Global weak solution for kinetic models of active swimming and passive suspensions

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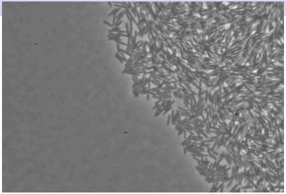
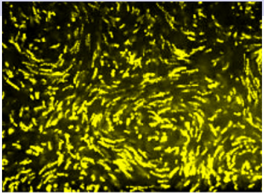
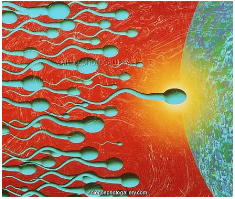
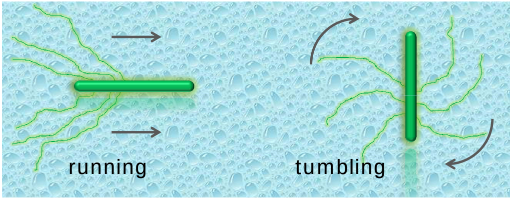
Swimming modeled by active rod-like and ellipsoidal particle suspensions

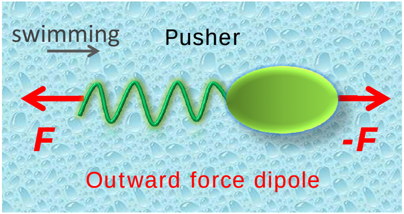
Modeling of swimming dynamics

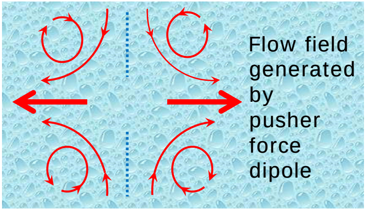
- Micro-fluid-mechanical/kinetic theory.

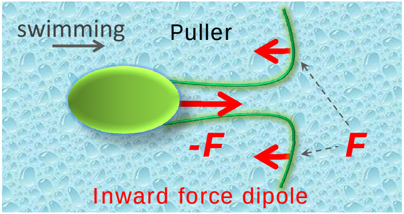
  Jeffery(1922), Kirkwood(1948), G.I. Taylor(1951), Lighthill(1960), Batchelor(1970), T.Y-T. Wu(1974), Purcell(1977), Doi, Edwards (1986), Childres, Graham, Shelley, Otto, Tzavaras, etc.

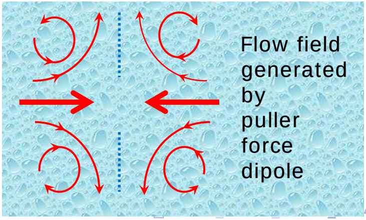
Bacterium acts as a force dipole, pusher and puller

- Almost no inertia, swimming at a terminal speed U_0 in n direction (experiencing least drag in this direction), along with the macroscopic velocity $u(x,t)$: $\dot{x} = u + U_0 n$
- Bacterium acts as a force dipole: $\begin{cases} \text{force of self-propulsion} & \boldsymbol{F} \\ \text{force against drag} & -\boldsymbol{F} \end{cases}$



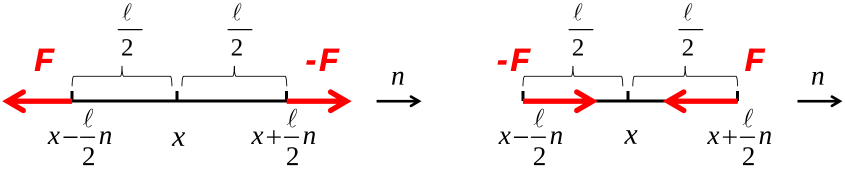






Stresslet exerted by bacterium on fluid

Kim and Karrila's book: *Microhydrodynamics: principles and selected applications.*1991



Pusher outward force dipole

Puller inward force dipole

- Force dipole:

  $$F = \pm|\boldsymbol{F}|n\left[\delta_{x+\frac{\ell}{2}n}(x) - \delta_{x-\frac{\ell}{2}n}(x)\right] \qquad + \text{ pusher}, - \text{ puller}$$

- Decomposition:

  $F = \nabla \cdot S + \nabla\psi$ in $\mathscr{D}'$, S is a symmetric traceless tensor

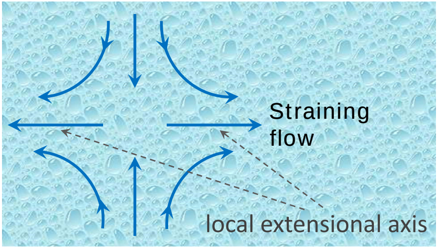
- Stresslet exerted by force dipole (multipole approximation)

  $$S = \mp(|\boldsymbol{F}|\ell)\left(n \otimes n - \tfrac{1}{d}\mathrm{Id}\right)\delta_x(x) \qquad - \text{ pusher}, + \text{ puller}$$

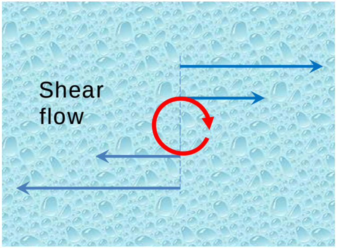
Local linear flow: symmetric and antisymmetric decomposition

