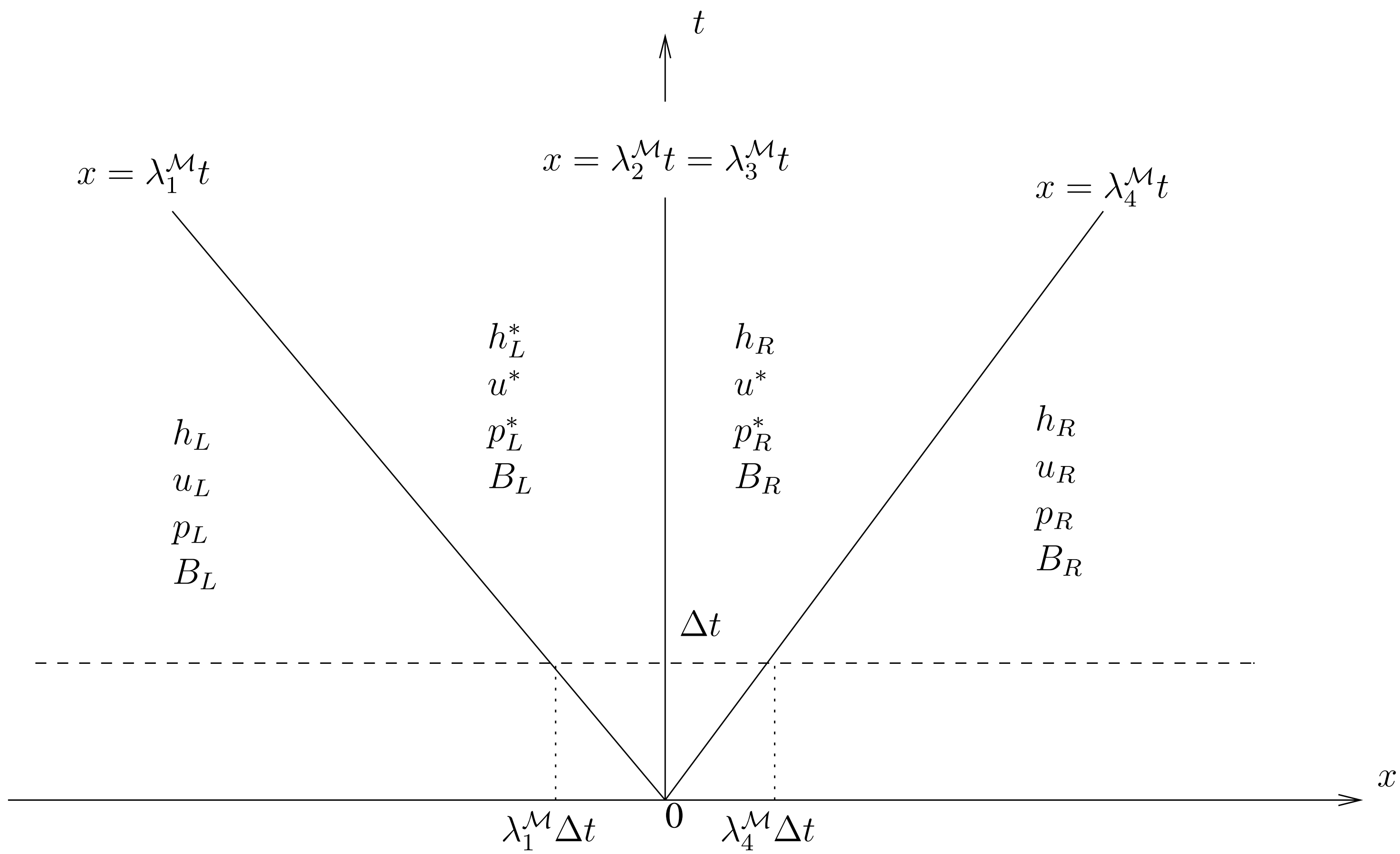
which vanishes when $\hat{u} \equiv 0$.

In other cases, the splitting method is not exact, but it is still quite accurate especially for small Δt.

Step 3a: First splitting step

Unlike the matrix $\mathcal{M}+\mathcal{L}$, the matrix $\mathcal{M}$ has a complete eigensystem.

Therefore, the system
$$\begin{cases} \mathbf{U}_t + \mathcal{M}\mathbf{U}_x = \mathbf{0} \\ \mathbf{U}(x,0) = \begin{cases} \mathbf{U}_L := (h_L, u_L, p_L, B_L)^T, & x < 0 \\ \mathbf{U}_R := (h_R, u_R, p_R, B_R)^T, & x > 0 \end{cases} \end{cases}$$
can be easily solved:



Step 3b: Second splitting step

The system $\mathbf{U}_t + \mathcal{L}\mathbf{U}_x = \mathbf{0}$ is in fact decoupled.

We thus need to solve 3 linear advection equations

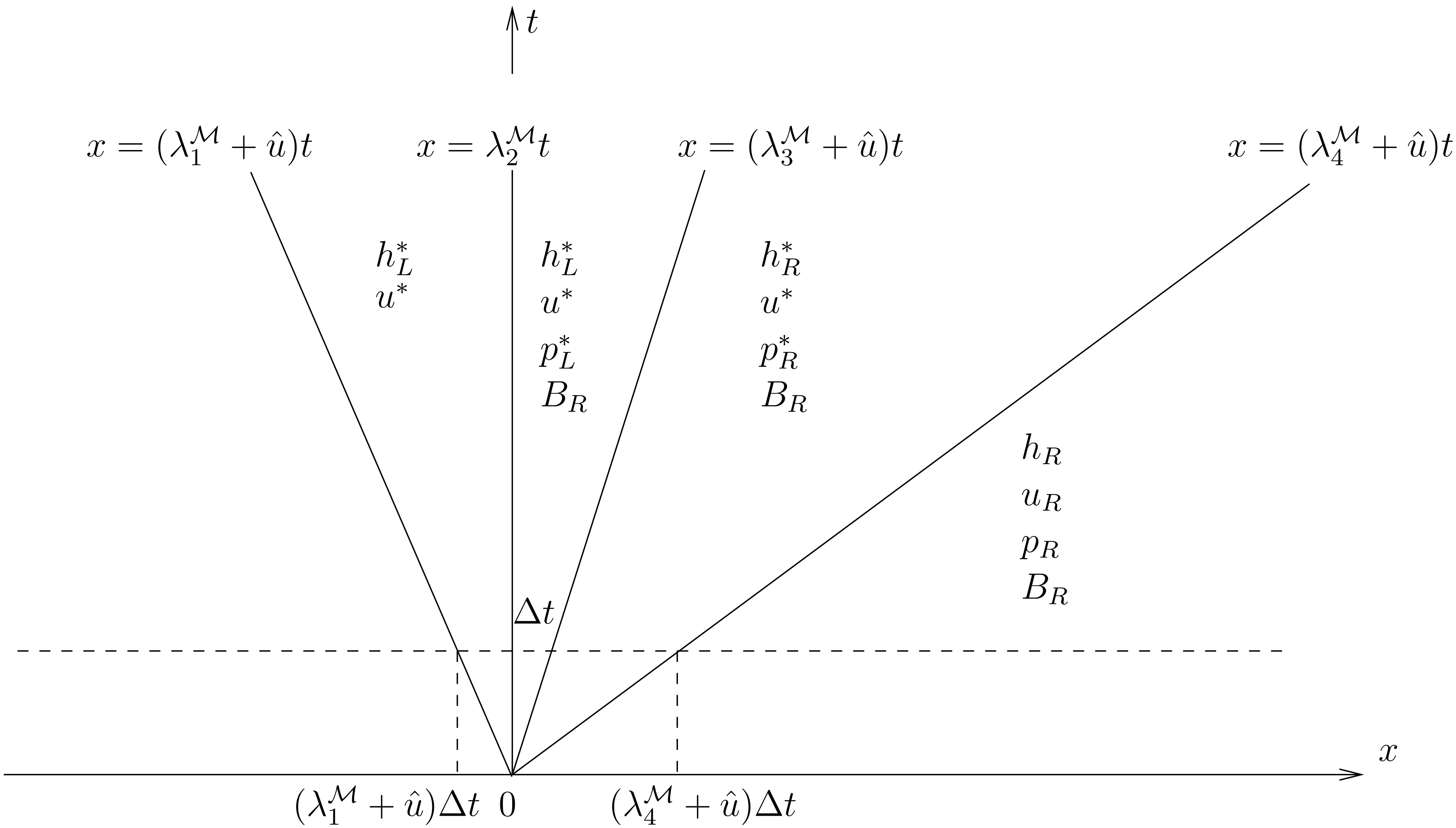
$$\begin{cases} h_t + \hat{u} h_x = 0 \\ u_t + \hat{u} u_x = 0 \\ p_t + \hat{u} p_x = 0 \end{cases}$$

