Boundary singularity for Boltzmann equation

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It is known for a simplified kinetic model, WKB model, that the flow velocity has a logarithmic singularity of $x \log x$ type near the planar boundary.

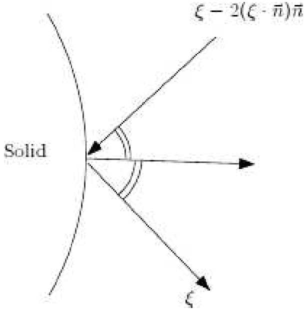
The computational results for Boltzmann equation support this.

In this talk, we shall confirm this phenomena by exact analysis for the thermal transportation problem of linearized Boltzmann equation.

In a diluted gas, thermal transpiration is a flow induced by temperature gradient of the boundary. It can exist without pressure gradient and can not be explained by fluid dynamics.

The first explanation in physics is by James Clerk Maxwell in the paper published in 1879. The key point is to introduce diffuse-reflection boundary condition.

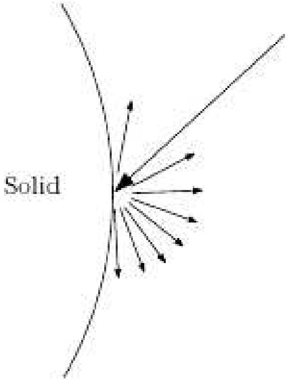
Boundary Conditions



Specular- reflection condition
$f(x,t,\xi) = f(x,t,\xi - 2(\xi \cdot \vec{n})\vec{n})$
for $\xi \cdot \vec{n} > 0$ on solid boundary

Thermal transpiration does not occur under this boundary condition.

Boundary Conditions



Diffuse-reflection condition

(i) The distribution leaving

the boundary is in thermal

equilibrium with the boundary.

(ii)There is no mass flux

across the boundary.

Thermal transpiration

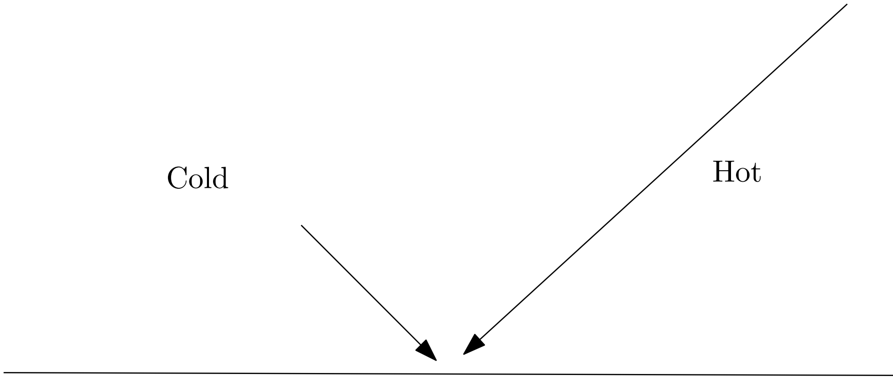
Without boundary, it is possible for a gas to have a temperature gradient without flow.

Cold

Hot

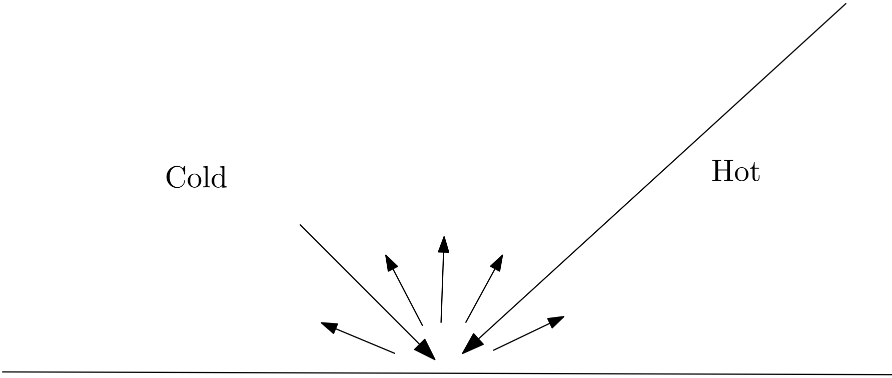
Thermal transpiration

We add a boundary with diffuse-reflection boundary condition.
Since the gas is very diluted, the molecules travel almost freely.