Local linear flow: $\nabla u = E + W$

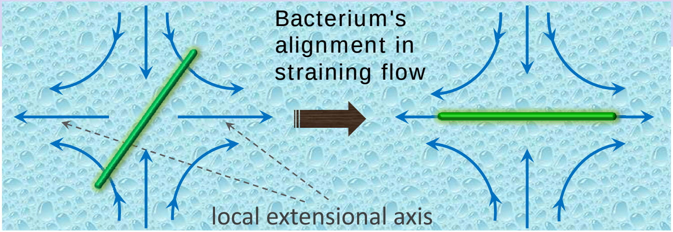
- Straining flow (symmetric part) $E = \frac{1}{2}(\nabla u + \nabla u^{\top})$



- Shear flow (antisymmetric part) $W = \frac{1}{2}(\nabla u - \nabla u^{\top})$



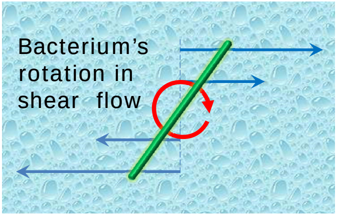
Change of bacterium’s direction under local linear flow

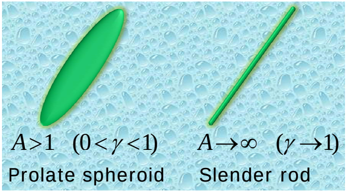


Bacterium’s direction changes as
(Jeffery equation, 1922)

$$\dot{n} = (\mathrm{Id} - n \otimes n)(\gamma E + W)n = -\nabla_n \phi + \frac{1}{2}\omega \times n$$

$$\phi = -\tfrac{1}{2}\gamma n \cdot En, \; \omega = \nabla \times u$$

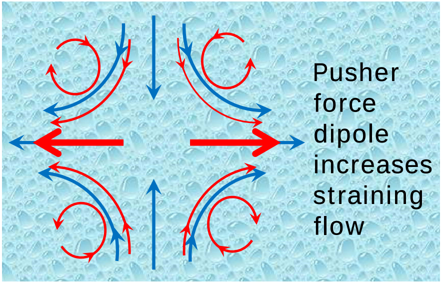




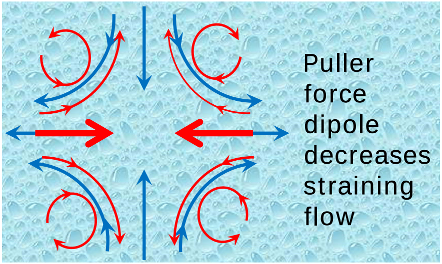
Bacterium’s shape parameter
$\gamma = \frac{A^2-1}{A^2+1}$, A is aspect ratio.
For $\gamma = 1$: $\dot{n} = (\mathrm{Id} - n \otimes n)\nabla u n$

Flow field generated by force dipole + background flow
Bacterium most of time is aligned with the straining flow

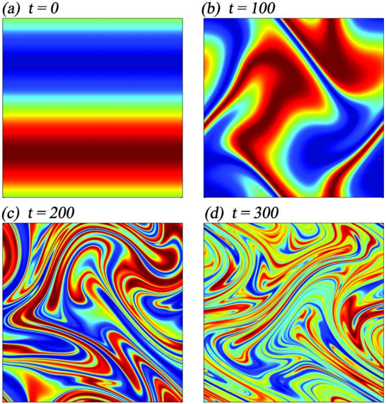
- Pusher force dipole increases straining flow and hence enhances flow mixing(called bio-mixing) (long-wavelength instability)



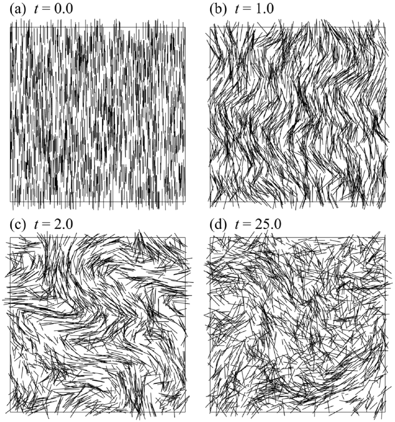
- Puller force dipole decreases straining flow and hence slows down background flow (stable: entropy-dissipation relation)



Pusher force dipole enhance flow mixing (bio-mixing), long-wavelength instability, numerical simulation



Saintillan-Shelley Phys. Fluids 2008



Saintillan-Shelley PRL 2007

Dynamics of swimmers, $D = U_0^2/6D_r$
G.L. Taylor dispersion theory, H. Brenner,1979,1980

- Dynamics of active motion

  $\dot{x}_i = u(x_i, t) + U_0 n_i$

  Change of direction

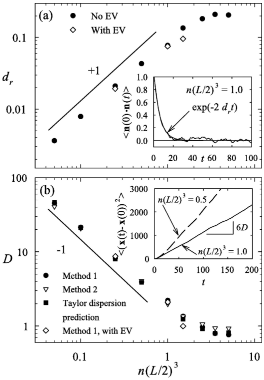
  $\dot{n}_i = (\mathrm{Id} - n_i \otimes n_i)(\gamma E + W) n_i$