subject to the piecewise constant initial data

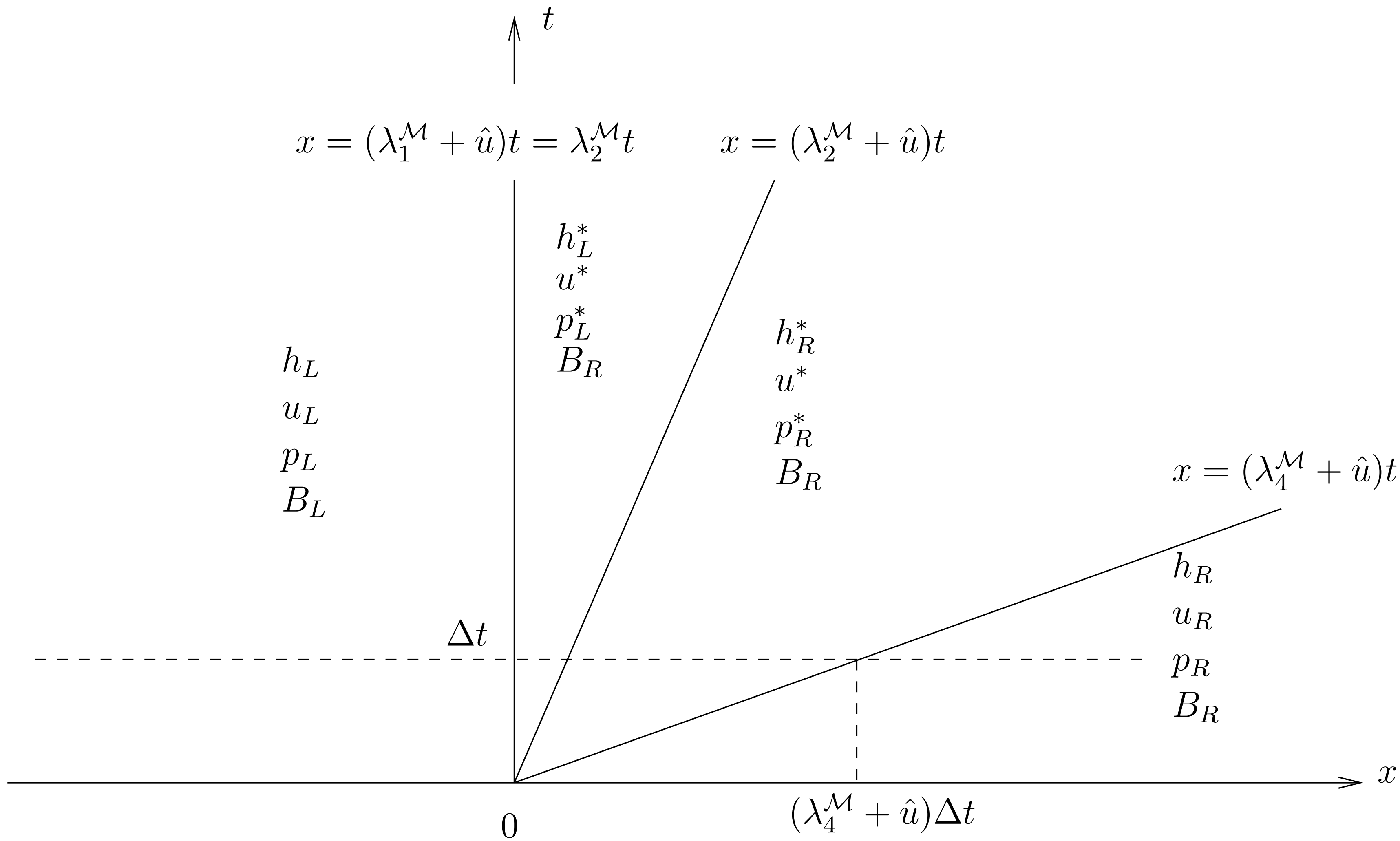
$$(h,u,p)^T(x,0) = \begin{cases} (h_L, u_L, p_L)^T, & x < \lambda_1^{\mathcal{M}} \Delta t \\ (h_L^*, u^*, p_L^*)^T, & \lambda_1^{\mathcal{M}} \Delta t < x < 0 \\ (h_R^*, u^*, p_R^*)^T, & 0 < x < \lambda_4^{\mathcal{M}} \Delta t \\ (h_R, u_R, p_R)^T, & x > \lambda_4^{\mathcal{M}} \Delta t \end{cases}$$

Assume that $\hat{u} > 0$. There are 3 cases depending on the sign of $\lambda_1^{\mathcal{M}} + \hat{u}$.

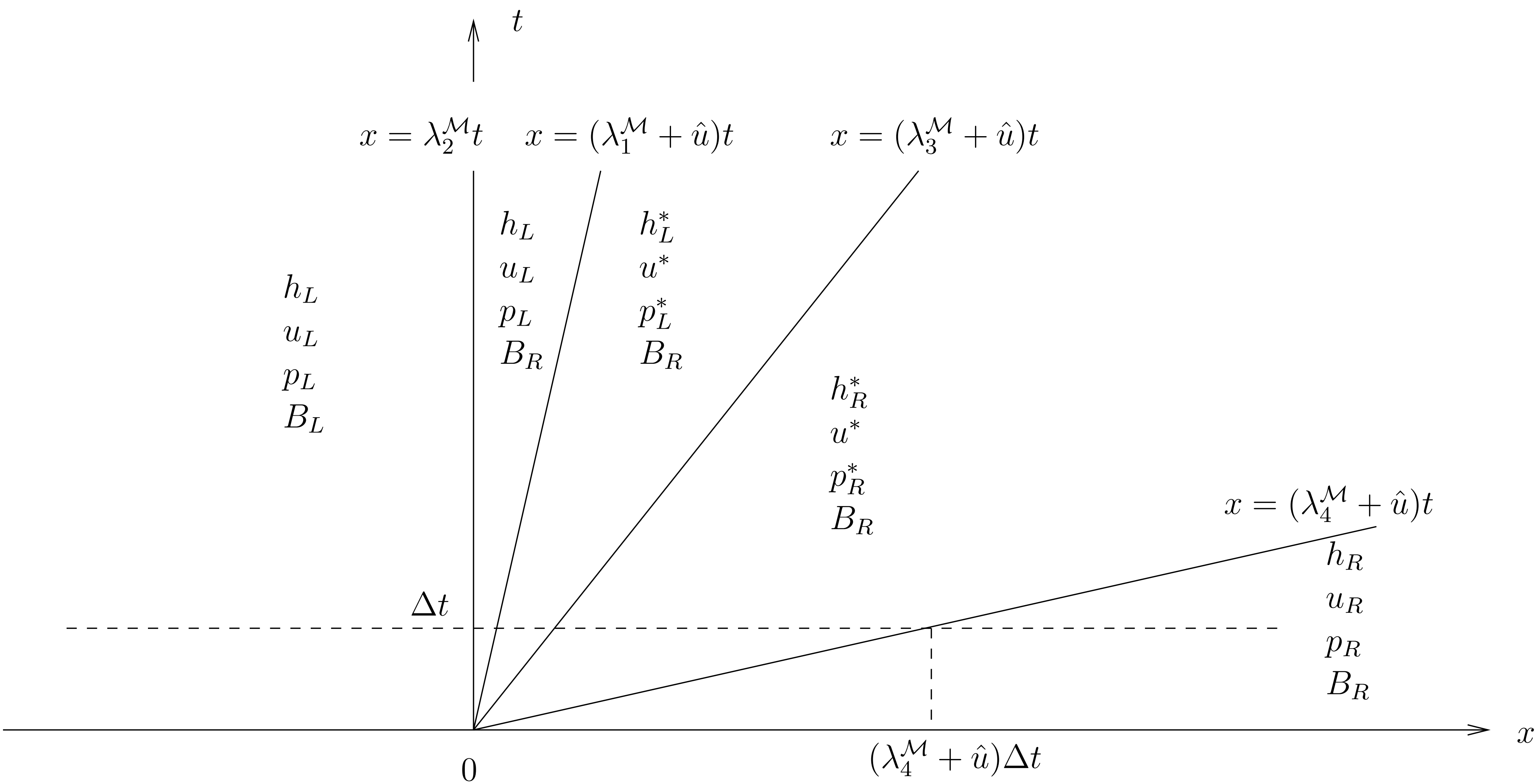
Case 1: $\lambda_1^{\mathcal{M}} + \hat{u} < 0$



