An Adaptive Moving Finite Volume Scheme for Shallow Water Equations with Dry and Complex Topography

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Outline

1. The Problem
2. Numerical Scheme
   - PDEs discretization
   - Adaptive Moving mesh methods
   - Solution algorithm
3. Numerical Examples
   - Still water flow over a smooth bed
   - Still water with a small perturbation
   - Analytical transcritical flow with shock
   - Numerical dam-break inundation with closed boundary

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Malpasset dambreak in France

Movie

Governing equation

Shallow water equations with bed slope and friction stress source terms

$$\begin{cases} h_t + \nabla \cdot (h\vec{u}) = 0, \\ h\vec{u} + \nabla \cdot (h\vec{u} \otimes \vec{u}) + \nabla \left(\frac{1}{2} g h^2\right) = -gh\nabla b - \tau_{\mathbf{b}}/\rho, \end{cases} \tag{1}$$

Manning friction model

$$\tau_b = (\tau_{bx}, \tau_{by}) = \rho C_f \vec{u}|\vec{u}| \tag{2}$$

where

$$C_f = gn^2/h^{1/3}$$

:

Still water equilibrium

$$\begin{cases} \vec{u} = 0, \\ w = h + b = \max(C, b) \end{cases} \tag{3}$$

water depth positivity

$$h \geq 0 \tag{4}$$

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The history Numerical method for SWEs

- Perthame B, Simeoni C. A kinetic scheme for the Saint-Venant system with a source term. *Calcolo* 2001; **38**(4):201–231.
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- Noelle S, Xing Y, Shu C. High-order well-balanced finite volume WENO schemes for shallow water equation with moving water. *Journal of Computational Physics* 2007; **226**(1):29–58.
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- Li J, Chen G. The generalized Riemann problem method for the shallow water equations with bottom topography. *International Journal for Numerical Methods in Engineering* 2006; **65**(6):834–862.
- Audusse E, Bouchut F, Bristeau M, Klein R, Perthame B. A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows. *Journal of Scientific Computation* 2004; **25**(6):2050–2065.
- Audusse E, Bristeau M. A well-balanced positivity preserving "second-order" scheme for shallow water flows on unstructured meshes. *Journal of Computational Physics* 2005; **206**(1):311–333.
- Berthon C, Foucher F. Efficient well–balanced hydrostatic upwind schemes for shallow–water equations. *Journal of Computational Physics* 2012; .
- . . .

PDEs discretization

Splitting scheme

$$\begin{cases} \widetilde{\vec{U}}_i = \vec{U}_i^n - \frac{\Delta t}{|E_i|} \sum_{j=1}^{3} \vec{F}_{\vec{n}_{ij}}(\vec{U})_{ij}^n |l_{ij}| + \Delta t \vec{S}_b(\vec{U})_i^n \\ (hu)_i^{n+1} = \widetilde{\vec{u}}_i - \Delta t S_f(\widetilde{\vec{U}}_i). \end{cases} \tag{5}$$

[1] Li J, Chen G. The generalized Riemann problem method for the shallow water equations with bottom topography. *International Journal for Numerical Methods in Engineering* 2006; **65**(6):834–862.

[2] Audusse E, Bristeau M. A well-balanced positivity preserving "second-order" scheme for shallow water flows on unstructured meshes. *Journal of Computational Physics* 2005; **206**(1):311–333.

Flux and Bed slop discretization

STEP 1: Hydrostatic reconstruction

$$\left\{\begin{aligned} b_{ij} &= \max(b_{ij}^L, b_{ij}^R), \\ h_{ij}^{L,*} &= \max(b_{ij}^L + h_{ij}^L - b_{ij}, 0), \\ h_{ij}^{R,*} &= \max(b_{ij}^R + h_{ij}^R - b_{ij}, 0) \end{aligned}\right. \Rightarrow \left\{\begin{aligned} \vec{U}_{ij}^L &= (h_{ij}^{L,*}, h_{ij}^{L,*}\vec{u}_{ij}^L) \\ \vec{U}_{ij}^R &= (h_{ij}^{R,*}, h_{ij}^{R,*}\vec{u}_{ij}^R) \end{aligned}\right. \tag{6}$$