- $dx_i = (u(x_i, t) + U_0 n_i)dt + \sqrt{2D}dB^i$

  $dn_i = (\mathrm{Id} - n_i \otimes n_i)(\gamma E + W) n_i dt + \sqrt{2D_r}dB_r^i$

  - $dB_r^i = (\mathrm{Id} - n_i \otimes n_i) \circ dB^i$
    is the spherical Brownian motion.
    Stochastic integral is in Stratonovich sense.
  - D is translational diffusivity
  - D_r is rotational diffusivity

- Taylor dispersion relation $D = U_0^2/6D_r$



Saintillan-Shelley
PRL 2007

Mean field limit , Fokker-Planck

- Recall
  - $\dot{x}_i = u(x_i, t) + U_0 n_i$
  - $\dot{n}_i = (\mathrm{Id} - n_i \otimes n_i)(\gamma E + W) n_i$
- Empirical distribution
  $f^N(x, n, t) := \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i, n_i)}(x, n)$
- Vlasov equation
  $\partial_t f^N + \nabla \cdot (u + U_0 n) f^N + \nabla_n \cdot (\mathrm{Id} - n \otimes n)(\gamma E + W) n f^N = 0$
- Fokker-Planck equation (add noise, $f^N \to f$)
  $\partial_t f + \nabla \cdot (u + U_0 n) f + \nabla_n \cdot (\mathrm{Id} - n \otimes n)(\gamma E + W) n f = D \, \Delta f + D_r \Delta_n f$

Stress tensor (Batchelor 1970,slender-body theory; Kirkwood formula), Navier-Stokes equation

- Recall stresslet for a bacterium (x_i, n_i)

  $S = \mp (|\boldsymbol{F}|\ell)\,(n_i \otimes n_i - \frac{1}{d}\mathrm{Id})\delta_{x_i}(x)$ $\qquad$ $-$ pusher , $+$ puller

- Average of stresslets for all bacteria $\left(\sigma_0 := \mp\left(\frac{|\boldsymbol{F}|\ell}{d}\right)\right)$

  $$\begin{aligned}\sigma^N &= \mp\left(\frac{|\boldsymbol{F}|\ell}{d}\right)\frac{1}{N}\sum_{i=1}^N (dn_i \otimes n_i - \mathrm{Id})\delta_{x_i}(x)\\ &= \sigma_0 \int_{\mathbb{S}} (dn \otimes n - \mathrm{Id})\left(\frac{1}{N}\sum_{i=1}^N \delta_{(n_i,\mathbf{x}_i)}(n,x)\right) dn\\ &= \sigma_0 \int_{\mathbb{S}} (dn \otimes n - \mathrm{Id}) f^N\, dn.\end{aligned}$$

- Stress exerted by the swimming of bacterium on fluid $(f^N \to f)$

  $\sigma = \sigma_0 \int_{\mathbb{S}} (dn \otimes n - \mathrm{Id}) f\, dn$ $\qquad$ $\sigma_0 < 0$, pusher; $\sigma_0 > 0$, puller

- Velocity u of fluid is governed by

  $\partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u + \nabla \cdot \sigma, \quad \nabla \cdot u = 0$

Kinetic model for swimming, Saintillan-Shelley 2008

- $\partial_t f + \nabla\cdot(u+\alpha n)f + \nabla_n \cdot (\mathrm{Id} - n\otimes n)(\gamma E + W)nf = D\,\Delta f + D_r \Delta_n f$
- $\mathrm{Re}\,(\partial_t u + u\cdot\nabla u) + \nabla p = \Delta u + \nabla\cdot\sigma, \quad \nabla\cdot u = 0$
- $\sigma = \beta \int_{\mathbb{S}} (dn\otimes n - \mathrm{Id}) f\, dn$
    - $\alpha > 0$, active; $\alpha = 0$, passive
    - $\beta < 0$, pusher; $\beta > 0$, puller
    - $0 < \gamma < 1$, prolate spheroidal; $\gamma \to 1$, slender rod-like
    - $\mathrm{Re} = 0$, Stokes model
- boundary conditions
    - no-flux $(\alpha nf - D\nabla f)\cdot\nu\big|_{\partial\Omega} = 0$
    - no-slip $u\big|_{\partial\Omega} = 0$

Entropy estimate, coupling terms

- $$\frac{d}{dt}\int_{\Omega\times\mathbb{S}} f\log f\,dn\,dx + 4\int_{\Omega\times\mathbb{S}}\left(D\left|\nabla\sqrt{f}\right|^2 + D_r\left|\nabla_n\sqrt{f}\right|^2\right)dn\,dx$$
$$= \alpha\int_{\Omega\times\mathbb{S}} n\cdot\nabla f\,dn\,dx + \int_{\Omega\times\mathbb{S}}\nabla_n f\cdot(\gamma E + W)n\,dn\,dx$$

- $$\frac{\mathrm{Re}}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx = -\beta\int_\Omega\int_{\mathbb{S}}(dn\otimes n - \mathrm{Id})fdn : \nabla u dx$$

- Since $Tr(E) = 0$ and $W^\top = -W$, one has
$$\int_{\mathbb{S}}\nabla_n f\cdot(\gamma E + W)n\,dn = \gamma\int_{\mathbb{S}}(dn\otimes n - \mathrm{Id})f\,dn : \nabla u$$