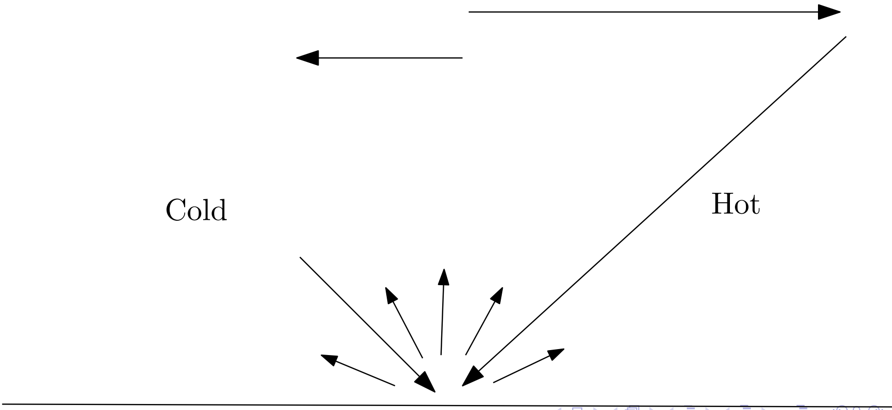
Thermal transpiration

When they hit the boundary, they scatter out isotropically because of the diffuse-reflection boundary condition.

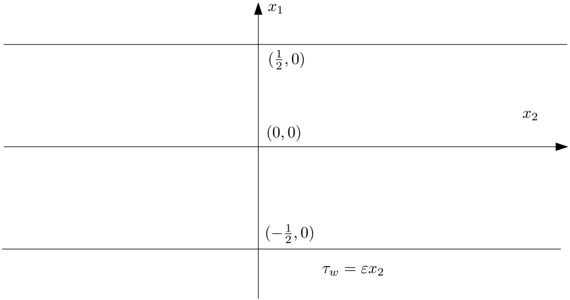


Thermal transpiration

Thus, there is a change of momentum in the horizontal direction. The total rate of change of momentum, the total force, is toward hot region.



Problem setup



We consider a highly diluted gas between two parallel plates and homogenous in x_3 direction. Perturbation of the wall temperature: $\tau_w = \varepsilon x_2$.

Problem setup

Gas region: stationary linearized Boltzmann equation

$$\sum_{i=1}^{2} \zeta_i \partial_{x_i} f(x,\zeta) = \frac{1}{\kappa} L(f) \text{ for } \kappa >> 1, \tag{1}$$

where κ is the Knudsen number, the ratio between mean free path and reference length, and L is the linearized collision operator.

We consider hard sphere potential. In this case,

$$L(f) = -\nu(|\zeta|) \cdot f + K(f), \tag{2}$$

where $\nu_0(1+\rho) < \nu(\rho) < \nu_1(1+\rho)$ and K integral operator with kernel

$$\begin{aligned} k(\zeta,\zeta_*) =& 2^{-\frac{1}{2}} \pi^{-1} |\zeta - \zeta_*|^{-1} \exp\left(\frac{|\zeta \times \zeta_*|^2}{|\zeta - \zeta_*|^2} - \frac{|\zeta|^2}{2} - \frac{|\zeta_*|^2}{2}\right) \\ & - 2^{-\frac{3}{2}} \pi^{-1} |\zeta - \zeta^*| \exp\left(-\frac{|\zeta|^2}{2} - \frac{|\zeta_*|^2}{2}\right) \end{aligned} \tag{3}$$

We consider the linearized diffuse-reflection boundary condition.