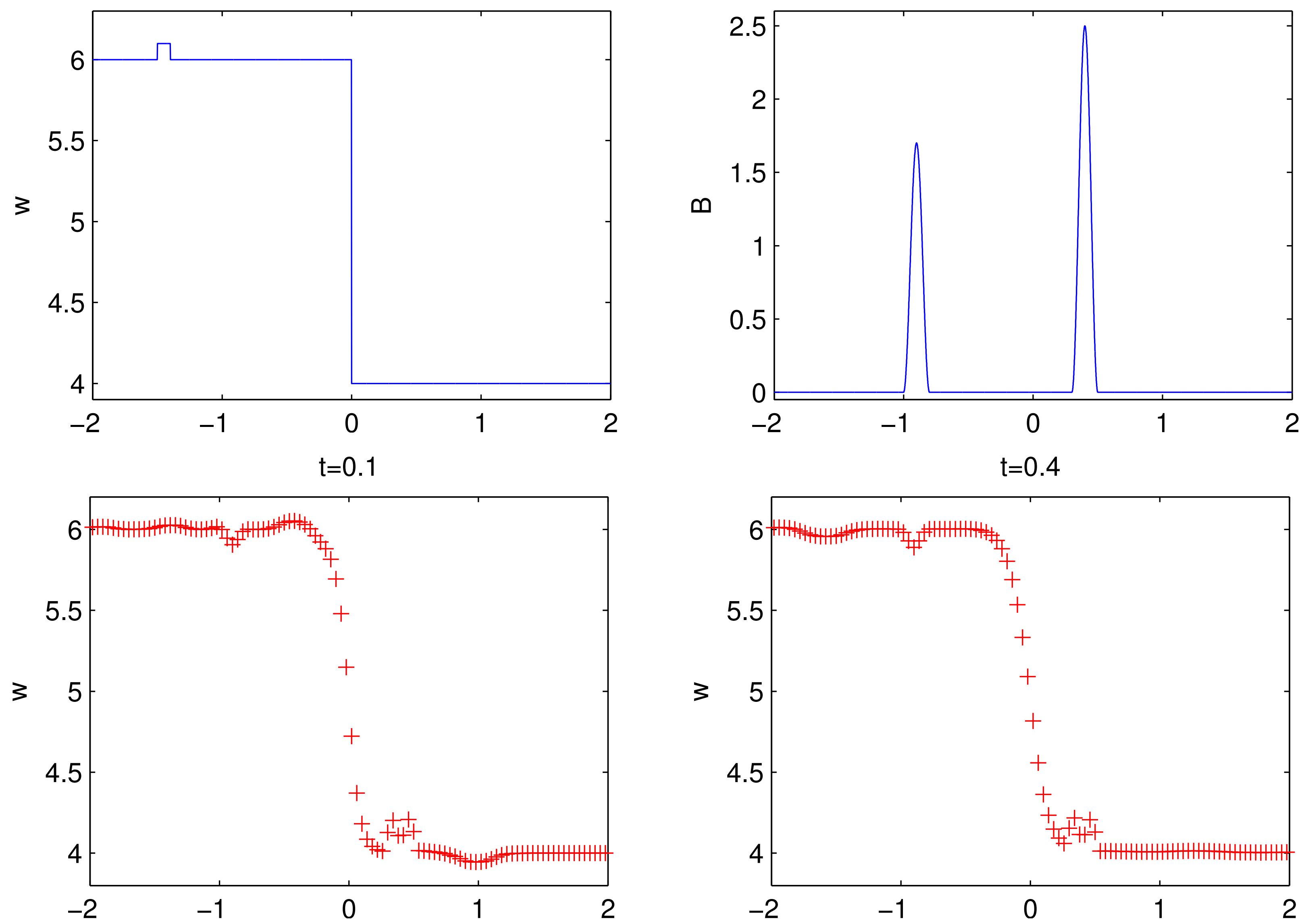
Case 2: $\lambda_1^{\mathcal{M}} + \hat{u} = 0$



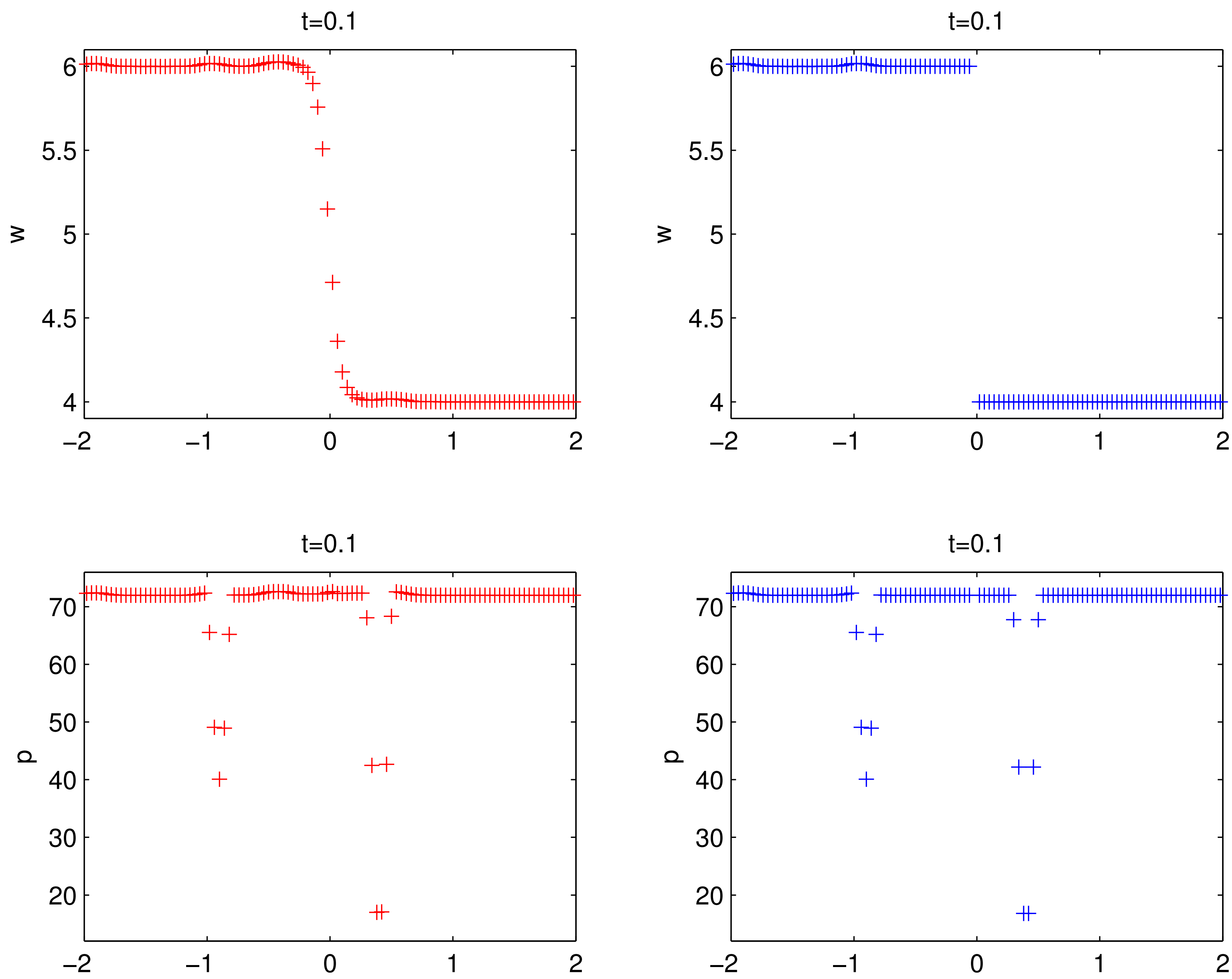
Case 3: $\lambda_1^{\mathcal{M}} + \hat{u} > 0$



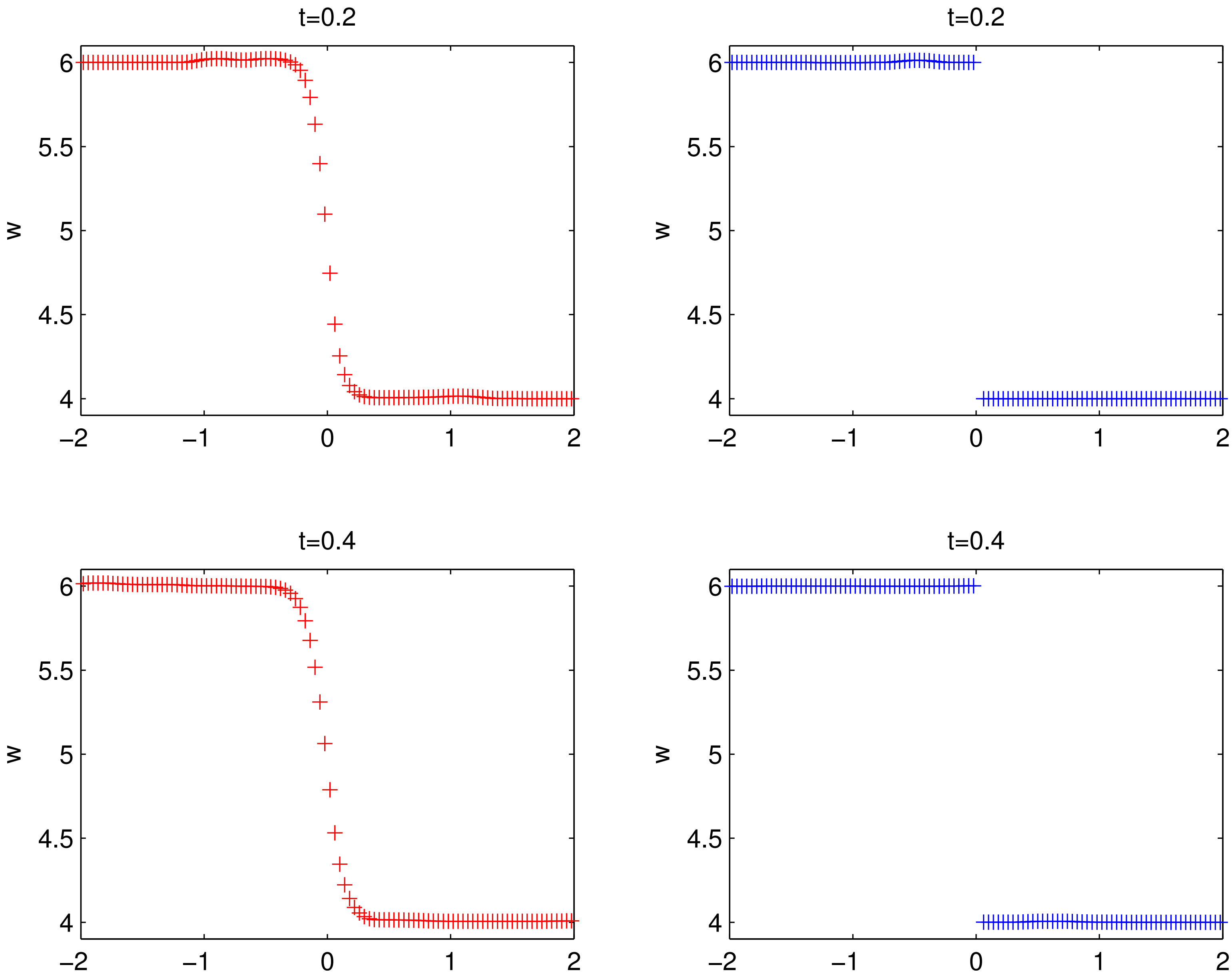
Small Perturbation of a Steady-State I – Revised