- $$\frac{d}{dt}\int_{\Omega\times\mathbb{S}} f\log f\,dn\,dx + 4\int_{\Omega\times\mathbb{S}}\left(D\left|\nabla\sqrt{f}\right|^2 + D_r\left|\nabla_n\sqrt{f}\right|^2\right)dn\,dx$$
$$= \alpha\int_{\Omega\times\mathbb{S}} n\cdot\nabla f\,dn\,dx + \gamma\int_\Omega\int_{\mathbb{S}}(dn\otimes n - \mathrm{Id})fdn : \nabla u dx$$

Puller ($\beta > 0$): stable, the two coupling terms cancel (recall $\gamma > 0$)

- $$\frac{d}{dt}\int_\Omega\left(\int_{\mathbb{S}} f\log f\,dn + \frac{\gamma Re}{2\beta}|u|^2\right)dx$$
$$+4\int_{\Omega\times\mathbb{S}}\left(D\left|\nabla\sqrt{f}\right|^2 + D_r\left|\nabla_n\sqrt{f}\right|^2\right)dndx + \frac{\gamma}{\beta}\int_\Omega|\nabla u|^2dx$$
$$= \alpha\int_{\Omega\times\mathbb{S}} n\cdot\nabla f\,dndx$$
$$= \alpha\int_{\Omega\times\mathbb{S}} n\cdot(2\nabla\sqrt{f}\sqrt{f})dndx$$
$$\leq 2D\int_{\Omega\times\mathbb{S}}\left|\nabla\sqrt{f}\right|^2 dndx + C\int_{\Omega\times\mathbb{S}} f dndx$$

Pushers ($\beta < 0$): long-wavelength instability, no cancelation of coupling terms, control total energy by density estimate

- $$\frac{d}{dt}\int_\Omega \left(\int_{\mathbb{S}} f \log f \, dn + \frac{\mathrm{Re}}{2}|u|^2\right) dx$$
  $$+4\int_{\Omega\times\mathbb{S}} \left(D\left|\nabla\sqrt{f}\right|^2 + D_r\left|\nabla_n\sqrt{f}\right|^2\right) dndx + \int_\Omega |\nabla u|^2 dx$$
  $$= \alpha\int_{\Omega\times\mathbb{S}} n\cdot\nabla f dndx + (\gamma-\beta)\int_{\Omega\times\mathbb{S}} (dn\otimes n - \mathrm{Id})f : \nabla u dndx$$
- $$\frac{d}{dt}\int_\Omega \left(\int_{\mathbb{S}} f \log f \, dn + \frac{\mathrm{Re}}{2}|u|^2\right) dx$$
  $$+2\int_{\Omega\times\mathbb{S}} \left(D\left|\nabla\sqrt{f}\right|^2 + D_r\left|\nabla_n\sqrt{f}\right|^2\right) dndx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx$$
  $$\leq C\left(\|\rho\|^2_{L^2(\Omega)} + 1\right) \qquad \text{where } \rho := \int_{\mathbb{S}} f dn$$
- $$\partial_t\rho + u\cdot\nabla\rho = D\Delta\rho - \alpha\nabla\cdot\int_{\mathbb{S}} nf\,dn,$$
  $$\|\rho\|_{L^2(\Omega)} \leq \|\rho_0\|_{L^2(\Omega)} e^{Ct}$$

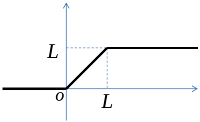
L^2-estimate for Stokes kinetic models

- Stokes equation $-\Delta u + \nabla p = \nabla \cdot \sigma, \quad \nabla \cdot u = 0$
  - $\int_\Omega |\nabla u|^2 dx = -\beta \int_{\Omega\times\mathbb{S}} (dn \otimes n - \mathrm{Id}) f : \nabla u dn dx$
  - $\|\nabla u\|^2_{L^2(\Omega)} \leq C\|\rho\|^2_{L^2(\Omega)} \leq Ce^{Ct}, \quad \|u\|_{H^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega;\, L^1(\mathbb{S}))}$
- $$\frac{1}{2}\frac{d}{dt}\int_{\Omega\times\mathbb{S}} |f|^2 dn dx + \int_{\Omega\times\mathbb{S}} \left(D|\nabla f|^2 + D_r|\nabla_n f|^2\right) dn dx$$
  $$= \alpha \int_{\Omega\times\mathbb{S}} nf \cdot \nabla f dn dx + \frac{\gamma}{2}\int_{\Omega\times\mathbb{S}} (dn \otimes n - \mathrm{Id}) f^2 : \nabla u dn dx$$
- $$\frac{d}{dt}\|f\|^2_{L^2(\Omega\times\mathbb{S})} + D\|\nabla f\|^2_{L^2(\Omega\times\mathbb{S})} + D_r\|\nabla_n f\|^2_{L^2(\Omega\times\mathbb{S})}$$
  $$\leq C\left(\|f\|^2_{L^2(\Omega\times\mathbb{S})} + \|f\|^2_{L^4(\Omega;L^2(\mathbb{S}))}\right)$$
- $$\frac{d}{dt}\|f\|^2_{L^2(\Omega\times\mathbb{S})} + \frac{D}{2}\|\nabla f\|^2_{L^2(\Omega\times\mathbb{S})} + \frac{D_r}{2}\|\nabla_n f\|^2_{L^2(\Omega\times\mathbb{S})} \leq C\|f\|^2_{L^2(\Omega\times\mathbb{S})}$$