$$\begin{aligned} &f(\pm\tfrac{1}{2}, x_2, \zeta) = \check{\sigma}\Phi_0 + \varepsilon x_2 \Phi_4, \text{ for } \zeta_1 \lesseqgtr 0, \\ &\check{\sigma} = -\tfrac{1}{2}\varepsilon x_2 - 2\sqrt{\pi}\int_{\zeta_1 \gtrless 0} |\zeta_1| E^{\frac{1}{2}} f(\zeta) d\zeta. \end{aligned} \tag{4}$$

Here,

$$E = \pi^{-\frac{3}{2}} e^{-|\zeta|^2}, \tag{5}$$

and,

$$\Phi_0 = E^{\frac{1}{2}},\ \Phi_4 = (|\zeta|^2 - \frac{3}{2})E^{\frac{1}{2}}. \tag{6}$$

- Chen, Chen, Liu, Sone(2007)

For large enough κ, there exists a solution in L^∞. Furthermore, the pressure is a constant in x_2 direction and

$$0 < c_1 \ln \kappa \le q_2 \le c_2 \ln \kappa, \tag{7}$$

where q_2 is the macroscopic velocity in x_2 direction

$$q_2 = \int_{\mathbb{R}^3} \phi(\zeta)\zeta_2 E^{\frac{1}{2}} d\zeta. \tag{8}$$

- Takata, Funagane(2010)

  Numerical result. Asymptotic formula for total flux.

Chen, Liu, Takata, 2010

Theorem (Existence with precise description on velocity distribution fuction.)

Theorem (Asymptotic of Flow Velocity)

Moreover, the macroscopic flow velocity is as follows.

$$
\begin{aligned}
\frac{1}{\varepsilon} q_2(x_1) = & \frac{1}{4\sqrt{\pi}} \Big(\ln \kappa + \frac{1}{2} - \gamma - C_T - |x_1 + \frac{1}{2}| \ln |x_1 + \frac{1}{2}| \\
& - |x_1 - \frac{1}{2}| \ln |x_1 - \frac{1}{2}| \Big) + O\Big(\frac{\ln^2 \kappa}{\kappa}\Big).
\end{aligned}
$$

Theorem (Singularity Near Boundary)

The derivative of the flow velocity q_2 tends to infinity logarithmically near the boundary:

$$
\frac{1}{\varepsilon} \partial_{x_1} q_2 = \Big(\frac{1}{4\sqrt{\pi}} + O\Big(\frac{\ln \kappa}{\kappa}\Big) \Big) \Big(\ln |\frac{1}{2} - x_1| - \ln |\frac{1}{2} + x_1| \Big) + O\Big(\frac{(\ln \kappa)^2}{\kappa}\Big). \quad (9)
$$

Special form of solution

We consider

$$f(x,\zeta)=\varepsilon(\psi(x_1,\zeta)+x_2(|\zeta|^2-\frac{5}{2})E^{\frac{1}{2}}). \tag{10}$$

Under this assumption, the boundary value problem becomes

$$\begin{cases} \zeta_1\partial_{x_1}\psi(x_1,\zeta)=\frac{1}{\kappa}L(\psi)-\zeta_2(|\zeta|^2-\frac{5}{2}) & \text{when } -\frac{1}{2}<x_1<\frac{1}{2}, \\ \psi(\pm\frac{1}{2},\zeta)=\sigma_\pm\Phi_0 & \text{when } \zeta_1\lessgtr 0, \\ \sigma_\pm=2\sqrt{\pi}\int_{\zeta_1\gtrless 0}|\zeta_1|E^{\frac{1}{2}}\psi(\pm\frac{1}{2},\zeta)d\zeta. & \end{cases} \tag{11}$$

Letting

$$2\phi = \psi(x_1, \zeta_1, \zeta_2, \zeta_3) + \psi(-x_1, -\zeta_1, \zeta_2, \zeta_3) - (\sigma_+ + \sigma_-)\Phi_0,$$

we have

$$\begin{cases} \zeta_1 \partial_{x_1} \phi(x_1, \zeta) = \frac{1}{\kappa} L(\phi) - \zeta_2(\frac{5}{2} - |\zeta|^2) & \text{when } -\frac{1}{2} < x_1 < \frac{1}{2}, \\ \phi(\pm\frac{1}{2}, \zeta) = 0 & \text{when } \zeta_1 \lessgtr 0. \end{cases} \tag{12}$$