STEP 2: Flux evaluation

$$\vec{F}_{\vec{n}_{ij}}(\vec{U})_{ij}^n = \vec{F}_{\vec{n}_{ij}}(\vec{U}_{ij}^L, \vec{U}_{ij}^R) + \frac{g}{2}((h_{ij}^L)^2 - (h_{ij}^{L,*})^2) \tag{7}$$

STEP 3: Bed slop discretization

$$\vec{S}_b(\vec{U})_i^n = -\frac{1}{|E_i|}\sum_{j=1}^{3}\begin{pmatrix} 0 \\ g\,\frac{h_{ij}^L + \bar{h}_i}{2}\left(b_{ij}^L - b_i\right)|l_{ij}|\vec{n}_{ij} \end{pmatrix}, \tag{8}$$

Friction term discretization

A partially implicit approach

$$\frac{(h\vec{u})^{n+1} - (\tilde{h}\tilde{\vec{u}})}{\Delta t} = -\frac{gn^2(h\vec{u})^{n+1}|\tilde{\vec{u}}|}{(\tilde{h})^{4/3}}, \tag{9}$$

then

$$(h\vec{u})^{n+1} = \frac{(\tilde{h}\tilde{\vec{u}})}{1 + \Delta t g n^2 |\tilde{\vec{u}}|/(\tilde{h})^{4/3}} = \frac{(\tilde{h}\tilde{\vec{u}})(\tilde{h})^{4/3}}{(\tilde{h})^{4/3} + \Delta t g n^2 |\tilde{\vec{u}}|}, \tag{10}$$

Properties

Well-balance, Positivity, Stable friction discretization.

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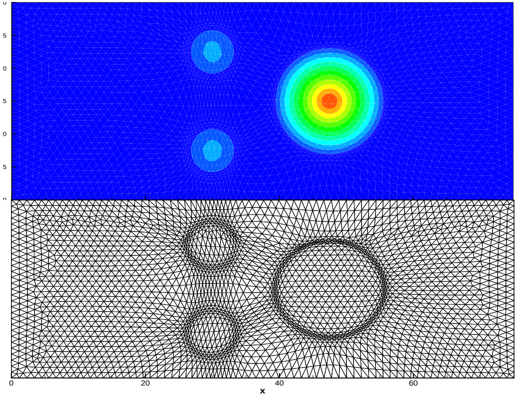
The need of Adaptive method

- Large variation of solution in the local domain.
- To see more detail information
- To capture the Wet/dry front



Classification of Adaptive method

- **Refinement method(h-method)**:
- **Polynomial method(p-method):**
- **Moving mesh method (r-method):**

The history of moving mesh method

- Winslow's moving mesh method based on variational principle;
  A. Winslow, *J. Comput. Phys.*, 1(1967), pp. 149-172.
- de Boor's equal-distribution principle;
  C. de Boor, Lecture notes in Math. 363 Springer-Verlag, Berlin,1973.
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- Harmonic map methods: R.Li, Tao. Tang and P.W.Zhang;
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- Tang H. Solution of the shallow-water equations using an adaptive moving mesh method. *International Journal for Numerical Methods in Fluids* 2004; **44**(7):789–810.
- Chen G, Tang H, Zhang P. Second-order accurate Godunov scheme for multicomponent flows on moving triangular meshes. *Journal of Scientific Computing* 2008; **34**(1):64–86.
- . . .

Main three components of moving mesh method

- **The Moving of meshes:** A mesh PDE decide a mapping from the logical domain to the physical domain. Solving it by FEM/FVM will get a new position of the meshes of the physical domain.
- **Monitor function:** To control the density of grid.
- **The interpolation of solution from old mesh to new mesh:**: conservation, maximum principle,divergence free. Still water preserving, Positivity

From the variational principle,

$$\tilde{E}(\boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{d}\int_{\Omega_l} (\tilde{\nabla}\boldsymbol{x}_i)^T G_i \tilde{\nabla}\boldsymbol{x}_i \ d\boldsymbol{\xi},$$

we get the Euler-Lagrange equation

$$\tilde{\nabla}\cdot(\omega\tilde{\nabla}\vec{x}) = 0.$$

It gives a mapping from the logical domain to the physical domain.