Approximate and converge to global weak solution

- Approximate problem preserving positivity
  - cut-off function: $\forall L > 1$,

$$Q^L(s) := \begin{cases} 0, & \text{if } s \le 0, \\ s, & \text{if } 0 \le s \le L, \\ L, & \text{if } s \ge L \end{cases}$$



  - semi-implicit scheme ($\tau = T/N$, $(0, T] = \bigcup_{k=1}^{N}((k-1)\tau, k\tau]$)
    - $$\frac{u_k^L - u_{k-1}^L}{\tau} + (u_{k-1}^L \cdot \nabla)u_k^L - \Delta u_k^L + \nabla p_k^L = \beta \nabla \cdot \int_{\mathbb{S}} (dn \otimes n - \mathrm{Id}) f_k^L dn; \quad \nabla \cdot u_k^L = 0$$
    - $$\frac{f_k^L - f_{k-1}^L}{\tau} + u_{k-1}^L \cdot \nabla f_k^L + \nabla \cdot \left[\alpha n Q^L(f_k^L)\right] - \Delta f_k^L - \Delta_n f_k^L = -\nabla_n \cdot \left[(\mathrm{Id} - n \otimes n)(\gamma E_k^L + W_k^L) n Q^L(f_k^L)\right]$$
  - Leray-Schauder fixed point theorem
- Compactness argument: taking $\tau = o(L^{-2}) \to 0$, Aubin-Lions-Simon lemma

Global existence theorem for weak entropy solution for any $\beta \in \mathbb{R}$, both pusher and puller

Let $d = 2, 3$, $u_0 \in L^2(\Omega)$, $f_0 \in L^2\left(\Omega;\ L^1(\mathbb{S})\right)$ with finite entropy. Then for any $\beta \in \mathbb{R}$, $\alpha \in [0, \infty)$ there is a global weak solution (u, f) which satisfies

- $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \cap H^1(0, T; H^{-2})$
- $f \geq 0, f \in L^\infty(0, T; L^2(\Omega; L^1(\mathbb{S}))), f \log f \in L^\infty(0, T; L^1(\Omega \times \mathbb{S}))$
- $\sqrt{f} \in L^2(0, T; H^1(\Omega \times \mathbb{S})), f \in H^1(0, T; H^{-4}(\Omega \times \mathbb{S}))$
- satisfy the energy/entropy inequality

Let $d = 2, 3$, $f_0 \geq 0$, $f_0 \in L^2(\Omega \times \mathbb{S})$. Then there exists a global weak solution (u, f) to Stokes kinetic model which satisfies

- $u \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$,
- $f \geq 0, f \in L^\infty(0, T;\ L^2(\Omega \times \mathbb{S})) \cap L^2(0, T;\ H^1(\Omega \times \mathbb{S}))$
- $f \in W^{1,4/d}(0, T; H^{-1}(\Omega \times \mathbb{S}))$

If $d = 2$, then the weak solution is unique

Passive suspension of dumbbell beads dimer (polymer)

- Dynamics of extensible dimer suspending in velocity field u

  $\dot{x} = u(x, t)$

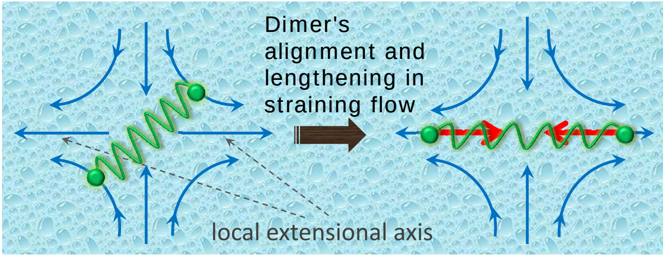
- In the presence of spring force $-\nabla_n U$ and neglecting inertia

  $\dot{n} = \nabla u n - \nabla_n U = -\nabla_n \phi + \frac{1}{2}\omega \times n - \nabla_n U$

  $(\phi = -\frac{1}{2} n \cdot E n, \omega = \nabla \times u)$

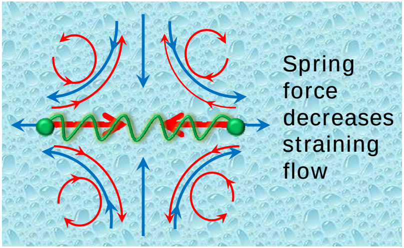
Dimer acts as puller, slowing down background flow

Recall $\dot{n} = \nabla u n - \nabla_n U = -\nabla_n \phi + \frac{1}{2}\omega \times n - \nabla_n U$



spring restore force

Acts as a force dipole against lengthening in straining flow. Similarly to puller, it slows down background flow (stable: relative entropy-dissipation relation)



Kinetic model for passive dumbbell beads
Main difficulty comes from the spring force

- $\partial_t f + u \cdot \nabla f + \nabla_n \cdot (\nabla u n - \nabla_n U) f = D\Delta f + D_r \Delta_n f$
- $\sigma = \beta \int_{\mathbb{R}^d} (\nabla_n U \otimes n - \mathrm{Id})\, f\, dn,\ \beta > 0$
- $\mathrm{Re}(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla p = \nabla \cdot \sigma, \quad \nabla u = 0$
  - For simplicity, let $D = D_r = \beta = \mathrm{Re} = 1$