Integral Equation and iteration Scheme

Isolate the source term

$$\begin{cases} \zeta_1 \partial_{x_1} \phi(x_1, \zeta) + \frac{1}{\kappa}\nu(|\zeta|)(\phi) = \frac{1}{\kappa}K(\phi) - \zeta_2(|\zeta|^2 - \frac{5}{2}), \\ \phi(\pm\frac{1}{2}, \zeta) = 0 \end{cases} \quad \text{when } \zeta_1 \lessgtr 0. \tag{13}$$

Integral equation

$$\phi(x_1, \zeta) = \begin{cases} \int_{-\frac{1}{2}}^{x_1} \frac{1}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}|x_1 - s|} (\frac{1}{\kappa}K(\phi) - \zeta_2(|\zeta|^2 - \frac{5}{2})E^{\frac{1}{2}}) ds & \zeta_1 > 0 \\ \int_{x_1}^{\frac{1}{2}} \frac{1}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}|x_1 - s|} (\frac{1}{\kappa}K(\phi) - \zeta_2(|\zeta|^2 - \frac{5}{2})E^{\frac{1}{2}}) ds & \zeta_1 < 0. \end{cases} \tag{14}$$

We rewrite $\phi = H(\frac{1}{\kappa}K(\phi) - \zeta_2(|\zeta|^2 - \frac{5}{2})E^{\frac{1}{2}})$

$$\phi = H(\frac{1}{\kappa}K(\phi) - \zeta_2(|\zeta|^2 - \frac{5}{2})E^{\frac{1}{2}})$$

We start iteration with 0 function, we have

$$\begin{array}{ll} 0 & \phi = 0 \\ 1 & \phi = H(\zeta_2(\frac{5}{2} - |\zeta|^2)E^{\frac{1}{2}}) \\ 2 & \phi = H(\zeta_2(\frac{5}{2} - |\zeta|^2)E^{\frac{1}{2}}) + H(\frac{1}{\kappa}K(H(\zeta_2(\frac{5}{2} - |\zeta|^2)E^{\frac{1}{2}}))) \\ \ldots & \end{array} \tag{15}$$

Iteration scheme

$$\begin{array}{l} \phi_0(x_1, \zeta) = H(\zeta_2(\frac{5}{2} - |\zeta|^2)E^{\frac{1}{2}}), \\ \phi_n(x_1, \zeta) = H(\frac{1}{\kappa}K(\phi_{n-1})). \end{array} \tag{16}$$

Formally $\phi = \phi_0 + \phi_1 + \phi_2 \ldots\ldots$

Existence of Solution with Precise Velocity Distribution

Theorem

For any $0 < a < \frac{1}{2}$, there exist $D_a, \kappa_0 > 0$ such that, for every $\kappa > \kappa_0$ there exists a solution ϕ for (12), and

$$|\phi(x_1,\zeta) - \phi_0(x_1,\zeta)| \leq D_a \frac{\ln\kappa}{\kappa} g(x_1,\zeta) e^{-a|\zeta|^2}, \tag{17}$$

where $g \geq 0$ has following property

$$|g(x_1,\zeta)| \leq \begin{cases} \frac{x_1+\frac{1}{2}}{|\zeta_1|} & \text{for } \zeta_1 \geq \frac{\nu(|\zeta|)|x_1+\frac{1}{2}|}{\kappa} \\ \frac{\kappa}{\nu(|\zeta|)} & \text{for } -\frac{\nu(|\zeta|)x_1-\frac{1}{2}|}{\kappa} < \zeta_1 < \frac{\nu(|\zeta|)|x_1+\frac{1}{2}|}{\kappa} \\ \frac{|x_1-\frac{1}{2}|}{|\zeta_1|} & \text{for } \zeta_1 \leq -\frac{\nu(|\zeta|)|x_1-\frac{1}{2}|}{\kappa}, \end{cases} \tag{18}$$