Well-Balanced vs. Well-Balanced-IT



Well-Balanced vs. Well-Balanced-IT



Pressure Oscillations – Revised

We redo the "pressure oscillation" test with the new central-upwind scheme with interface tracking

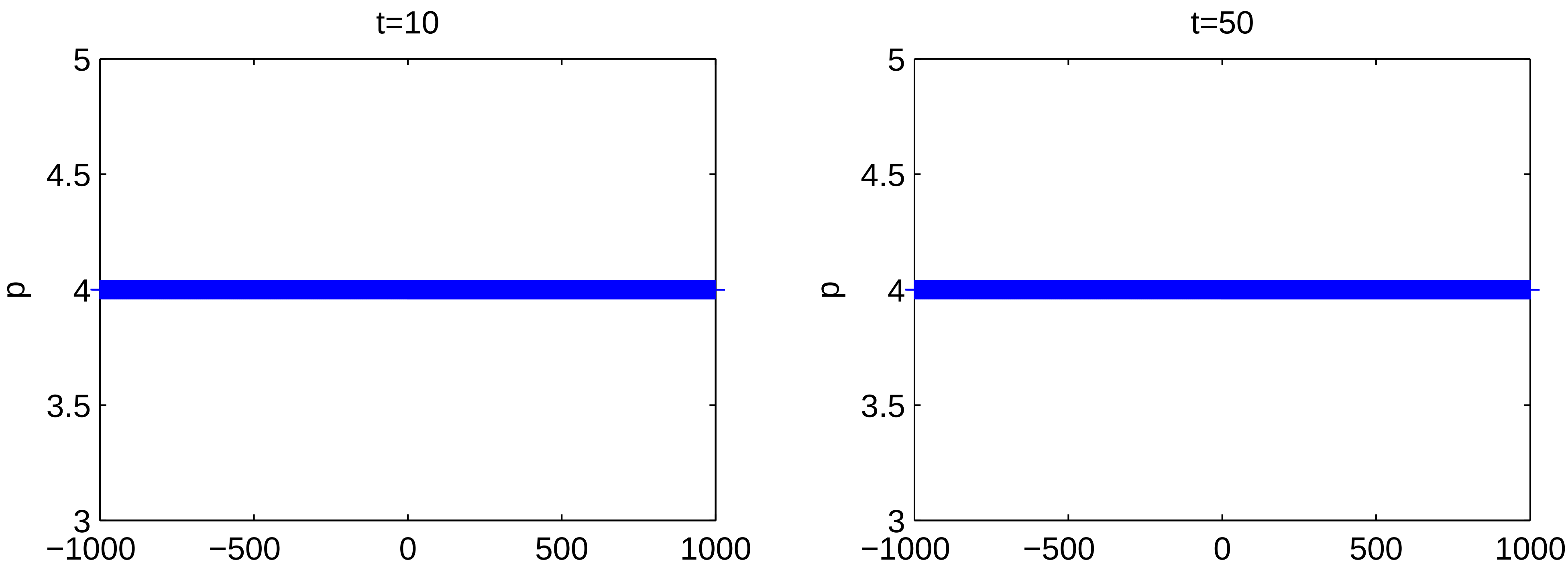
- The initial condition is

$$\begin{cases} u_l = 4,\ \theta_l = 1,\ h_l = 2\sqrt{2} & \text{if } x < 0 \\ u_r = 4,\ \theta_r = 8,\ h_r = 1 & \text{if } x > 0 \end{cases}$$

- Notice that $p = 4g$ for all x, thus initially there is no pressure jump

- $g = 1$

- The bottom topography is flat


t=10
t=50
p
5
4.5
4
3.5
3
−1000
−500
0
500
1000

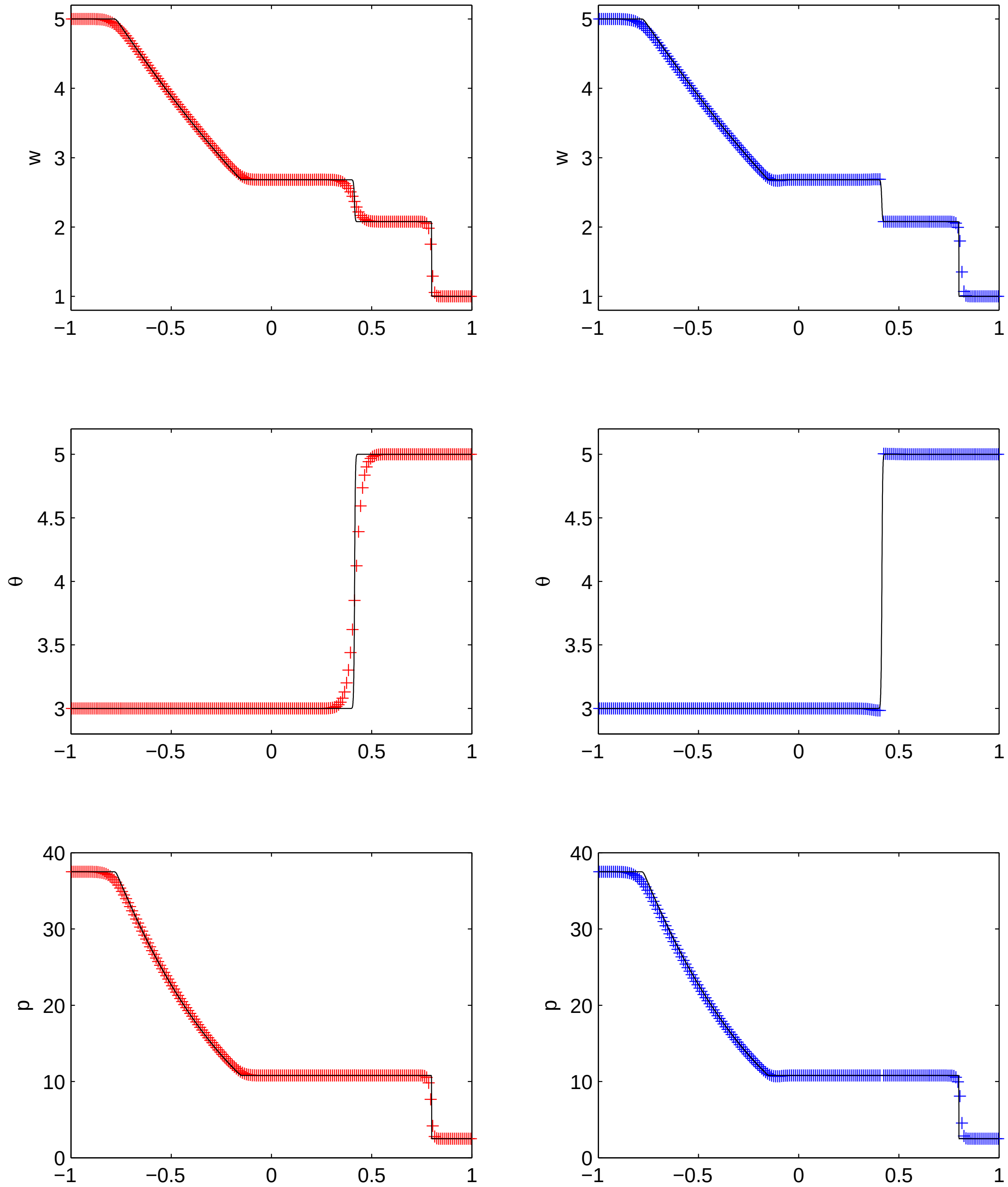
Dam Break over a Flat Bottom

- Flat bottom topography:

$$B(x) \equiv 0$$

- Initial data:

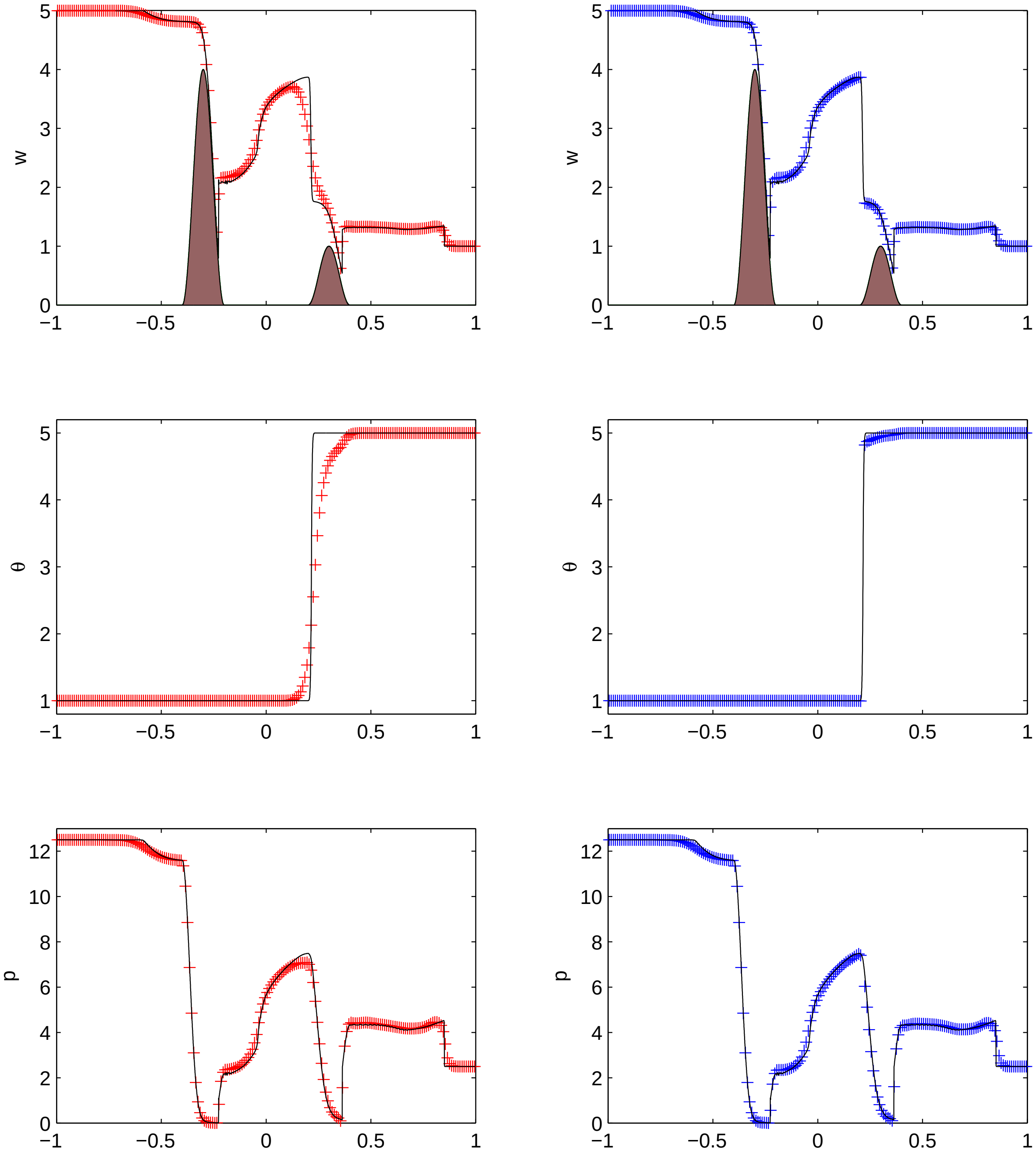
$$\begin{cases} u_l = 0,\ \theta_l = 3,\ h_l = 5 & \text{if} \quad x < 0 \\ u_r = 0,\ \theta_r = 5,\ h_r = 1 & \text{if} \quad x > 0 \end{cases}$$


w
θ
p

Dam Break over a Nonflat Bottom

$$(w,u,\theta)^T(x,0)=\begin{cases}(5,0,1)^T, & x<0\\ (1,0,5)^T, & x>0\end{cases}$$

$$B(x)=\begin{cases} 2(\cos(10\pi(x+0.3))+1), & -0.4\le x\le -0.2\\ 0.5(\cos(10\pi(x-0.3))+1), & 0.2\le x\le 0.4\\ 0, & \text{otherwise}\end{cases}$$



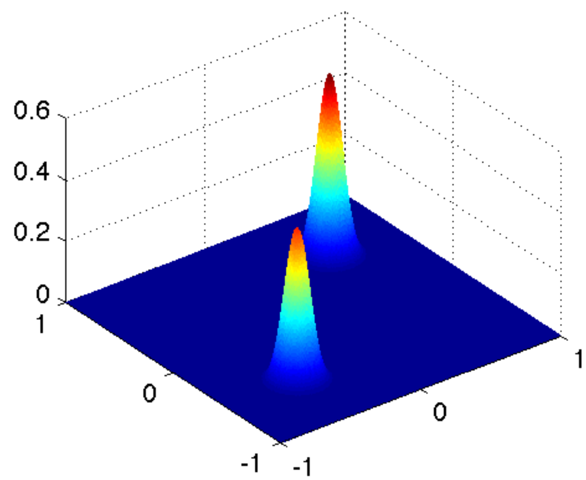
w
θ
p

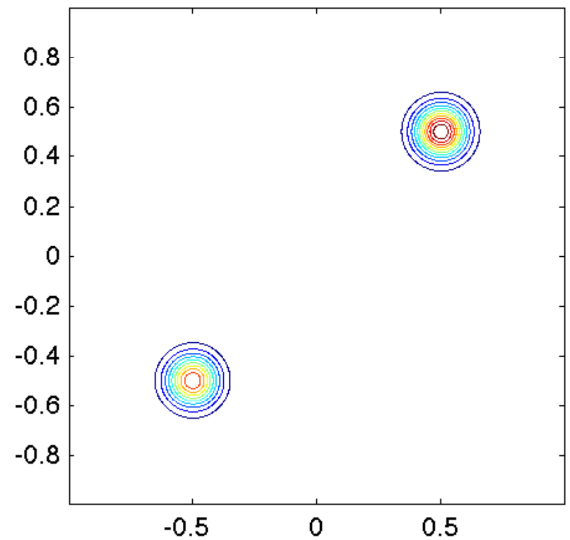
Two-Dimensional Examples

Steady State

Bottom topography consists of two Gaussian shaped humps:

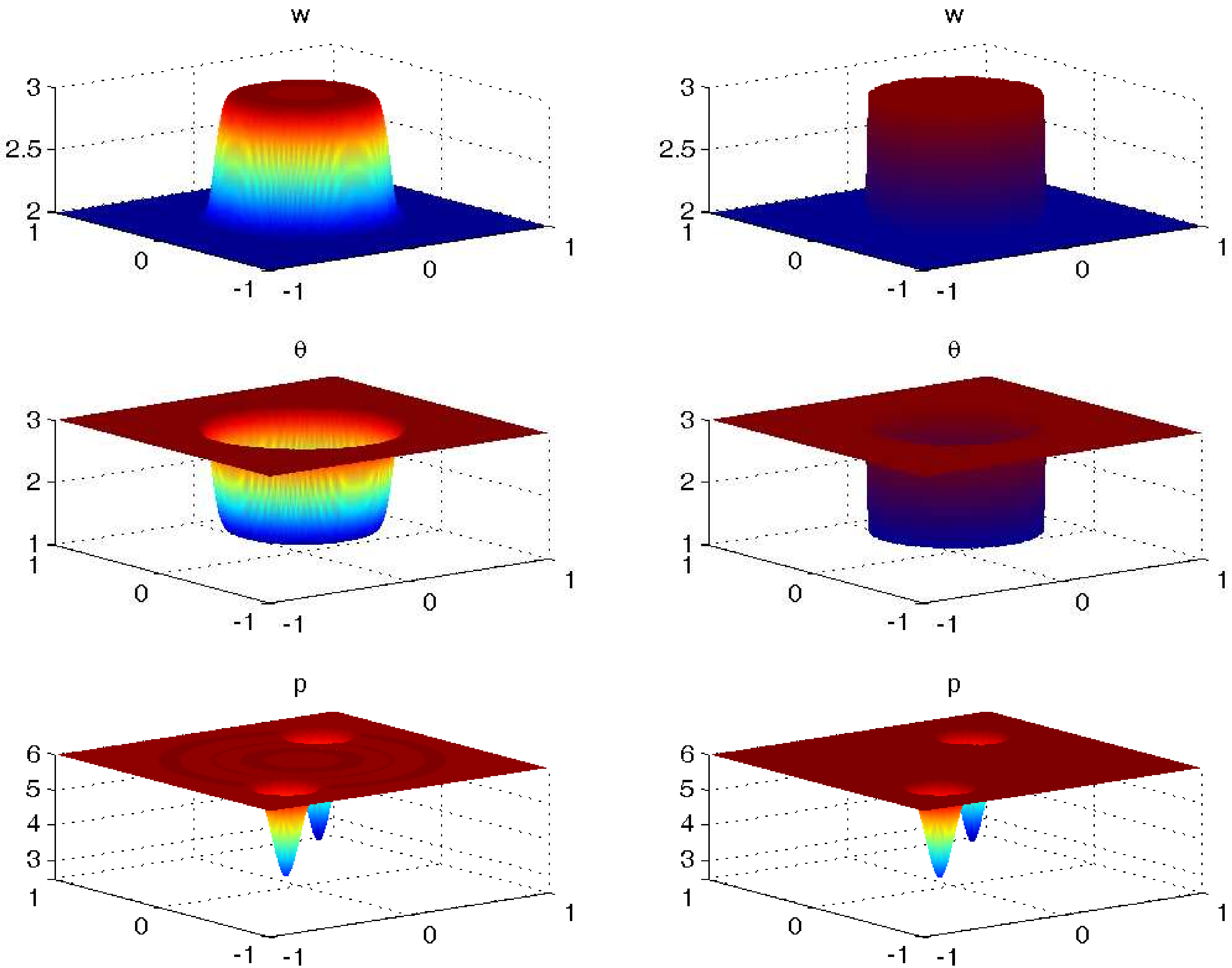
$$B(x,y)=\begin{cases}0.5\exp(-100((x+0.5)^2+(y+0.5)^2)), & x<0\\ 0.6\exp(-100((x-0.5)^2+(y-0.5)^2)), & x>0\end{cases}$$