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[1] A. Winslow, *J. Comput. Phys.*, 1(1967), pp. 149-172.

[2] H.Z. Tang and T. Tang, *SIAM J. Numer. Anal.*, 41(2001), pp. 487-515.

[3] G.X. Chen, H.Z. Tang and P.W. Zhang, *J. Sci. Comput.*, 34(2008), pp. 64-86.

FEM for the grid PDE

Let the meshes on Ω_l is $\mathcal{T}_l$, We choose the trial function space

$$V_h = \left\{ v \in C^0(\Omega_l) : v|_E \in P^1(E), \quad \forall E \in \mathcal{T}_l, \text{and} v|_{\partial\Omega_l} = \vec{x}^n|_{\partial\Omega_l} \right\},$$

and the test function space

$$V_{h,0} = \left\{ v \in C^0(\Omega_l) : v|_E \in P^1(E), \quad \forall E \in \mathcal{T}_l, \text{and} v|_{\partial\Omega_l} = 0 \right\}.$$

Find $\vec{x}_h(\vec{\xi}) \in V_h$ such that

$$\int_{\Omega_l} \omega \tilde{\nabla} \vec{x}_h(\vec{\xi}) \cdot \tilde{\nabla} v(\vec{\xi}) d\xi = 0, \quad \forall v \in V_{h,0}.$$

Let $\vec{x}_h(\vec{\xi}) = \sum_{j=1}^{nd} \vec{x}_i \phi^i(\vec{\xi})$, then

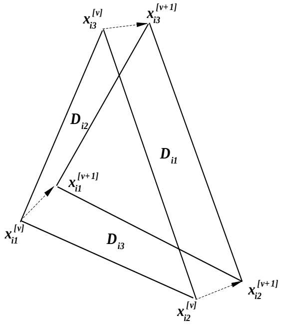
$$\sum_{j=1}^{nd} \vec{x}_j \int_{\Omega_l} \omega \tilde{\nabla} \phi^j(\vec{\xi}) \cdot \tilde{\nabla} \phi^i(\vec{\xi}) d\xi = 0, \quad i = 1, 2, \cdots, ndi,$$

let

$$a_{ij} = \int_{\Omega_l} \omega \tilde{\nabla} \phi^j(\vec{\xi}) \cdot \tilde{\nabla} \phi^i(\vec{\xi}) d\xi.$$

Relax Jacobi iterations method

$$
\begin{cases}
\hat{\vec{x}}_i & = & -\sum\limits_{j\neq i} \vec{x}_j^{[\nu]} \cdot a_{ij}/a_{ii}, \\
\\
\vec{x}_i^{[\nu+1]} & = & \vec{x}_i^{[\nu]} + \theta\left(\hat{\vec{x}}_i - \vec{x}_i^{[\nu]}\right),
\end{cases}
$$

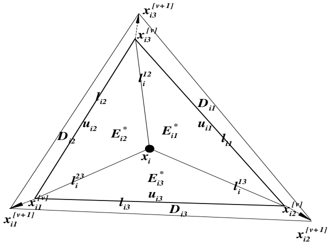


To avoid the grid winding:CFL type condition

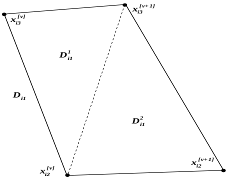
$$\max f(\tau) = \tau$$
$$\text{s.t.} - \min\{|D_{ij}(\tau)|, 0\} < |E^*_{ij}| := \frac{|E_i^{[\nu]}|}{\Lambda(d)}, \quad \forall i, j,$$