and ϕ_0 is given as follow:

$$\phi_0(x_1,\zeta) = \begin{cases} \int_{-\frac{1}{2}}^{x_1} \frac{1}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}|x_1-s|} \zeta_2(\frac{5}{2} - |\zeta|^2) E^{\frac{1}{2}}) ds & \text{for } \zeta_1 > 0 \\ \int_{x_1}^{\frac{1}{2}} \frac{1}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}|x_1-s|} \zeta_2(\frac{5}{2} - |\zeta|^2) E^{\frac{1}{2}}) ds & \text{for } \zeta_1 < 0. \end{cases}$$

Proof of singularity

For the first term, from the explicit formula and asymptotic for some special functions, we have

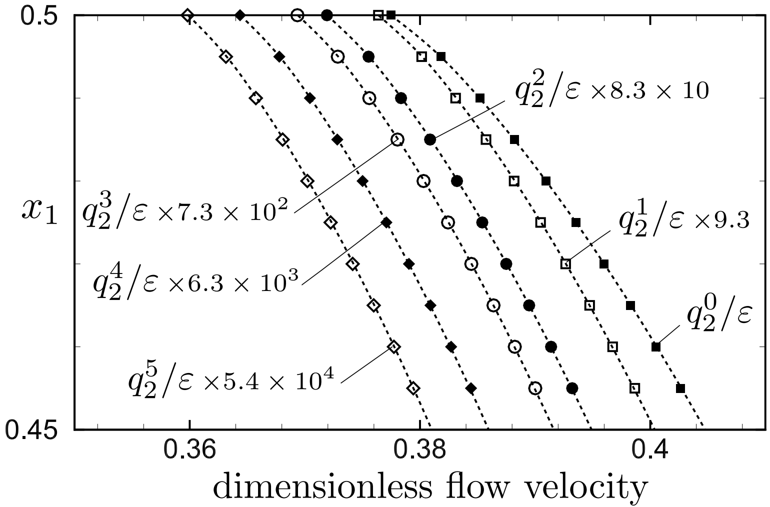
$$\frac{1}{\varepsilon}\partial_{x_1} q_2^0 = \frac{1}{4\sqrt{\pi}}(\ln|\frac{1}{2} - x_1| - \ln|\frac{1}{2} + x_1|) + O(\frac{1}{\kappa}). \tag{20}$$

The goal is to prove the remainder do not change the trend. The following lemma will do the job.

Lemma

$$\frac{1}{\varepsilon}|\frac{\partial}{\partial x_1} q_2^n(x_1)| \leq C(C_a \frac{\ln\kappa}{\kappa})^n(\ln\kappa + \left|\ln|\frac{1}{2} + x_1|\right| + \left|\ln|\frac{1}{2} - x_1|\right|) \tag{21}$$

Numerical demonstration



Calculated and plotted by Funagane

$$\begin{aligned}\frac{1}{\varepsilon}q_2^n(x_1) &= \int \zeta_2 E^{\frac{1}{2}}\phi_n d\zeta \qquad (22)\\ &= \int_{\zeta_1>0}\int_{-\frac{1}{2}}^{x_1}\zeta_2\frac{1}{|\zeta_1|}e^{-\frac{\nu}{\kappa|\zeta_1|}(x_1-s)}E^{\frac{1}{2}}\frac{1}{\kappa}K(\phi_{n-1})(s)dsd\zeta\\ &+ \int_{\zeta_1<0}\int_{x_1}^{\frac{1}{2}}\zeta_2\frac{1}{|\zeta_1|}e^{-\frac{\nu}{\kappa|\zeta_1|}|x_1-s|}E^{\frac{1}{2}}\frac{1}{\kappa}K(\phi_{n-1})(s)dsd\zeta.\end{aligned}$$