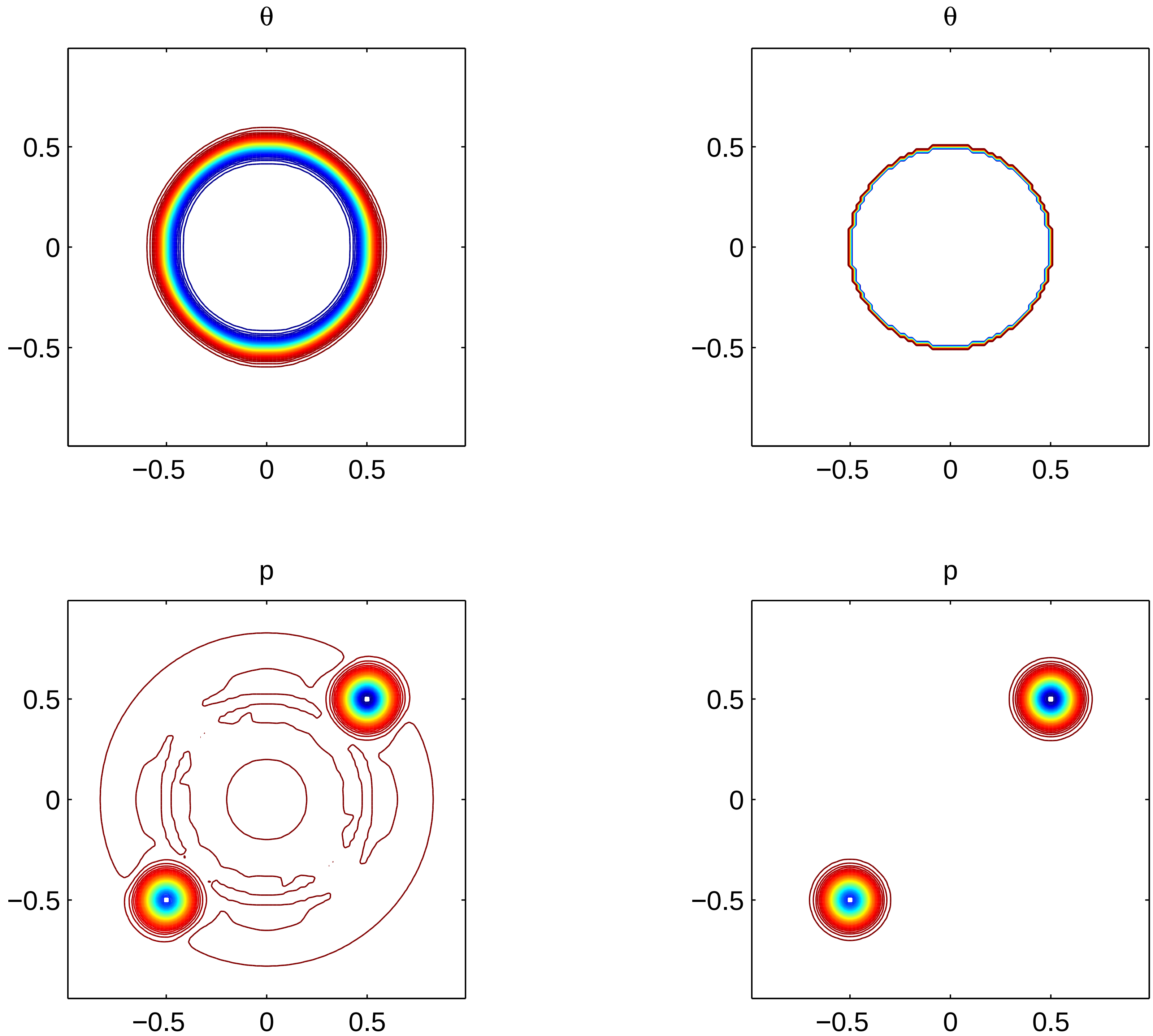




Initial data: two "lake at rest" states of type I connected through the temperature jump corresponding to the "lake at rest" state of type II:

$$(w,u,v,\theta)^T(x,y,0)=\begin{cases}(3,0,0,\frac{4}{3})^T, & x^2+y^2<0.25\\ (2,0,0,3)^T, & \text{otherwise}\end{cases}$$


w
w
θ
θ
p
p


θ
θ
p
p
0.5
0
−0.5
−0.5
0
0.5

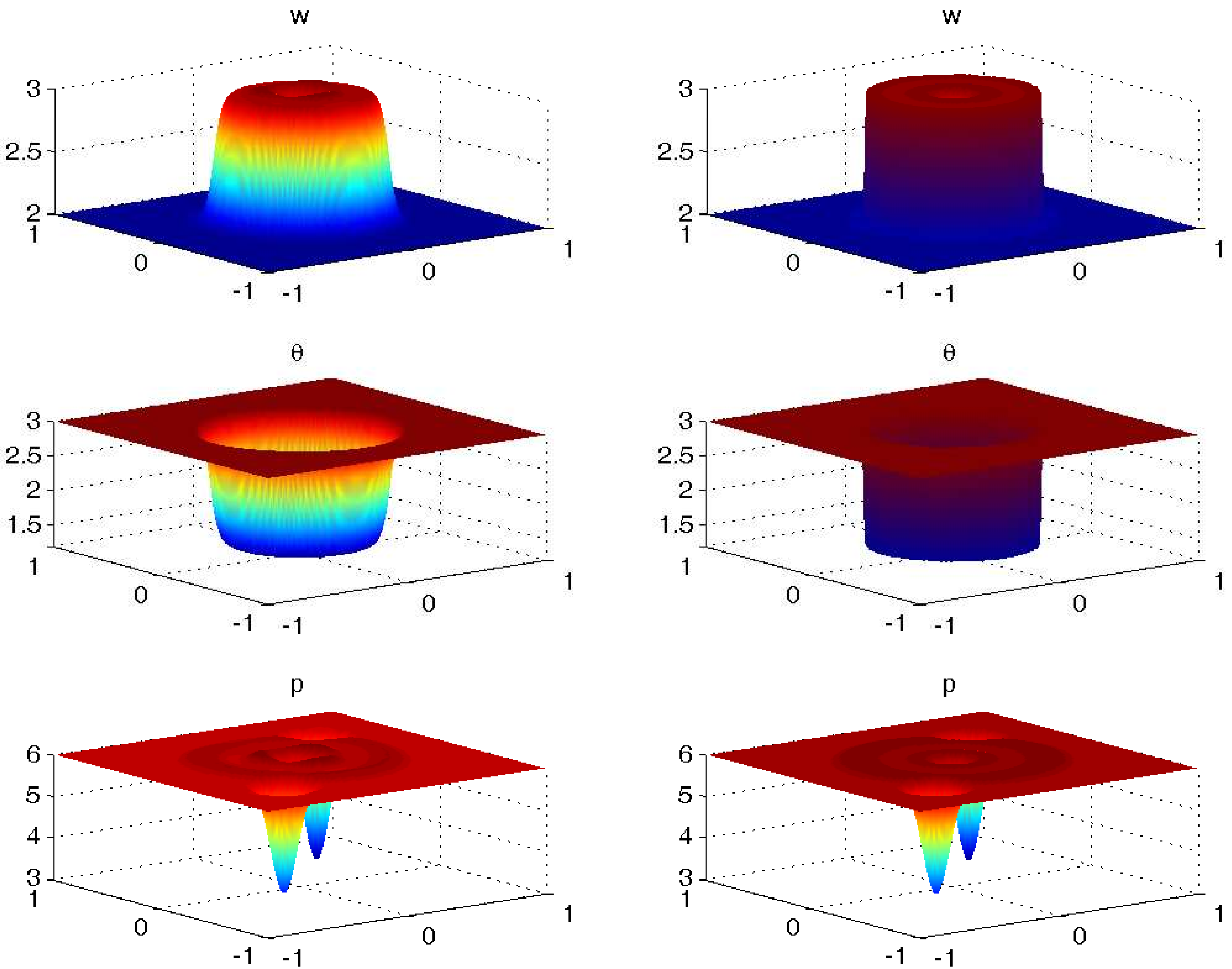
Small Perturbation of a Steady-State Solution

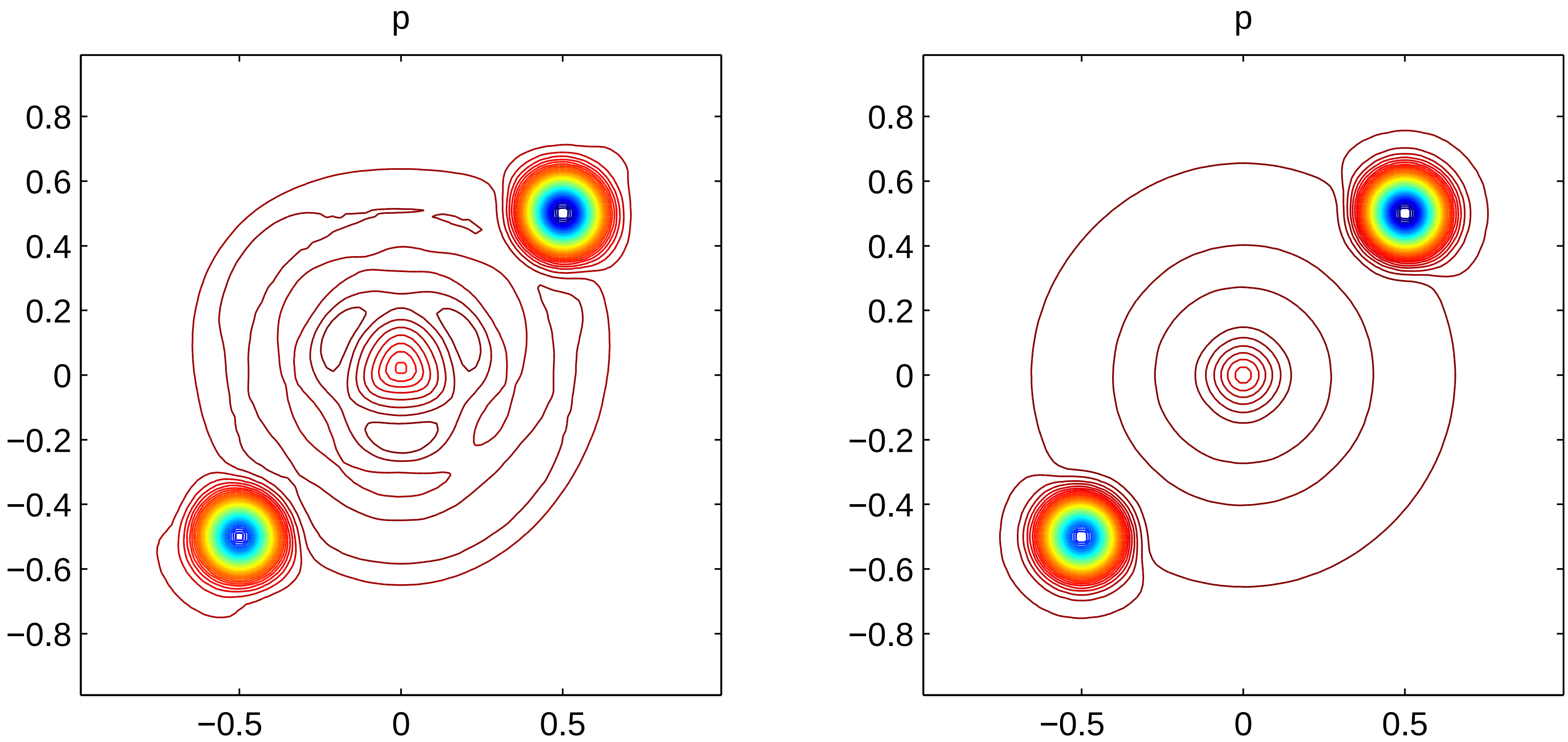
The same setting with perturbed initial condition inside a small annulus near the center of the computational domain:

$$(w,u,v,\theta)^T(x,y,0) = \begin{cases} (3+0.1,0,0,\frac{4}{3})^T, & 0.01 < x^2+y^2 < 0.09 \\ (3,0,0,\frac{4}{3})^T, & 0.09 < x^2+y^2 < 0.25 \\ & \text{ or } x^2+y^2 < 0.01 \\ (2,0,0,3)^T, & \text{otherwise} \end{cases}$$

- WB scheme:
  - the w- and θ-components of the solutions get smeared
  - the p-component develops circular-shape pressure oscillation, which interact with the perturbation. This interaction leads to the appearance of parasitic waves

- WB-IT scheme:
  - the w- and θ-components of the solutions shrply resolved
  - captures the perturbation in the p-component much more accurate


w
3
2.5
2
1
0
-1
-1
0
1
w
3
2.5
2
1
0
-1
-1
0
1
θ
3
2.5
2
1.5
1
0
-1
-1
0
1
θ
3
2.5
2
1.5
1
0
-1
-1
0
1
p
6
5
4
3
1
0
-1
-1
0
1
p
6
5
4
3
1
0
-1
-1
0
1


p
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−0.5
0
0.5
p
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−0.5
0
0.5

Radial Dam Break over the Flat Bottom

The initial condition is

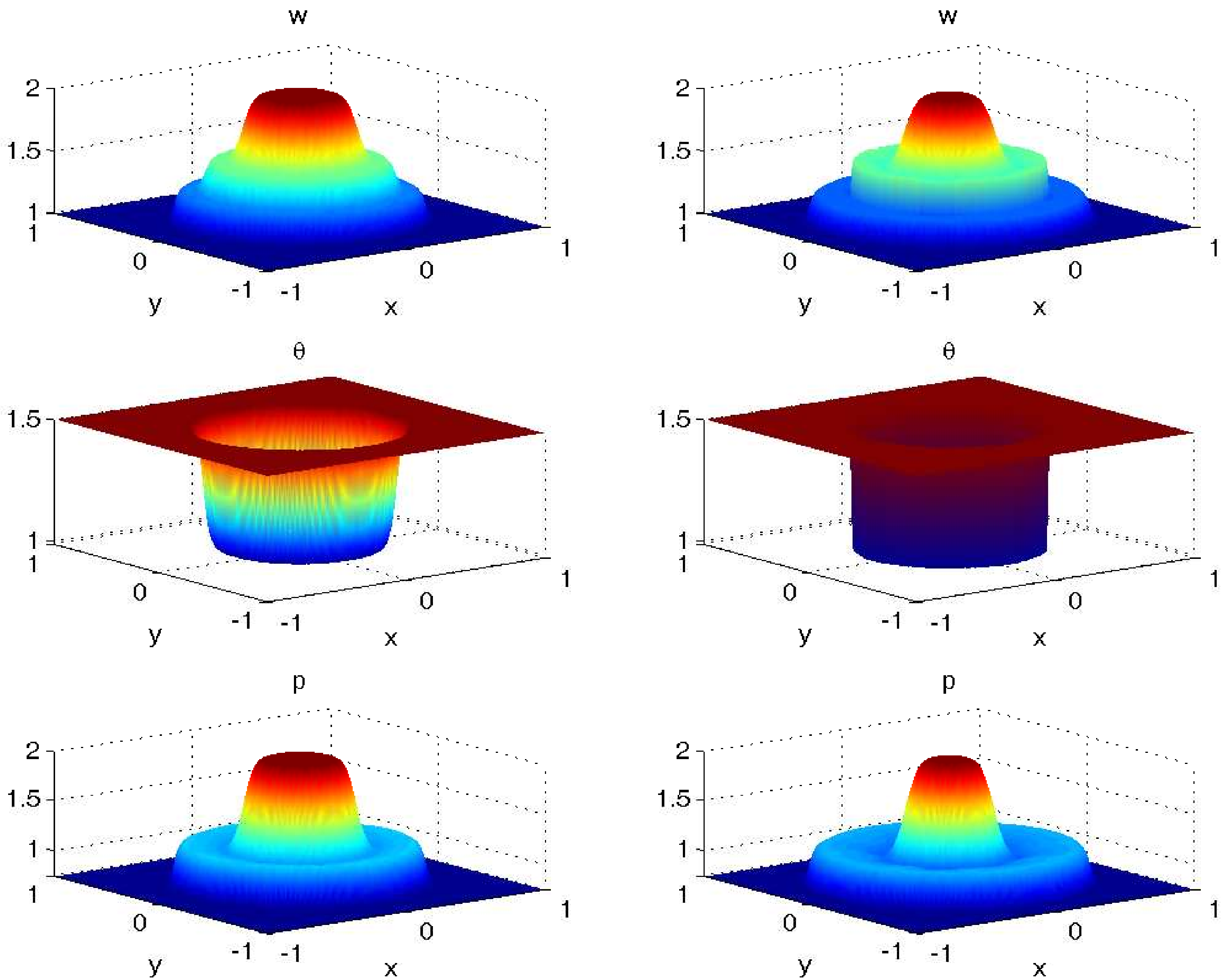
$$(w,u,v,\theta)^T(x,y,0)=\begin{cases}(2,0,0,1)^T, & x^2+y^2<0.25\\ (1,0,0,1.5)^T, & \text{otherwise}\end{cases}$$

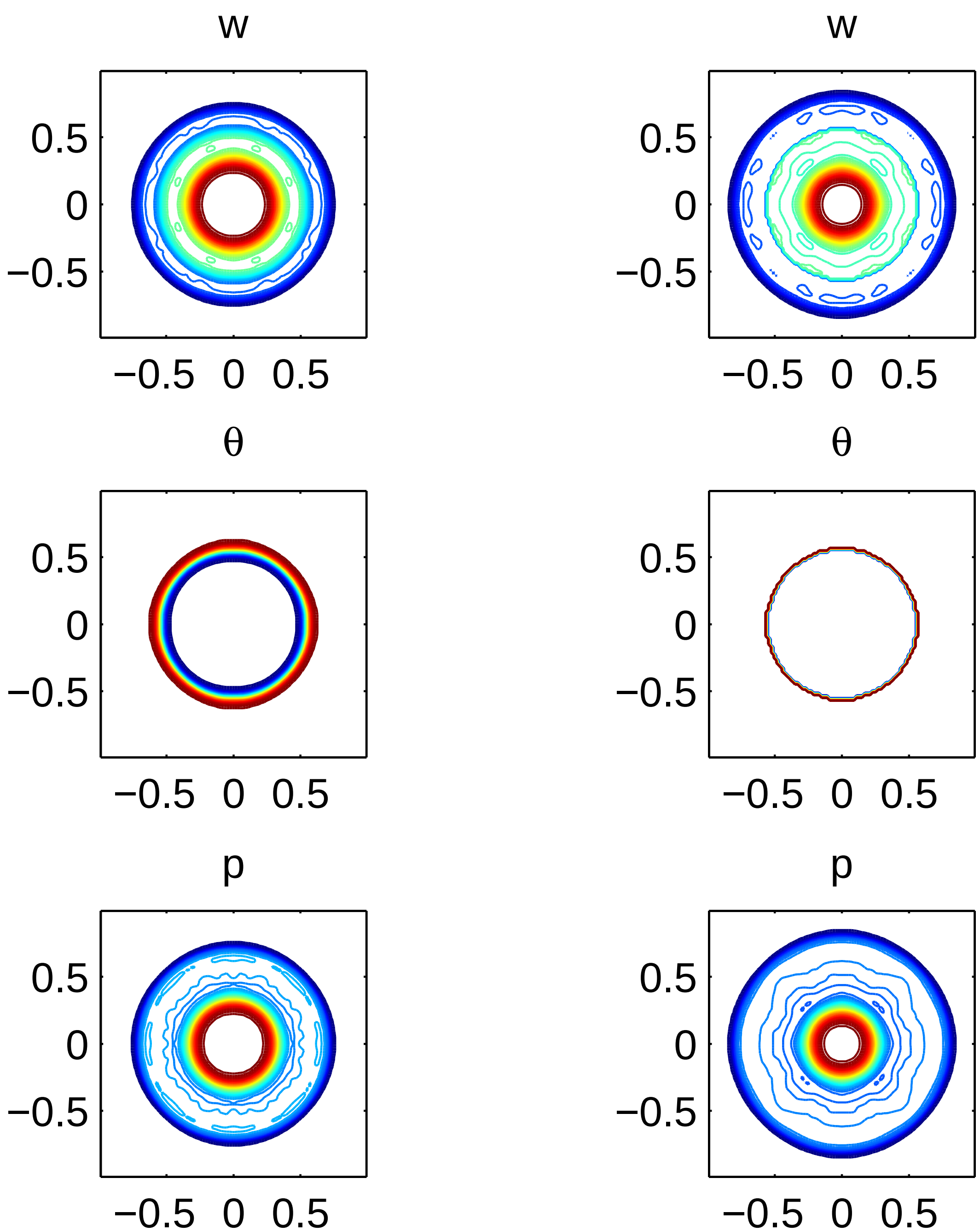
When the dam is removed, a shock wave travels radially outwards, and a rarefaction wave moves inward with a contact wave remaining between them.