$\Rightarrow$

$$|E_i^{[\nu+1]}| = |E_i^{[\nu]}| + \sum_{j=1}^{\Lambda(d)} |D_{ij}| = \sum_{j=1}^{\Lambda(d)} \left(\frac{1}{\Lambda(d)}|E_i^{[\nu]}| + |D_{ij}|\right) > 0.$$



$$
\begin{aligned}
& |D_{i1}| = |D_{i1}^1| + |D_{i1}^2| \\
= \quad & \frac{1}{2}\left\langle \vec{x}_{i3}^{[\nu+1]} - \vec{x}_{i2}^{[\nu]}, \vec{x}_{i3}^{[\nu]} - \vec{x}_{i2}^{[\nu]} \right\rangle + \frac{1}{2}\left\langle \vec{x}_{i2}^{[\nu+1]} - \vec{x}_{i2}^{[\nu]}, \vec{x}_{i3}^{[\nu+1]} - \vec{x}_{i2}^{[\nu]} \right\rangle \\
= \quad & \frac{1}{2}\left\langle \vec{x}_{i2}^{[\nu+1]} - \vec{x}_{i3}^{[\nu]}, \vec{x}_{i3}^{[\nu+1]} - \vec{x}_{i2}^{[\nu]} \right\rangle .
\end{aligned}
$$

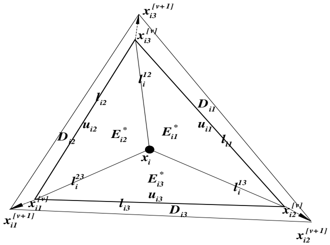


[1] J.Q. Han and H.Z. Tang, *J. Comput. Phys.*, 220(2007), pp. 791-812.

[2] G.X. Chen, H.Z. Tang and P.W. Zhang, *J. Sci. Comput.*, 34(2008), pp. 64-86.

Properties of $|D_{ij}|$

$$\begin{cases} |D_{ij}| & = -|D_{ji}|, \\ & \\ |E_i^{[\nu+1]}| & = |E_i^{[\nu]}| + \displaystyle\sum_{j=1}^{\Lambda(d)} |D_{ij}|. \end{cases}$$



Update the average of the solution: Conservative

$$|E_i^{[\nu+1]}|\overline{u}_{E_i^{[\nu+1]}} = |E_i^{[\nu]}|\overline{u}_{E_i^{[\nu]}} + \sum_{j=1}^{\Lambda(d)} \widetilde{\vec{F}}(u_{ij}^L, u_{ij}^R),$$

$$\widetilde{\vec{F}}(u_{ij}^L, u_{ij}^R) = \max\{|D_{ij}|, 0\}u_{ij}^R + \min\{|D_{ij}|, 0\}u_{ij}^L,$$

$$\widetilde{\vec{F}}(u_{ij}^L, u_{ij}^R) = -\widetilde{\vec{F}}(u_{ij}^R, u_{ij}^L)$$

$\Rightarrow$

$$\sum_i |E_i^{[\nu]}|\overline{u}_{E_i^{[\nu]}} = \sum_i |E_i^{[\nu+1]}|\overline{u}_{E_i^{[\nu+1]}}.$$

$$\widetilde{\mathcal{F}}(u_{ij}^L, u_{ij}^R) = \max\{|D_{ij}|, 0\}\mathcal{S}(u_{ij}^R) + \min\{|D_{ij}|, 0\}\mathcal{S}(u_{ij}^L).$$

Positivity

$$
\begin{aligned}
|E_i^{[\nu+1]}|\overline{h}_{E_i^{[\nu+1]}} &= |E_i^{[\nu]}|\overline{h}_{E_i^{[\nu]}} + \sum_{j=1}^{\Lambda(d)} (\max\{|D_{ij}|, 0\} h_{ij}^R + \min\{|D_{ij}|, 0\} h_{ij}^L), \\
&= \sum_{j=1}^{\Lambda(d)} \frac{|E_i^{[\nu]}|}{\Lambda(d)} \overline{h}_{ij}^L + \sum_{j=1}^{\Lambda(d)} (\max\{|D_{ij}|, 0\} h_{ij}^R + \min\{|D_{ij}|, 0\} h_{ij}^L), \\
&= \sum_{j=1}^{\Lambda(d)} (\frac{|E_i^{[\nu]}|}{\Lambda(d)} + \min\{|D_{ij}|, 0\}) h_{ij}^L + \sum_{j=1}^{\Lambda(d)} \max\{|D_{ij}|, 0\} h_{ij}^R, \\
&\geq 0
\end{aligned}
$$