- Interaction operator
  $\Delta_n f + \nabla_n \cdot (\nabla_n U - \nabla u n) f = \nabla_n \cdot \left(\nabla_n f + f \nabla_n U + f \nabla_n \phi - \frac{1}{2} f \omega \times n \right)$
  For co-rotational, ∇u is replaced by W and ϕ is absent
- Linear Fokker-Planck operator
  $\nabla_n \cdot (\nabla_n f + f \nabla_n U) = \nabla_n \cdot \left(M \nabla_n \frac{f}{M} \right),\ M(n) := \frac{e^{-U(n)}}{\int_{\mathbb{R}^d} e^{-U(n)} dn}$
  gives a relative entropy in weighted spaces $L^2_M(\mathbb{R}^d)$

Relative entropy estimate

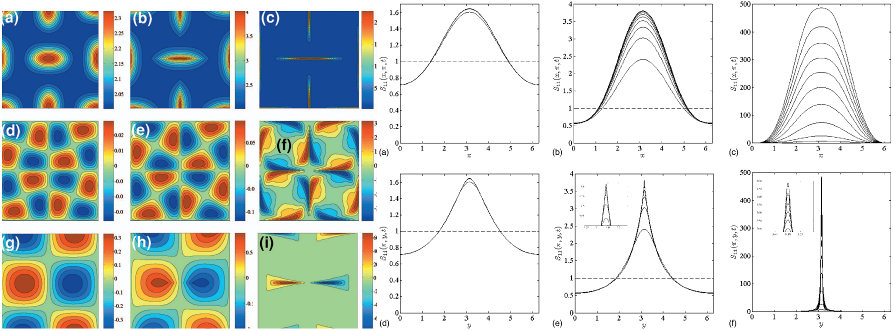
- $\hat{f} := \dfrac{f}{M}$, equivalent model with initial-boundary conditions
  - $M[\partial_t \hat{f} + u \cdot \nabla \hat{f} - \Delta \hat{f}] + \nabla_n \cdot \left(M \nabla u n \hat{f} \right) = \nabla_n \cdot \left(M \nabla_n \hat{f} \right)$
    $$\sigma = \int_{\mathbb{R}^d} (\nabla_n U \otimes n - \mathrm{Id}) \hat{f} \, M dn$$
    $$\partial_t u + u \cdot \nabla u + \nabla p = \Delta u + \nabla \cdot \sigma, \quad \nabla \cdot u = 0$$
  - $\left| M \nabla_n \hat{f} - M \nabla u n \hat{f} \right| \to 0$ as $|n| \to \infty$
    $$M \nabla \hat{f} \cdot \nu|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = 0$$
    $$\hat{f}|_{t=0} = \frac{f_0}{M}, \; u|_{t=0} = u_0$$
- $$\frac{d}{dt} \int_\Omega \left(\int_{\mathbb{R}^d} \hat{f} \log \hat{f} \, M dn + \frac{1}{2} |u|^2 \right) dx$$
  $$+4 \int_{\Omega \times \mathbb{R}^d} \left(\left| \nabla \sqrt{\hat{f}} \right|^2 + \left| \nabla_n \sqrt{\hat{f}} \right|^2 \right) M dn \, dx + \int_\Omega |\nabla u|^2 dx = 0$$

Spring potential of Hookean dumbbell model
linear Hookean

- Spring potential $U(n) = V(\frac{1}{2}|n|^2)$, hence $\nabla_n U(n) = V'(\frac{1}{2}|n|^2)n$
- Linear Hookean: $V(s) = s$ in $[0, \infty)$
  - Maxwellian $M(n) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|n|^2}{2}}$, normalized Gaussian
  - the micro-macro model has a macroscopic 2th-moment-closure (called Oldroyd-B model when $\rho = 1$)

$$\sigma_t + u \cdot \nabla\sigma - \Delta\sigma + 2\sigma = (\nabla u)\sigma + \sigma(\nabla u)^\top + \rho\,(\nabla u + (\nabla u)^\top)$$
$$\partial_t \rho + u \cdot \nabla\rho - \Delta\rho = 0$$
$$\partial_t u + u \cdot \nabla u + \nabla p = \Delta u + \nabla \cdot \sigma, \quad \nabla \cdot u = 0$$

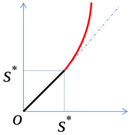
Emergence of singular structures in Oldroyd-B fluids, B. Thomases and M. Shelley, Phy Fuids 2007



Spring potential of Hookean dumbbell model super-linear Hookean

- Spring potential $U(n) = V(\frac{1}{2}|n|^2)$, hence $\nabla_n U(n) = V'(\frac{1}{2}|n|^2)n$
- Linear Hookean: $V(s) = s$ in $[0, \infty)$
  - Maxwellian $M(n) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|n|^2}{2}}$, normalized Gaussian
- Super-linear Hookean:

$$V(s) = \begin{cases} \text{linear,} & 0 \le s \le s^* \\ \text{super-linear,} & s > s^* \end{cases} \quad (s^* > 0)$$