- WB scheme: the solution is overly smeared both at the center of the computational domain and especially in the contact region; also notice a small circular wave at the top of the external ring in p, which is a numerical artifact caused by inaccurate resolution of the temperature jump

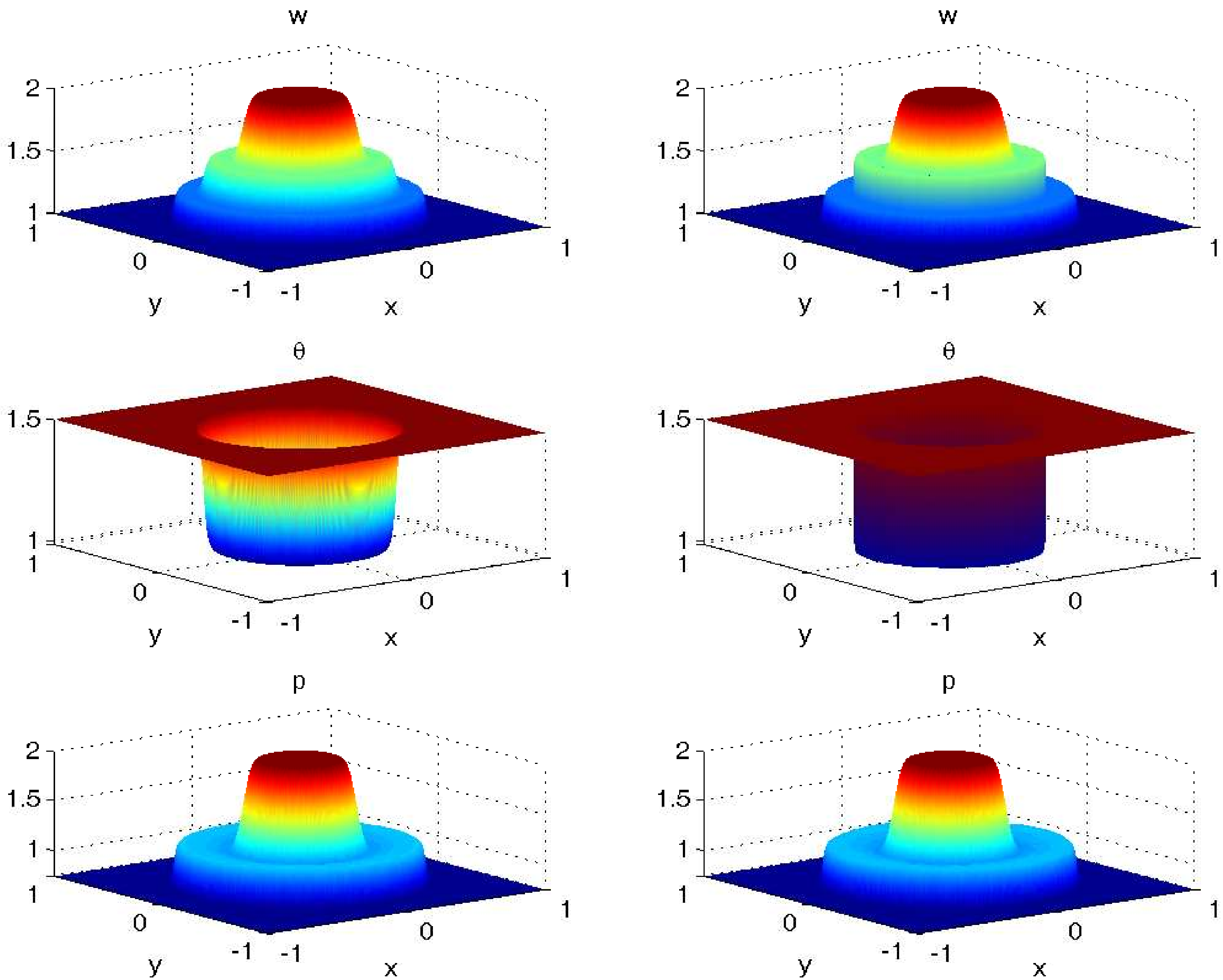
- WB-IT scheme achieves a very sharp resolution

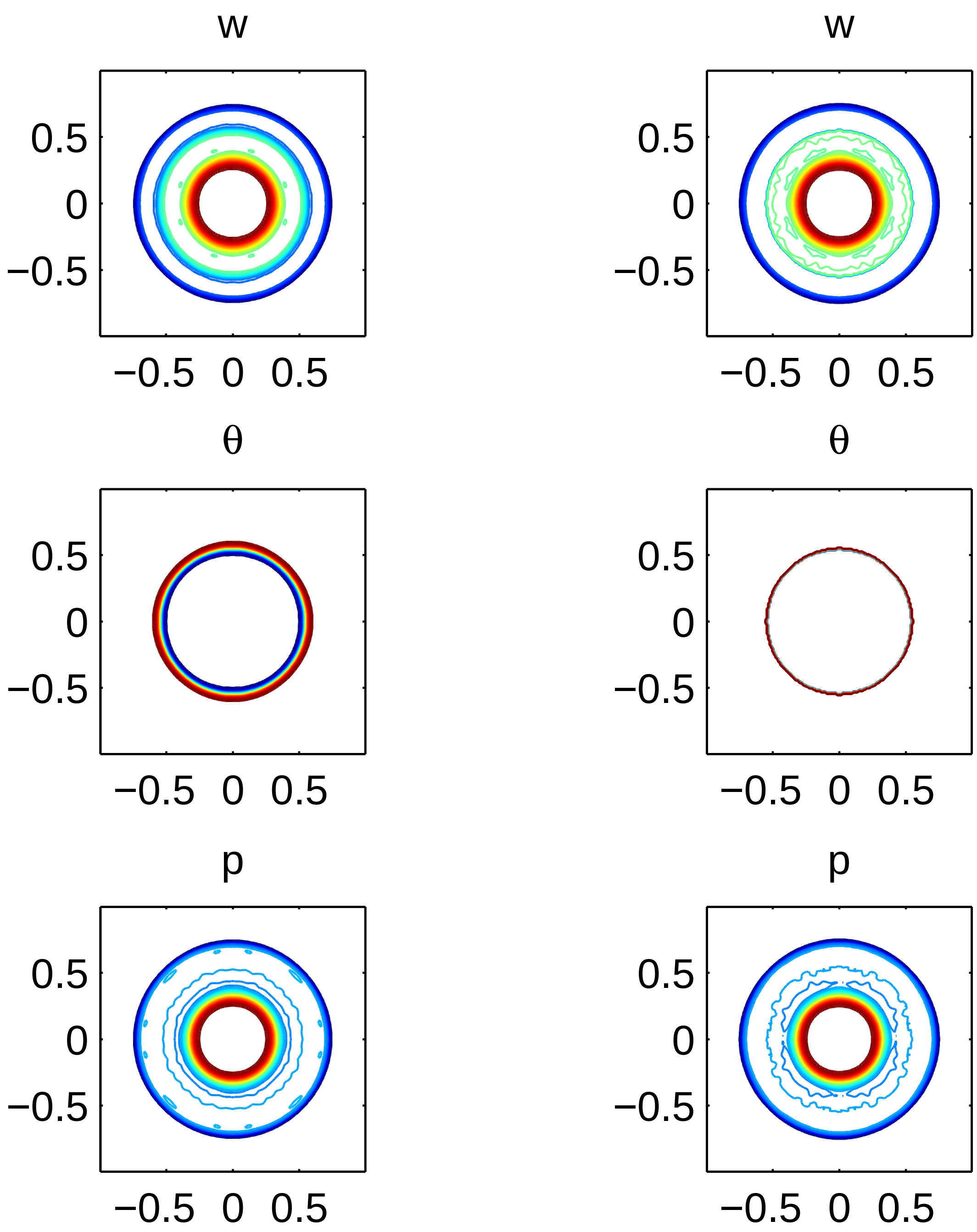
$$\Delta x = \Delta y = 2/100$$




w
0.5
0
−0.5
−0.5 0 0.5
w
0.5
0
−0.5
−0.5 0 0.5
θ
0.5
0
−0.5
−0.5 0 0.5
θ
0.5
0
−0.5
−0.5 0 0.5
p
0.5
0
−0.5
−0.5 0 0.5
p
0.5
0
−0.5
−0.5 0 0.5

$$\Delta x = \Delta y = 2/200$$





w
w
θ
θ
p
p

THANK YOU!