Still water preserving

$$
\begin{aligned}
|E_i^{[\nu+1]}|\overline{q}_{E_i^{[\nu+1]}} &= |E_i^{[\nu]}|\overline{q}_{E_i^{[\nu]}} + \sum_{j=1}^{\Lambda(d)} (\max\{|D_{ij}|, 0\} q_{ij}^R + \min\{|D_{ij}|, 0\} q_{ij}^L), \\
&= |E_i^{[\nu]}| C + \sum_{j=1}^{\Lambda(d)} (\max\{|D_{ij}|, 0\} C + \min\{|D_{ij}|, 0\} C), \\
&= (|E_i^{[\nu]}| + \sum_{j=1}^{\Lambda(d)} (\max\{|D_{ij}|, 0\} + \min\{|D_{ij}|, 0\}))\, C,
\end{aligned}
$$

$\Rightarrow$

$$
\overline{q}_{E_i^{[\nu+1]}} = C
$$

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Solution algorithm

Step 1: Adaptive moving of triangular meshes

- A1 Adjust the cells $\vec{x}^{[\nu]}$ to the new position $\vec{x}^{[\nu+1]}$.
- A2 Update the conservative flow variables $\vec{U}_i^{[\nu+1]}$.
- A3 Stop criterion: $\left|\vec{x}_i^{[\nu+1]} - \vec{x}_i^{[\nu]}\right| < \epsilon$ .or. $\nu = 5$

Step 2: Update $\vec{U}_i^{n+1}$ over the new fixed meshes $\vec{x}_i^{n+1}$:.

- B1 Update the solution $\widetilde{\vec{U}}_i$ without the friction .
- B2 Update the solution $\vec{U}_i^{n+1}$ for friction term .

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Arc-Length type Monitor function

$$\omega_i = \sqrt{1 + \alpha \tilde{\omega}_i^2(\beta, h+b)}$$

,

$$\tilde{\omega}_i(\beta, h+b) =: \min\left\{1, |\nabla_\xi(h+b)|_i \Big/ \beta \max_i |\nabla_\xi(h+b)|_i\right\}$$

$$\alpha = 40.0 \quad \beta = 0.25$$

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Still water flow over a smooth bed

Bottom topograph

$$B(x,y) = 0.8\exp\left\{-50\left[(x-0.5)^2 + (y-0.5)^2\right]\right\}, (x,y) \in [0,1]\times[0,1], \quad (11)$$

Initial condition

$$\begin{cases} h(x,y) = 1 - B(x,y), \\ \vec{u}(x,y,0) = 0. \end{cases} \quad (12)$$

Results at time $t = 1.7s$

Algorithm	$\lvert h+b-1\rvert$	$\lvert u\rvert$	$\lvert v\rvert$
Fixed meshes	2.00E-15	4.06E-14	4.44E-14
Moving meshes	7.22E-15	1.39E-14	1.77E-14