  - $V \in W^{2,\infty}_{loc}\big([0,\infty); \mathbb{R}_{\ge 0}\big)$; convex in $[0,\infty)$; $V(s) = s$, $s \in [0, s^*]$; $\lim_{s\to\infty} V(s)/s = \infty$; $V'(s) \le e^{V(s)/4}$ $(\forall s >> 1)$
  - Maxwellian $M(n) = Ce^{-U(n)}$, $U(n) >> |n|^2$ at far-field
  - $\int_{\mathbb{R}^d} |n|^2 |\nabla_n U(n)|^2 M dn < \infty$ implies $\| \int_{\mathbb{R}^d} \nabla_n U \otimes n \hat{f} M dn \|_{L^2(\Omega)} \le C \|\hat{f}\|_{L^2_M(\Omega\times\mathbb{R}^d)}$ (approximate problem)

Weak compactness of $\sigma(x,t)$ in $L^1((0,T)\times\Omega)$

- $$\int_{(0,T)\times\Omega} \sigma : \nabla v dx dt = \int_{(0,T)\times\Omega}\int_{\mathbb{R}^d} \nabla_n U \otimes n\hat{f}\, M dn : \nabla v dx dt$$
- $$\int_{\mathbb{R}^d} \nabla_n U \otimes n\hat{f}\, M dn : \nabla v = \int_{\mathbb{R}^d} \nabla_n \hat{f} \otimes n\, M dn : \nabla v$$
  $$= 2\int_{\mathbb{R}^d} \nabla_n \sqrt{\hat{f}} \otimes \mathbf{n} \sqrt{\hat{f}}\, M dn : \nabla v \quad (\nabla\cdot v = 0)$$
- $$\int_{(0,T)\times\Omega} \sigma : \nabla v dx dt = 2\int_{(0,T)\times\Omega}\int_{\mathbb{R}^d} \nabla_n \sqrt{\hat{f}} \otimes \mathbf{n} \sqrt{\hat{f}} M dn : \nabla v dx dt$$
- Entropy estimate implies that $\|\sqrt{\hat{f}}\|_{L^2(0,T;H^1_M(\Omega\times\mathbb{R}^d))} \le C$, hence $\nabla_n \sqrt{\hat{f}}$ has weak compactness in $L^2(0,T;L^2_M(\Omega\times\mathbb{R}^d))$
- Note that $\mathbf{n} \in \mathbb{R}^d$, while test function $v(x,t) \in C_0^\infty((0,T)\times\Omega)$
- Need the compactness of $\sqrt{\hat{f}}$ in $L^2(0,T;L^2_{M(1+|\mathbf{n}|^2)}(\Omega\times\mathbb{R}^d))$
- QUESTION: $H^1_M(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^2_{M(1+|\mathbf{n}|^2)}(\mathbb{R}^d)$??

Compact and noncompact embedding theorem

- Super-linear Hookean: $M(n) = e^{-U(n)}$, $U(n) >> |n|^2$ at infinity

$$H^1_M(\mathbb{R}^d) \hookrightarrow \hookrightarrow L^2_{M(1+|n|^2)}(\mathbb{R}^d)$$

  - We CAN get global weak solution for super-linear Hookean

- Linear Hookean: $M(n) = e^{-\frac{|n|^2}{2}}$

$$H^1_M(\mathbb{R}^d) \hookrightarrow\not\hookrightarrow L^2_{M(1+|n|^2)}(\mathbb{R}^d)$$

  - We CANNOT get global weak solution for linear Hookean

Embedding theorem: $H^1_M(\mathbb{R}^d) \hookrightarrow L^2_{M(1+U(n))}(\mathbb{R}^d)$

$M = e^{-U}$: $\begin{cases} \text{super-linear} & U(n) >> |n|^2 \text{ at far-field} \\ \text{linear} & U(n) = \frac{|n|^2}{2} \end{cases}$

- Uniform convex (Bakry-Emery) condition: $D^2 U(n) \geq \mathrm{Id}$ on $\mathbb{R}^d$
- Logarithmic Sobolev inequality: $\forall \varphi \in H^1_M$,

$$\int_{\mathbb{R}^d} |\varphi|^2 \log\left(|\varphi|^2\right) M dn \leq 2\|\nabla_n \varphi\|^2_{L^2_M} + \|\varphi\|^2_{L^2_M} \log\left(\|\varphi\|^2_{L^2_M}\right)$$

- Fenchel-Young inequality $rs \leq r(\log r - 1) + e^s$, $\forall r, s \geq 0$ (where $r(\log r - 1)$ and e^s are Legendre dual functions)

$$\int_{\mathbb{R}^d} |\varphi|^2 U(n) M dn = 2 \int_{\mathbb{R}^d} |\varphi|^2 \frac{U(n)}{2} M dn$$

$$\leq 2 \int_{\mathbb{R}^d} |\varphi|^2 \log\left(|\varphi|^2\right) M dn + 2 \int_{\mathbb{R}^d} e^{U(n)/2} M dn$$

$$\leq 4\|\nabla_n \varphi\|^2_{L^2_M} + 2\|\varphi\|^2_{L^2_M} \log\left(\|\varphi\|^2_{L^2_M}\right) + 4 \int_{\mathbb{R}^d} e^{-U(n)/2} dn$$