We deal with q_{2+}^n first, the other term can be treated similarly. We integrate by parts to obtain

$$\frac{1}{\varepsilon}q_{2+}^n(x_1) = -\int_{\zeta_1>0}\int_{-\frac{1}{2}}^{x_1}\zeta_2\frac{1}{\nu}e^{-\frac{\nu}{\kappa|\zeta_1|}(x_1-s)}E^{\frac{1}{2}}\partial_s K(\phi_{n-1})(s)dsd\zeta$$

$$+\int_{\zeta_1>0}\zeta_2\frac{1}{\nu}E^{\frac{1}{2}}(K(\phi_{n-1})(x_1) - e^{-\frac{\nu}{\kappa|\zeta_1|}(x_1+\frac{1}{2})}K(\phi_{n-1})(-\frac{1}{2}))d\zeta.$$

Differentiate the formula, we obtain

$$\frac{1}{\varepsilon}\partial_{x_1} q_{2+}^n(x_1) = \int_{\zeta_1>0} \zeta_2 \frac{1}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}(x_1+\frac{1}{2})} E^{\frac{1}{2}} \frac{1}{\kappa} K(\phi_{n-1})(-\frac{1}{2}) d\zeta$$
$$+ \int_{\zeta_1>0} \int_{-\frac{1}{2}}^{x_1} \zeta_2 \frac{1}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}(x_1-s)} E^{\frac{1}{2}} \frac{1}{\kappa} \partial_s K(\phi_{n-1})(s) ds d\zeta = SI_n + DI_n.$$

For the single integral part, using Lemma (**??**) and (**??**), we have the following estimate

$$|\frac{1}{\kappa} K(\phi_{n-1})| < C_a'(C_a \frac{\ln\kappa}{\kappa})^n e^{-a|\zeta|^2}. \tag{23}$$

We use the estimate $\nu(\zeta) > \nu_0$ and integrate over ζ_2, ζ_3 to obtain

$$SI_n \leq C_a'(C_a \frac{\ln \kappa}{\kappa})^n \int_{\zeta_1 > 0} e^{-\frac{\nu_0}{\kappa|\zeta_1|}|\frac{1}{2}+x_1|} \frac{1}{|\zeta_1|} e^{-(a+\frac{1}{2})|\zeta_1|^2} d\zeta_1, \quad (24)$$

We break the domain of the integral into tree parts $(0, \frac{\nu_0|\frac{1}{2}+x_1|}{\kappa})$, $(\frac{\nu_0|\frac{1}{2}+x_1|}{\kappa}, \frac{1}{\sqrt{a+\frac{1}{2}}})$, and $(\frac{1}{\sqrt{a+\frac{1}{2}}}, \infty)$. and obtain

$$|SI_n| \leq C_a'(C_a \frac{\ln \kappa}{\kappa})^n (\ln \kappa + \big| \ln |\frac{1}{2} + x_1| \big|) \quad (25)$$

For the double integral part, we can prove the following lemma by induction.

Lemma

$$\partial_s K(\phi_n) \leq C(C_a \frac{\ln\kappa}{\kappa})^n (\ln\kappa + \big|\ln|\frac{1}{2}+s|\big| + \big|\ln|\frac{1}{2}-s|\big|). \quad (26)$$

Thus,

$$|DI_n| \leq C(C_a \frac{\ln\kappa}{\kappa})^{n-1} \frac{1}{\kappa}$$
$$\cdot \int_{\zeta_1>0} \int_{-\frac{1}{2}}^{x_1} \frac{\zeta_2}{|\zeta_1|} e^{-\frac{\nu}{\kappa|\zeta_1|}(x_1-s)} E^{\frac{1}{2}} (\ln\kappa + \big|\ln|s+\frac{1}{2}|\big| + \big|\ln|s-\frac{1}{2}|\big|) ds d\zeta$$
$$\leq \frac{(C_a \ln\kappa)^{n-1}}{\kappa^n}$$
$$\cdot \int_{-\frac{1}{2}}^{x_1} (\ln\kappa + |\ln(x_1-s)|)(\ln\kappa + \big|\ln|\frac{1}{2}+s|\big| + \big|\ln|\frac{1}{2}-s|\big| ds$$
$$\leq C_a'(C_a \frac{\ln\kappa}{\kappa})^n \ln\kappa.$$

Thank You!