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Bottom topograph

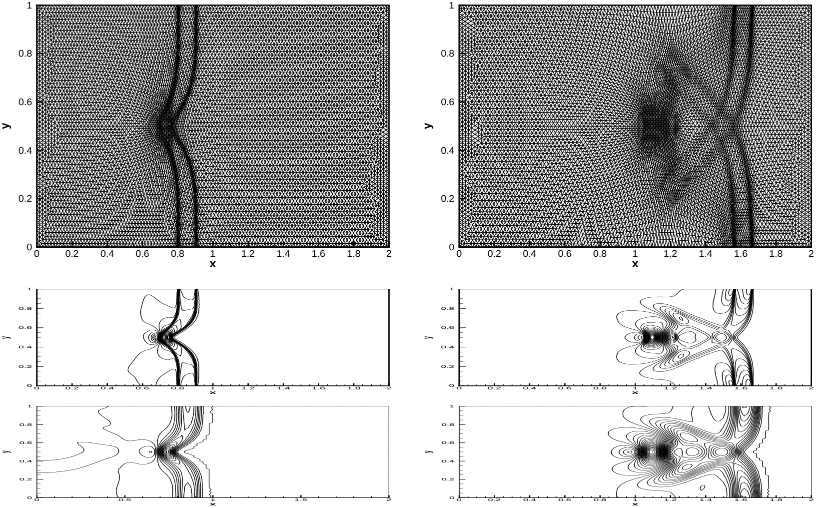
$$B\left(x,y\right)=0.8\exp\left\{-5(x-0.9)^2+50(y-0.5)^2\right\},(x,y)\in[0,2]\times[0,1]. \qquad (13)$$

Still Initial condition

$$h\left(x,y\right)=\begin{cases}1{+}\varepsilon-B\left(x,y\right), & \text{if } x\in[0.05,0.15]\,\mathrm{m},\\ 1-B\left(x,y\right), & \text{Otherwise},\end{cases} \qquad (14)$$

where $\varepsilon=0.01$

Numerical Results at $t = 0.24s$ and $0.48s$



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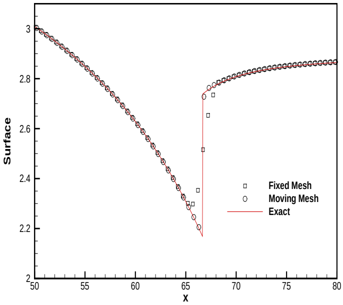
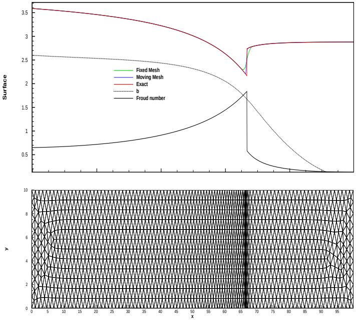
Initial and Boundary conditions

$$\text{IC}: q = 2m^2/s; h(x) = \max(h_{st}(100) + B(100) - B(x), 0.0)$$

$$\text{BC}: \text{upstream}: q = 2; \quad \text{downstream}: h_{st}(100)$$





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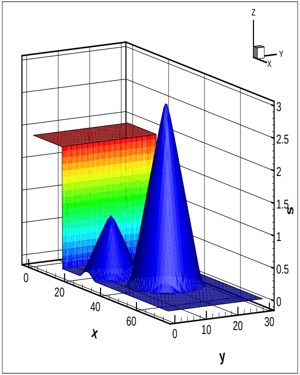
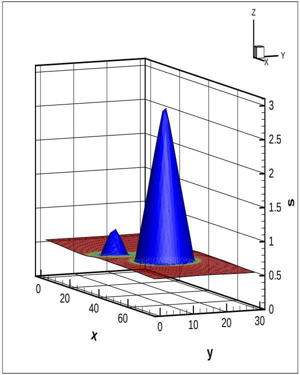
Initial and Boundary conditions

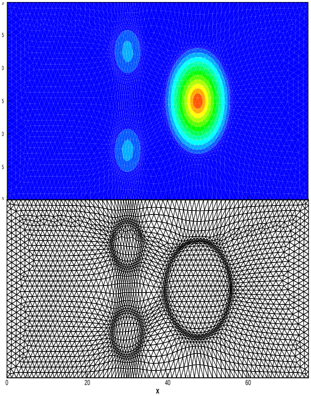
$$\text{IC}: h(x) = 1.875 \quad x < 16$$

$$\text{BC}: \text{ reflection boundary condition}$$





Close Channel dam-break.



movie

Malpasset dambreak in France

Movie

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