- $H^1_M \subset L^2_{M(1+U(n))}$; $\|\varphi\|_{H^1_M} \leq 1 \Rightarrow \|\varphi\|_{L^2_{M(1+U(n))}} \leq C$
- $\forall\ 0 \neq \psi \in H^1_M$, $\left\|\frac{\psi}{\|\psi\|_{H^1_M}}\right\|_{H^1_M} \leq 1 \Rightarrow \left\|\frac{\psi}{\|\psi\|_{H^1_M}}\right\|_{L^2_{M(1+U(n))}} \leq C$
- $\forall \psi \in H^1_M$, $\|\psi\|_{L^2_{M(1+U(n))}} \leq C\|\psi\|_{H^1_M}$

Compact embedding theorem for super-linear Hookean $M(n) = e^{-U(n)}$, $U(n) >> |n|^2$: $H^1_M(\mathbb{R}^d) \hookrightarrow \hookrightarrow L^2_{M(1+|n|^2)}(\mathbb{R}^d)$

- $H^1_M(\mathbb{R}^d) \hookrightarrow L^2_{M(1+U(n))}(\mathbb{R}^d)$ (proved just now)
- $\{\varphi_k\}$ is bdd in $H^1_M(\mathbb{R}^d) \Rightarrow \{\varphi_k\}$ is bdd in $L^2_{M(1+U(n))}(\mathbb{R}^d)$
- $U(n) >> |n|^2$ at infinity

  $\Rightarrow \{\varphi_k\}$ is uniform integrable at far-field in $L^2_{M(1+|n|^2)}(\mathbb{R}^d)$.
- Diagonal argument

  $L^2_{M(1+|n|^2)}$ $\begin{cases} \text{uniform integrability at far-field} \\ \text{Rellich-Kondrachov theorem on bdd domain} \end{cases}$

  $\Rightarrow \{\varphi_k\}$ has a convergent subsequence in $L^2_{M(1+|n|^2)}(\mathbb{R}^d)$.

Noncompact embedding theorem for linear Hookean $M = e^{-\frac{|n|^2}{2}}$: $H^1_M(\mathbb{R}^d) \not\hookrightarrow\hookrightarrow L^2_{M(1+|n|^2)}(\mathbb{R}^d)$ (Prove by contradiction)

Suppose $H^1_M \hookrightarrow\hookrightarrow L^2_{M(1+|n|^2)}$

- $H^1_M \equiv H^1_M \cap L^2_{M(1+|n|^2)}$ $(H^1_M \hookrightarrow L^2_{M(1+|n|^2)})$

$\Longleftrightarrow H^1_M \cap L^2_{M(1+|n|^2)} \hookrightarrow\hookrightarrow L^2_{M(1+|n|^2)}$

- $\|\phi_j\|_{L^2_{M(1+|n|^2)}} = \left\| \psi_j := \sqrt{M}\,\phi_j \right\|_{L^2_{(1+|n|^2)}}$ (compactness equivalence)
- $\|\phi_j\|_{H^1_M \cap L^2_{M(1+|n|^2)}} \leq C$ if and only if $\|\psi_j\|_{H^1 \cap L^2_{(1+|n|^2)}} \leq C$

$\Longleftrightarrow H^1 \cap L^2_{(1+|n|^2)} \hookrightarrow\hookrightarrow L^2_{(1+|n|^2)}$

- $\|\psi\|_{L^2_{(1+|n|^2)}} = \|\mathcal{F}\psi\|_{H^1}$
- $\|\psi\|_{H^1 \cap L^2_{(1+|n|^2)}} = \|\mathcal{F}\psi\|_{H^1 \cap L^2_{(1+|n|^2)}}$ (Parseval-type identity)

$\Longleftrightarrow H^1 \cap L^2_{(1+|n|^2)} \hookrightarrow\hookrightarrow H^1$ A CONTRADICTION!

(This is impossible for sequence with compact support)

Global existence theorem for super-linear Hookean dumbbell model

Let $d = 2, 3$, $u_0 \in L^2(\Omega)$, $f_0 \in L^\infty\big(\Omega;\ L^1(\mathbb{R}^d)\big)$, $\hat{f}_0 \log \hat{f}_0 \in L^1_M(\Omega \times \mathbb{R}^d)$. Then there exists a global weak solution (u, f) to super-linear Hookean model which satisfies

- $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-3}(\Omega))$
- $f \in L^\infty((0, T) \times \Omega; L^1(\mathbb{R}^d)) \cap H^1(0, T; H^{-2-d}(\Omega \times \mathbb{R}^d))$
- $f \geq 0, \hat{f} \log \hat{f} \in L^\infty(0, T; L^1_M(\Omega \times \mathbb{R}^d))$
- $\nabla \sqrt{\hat{f}}, \nabla_n \sqrt{\hat{f}} \in L^2_M\big((0, T) \times \Omega \times \mathbb{R}^d\big)$
- satisfy energy/relative entropy inequality

Some most relevant references on existence theorems

- Barrett and Süli. *Math. Models Methods Appl. Sci.* 2011
  Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: Finitely extensible nonlinear bead-spring chains.
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- L. Zhang, H. Zhang and P. Zhang. *Commun. Math. Sci.* 2008
  Global existence of weak solutions to the regularized Hookean dumbbell model.

Thank You!