Boundary Layer problem: Navier-Stokes equations and Euler equations

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1. *Statement of the problem - Vanishing viscosity limit*

2. *First main result*
    - Estimates, independent of the viscosity
    - Trace value of the vorticity
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    - Strong convergence

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Navier-Stokes equations. Dirichlet condition

$$\begin{cases} \mathbf{v}_t + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \nabla p = \nu \Delta \mathbf{v} & \text{in} \quad \Omega_T := \Omega \times (0, T), \\ \operatorname{div} \mathbf{v} = 0 \quad \text{in} \quad \Omega_T := \Omega \times (0, T), \\ \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{cases}$$

Homogeneous Dirichlet boundary condition

$$\mathbf{v} = \mathbf{0} \qquad \text{on} \quad \Gamma_T := \Gamma \times (0, T).$$

- Bucur D., Feireisl E., Necasova S.,
  Boundary Behavior of Viscous Fluids: influence of wall roughness ..., *Arch. Rational Mech. Anal.*, **197**, (2010).

- Priezjev N.V., Troian S.M.,
  Influence of periodic wall roughness on the slip behavior at liquid/solid interfaces: ..., *J. Fluid Mech.*, **554**, (2006).

Navier-slip condition

Flux of the fluid through the boundary Γ

$$\mathbf{v}\cdot \mathrm{n} = a \qquad \text{on} \qquad \Gamma_T \qquad \text{with} \qquad \int_\Gamma a\, d\mathbf{x} = 0, \qquad \forall t \in [0, T],$$

Navier´s slip boundary condition

$$2D(\mathbf{v})\mathrm{n}\cdot \mathrm{s} + \alpha \mathbf{v}\cdot \mathrm{s} = \beta \qquad \text{on} \qquad \Gamma_T,$$

$D(\mathbf{v}) = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$ - the rate-of-strain tensor;
n and s - the external normal and the tangent vector to Γ.
α, β describe physical properties of Γ.

Vorticity formulation

The Navier-Stokes equations in terms of the vorticity $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$ and the velocity $\mathbf{v}$

$$\begin{cases} \partial_t \omega + \mathrm{div}\,(\mathbf{v}\omega) = \nu \Delta \omega, \qquad \mathrm{div}\mathbf{v} = 0 \qquad \text{in} \quad \Omega_T, \\ \omega(\mathbf{x}, 0) = \omega_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \qquad \text{with} \quad \omega_0 := \mathrm{rot}\,(\mathbf{v}_0)\,, \end{cases}$$

$$\mathbf{v} \cdot \mathrm{n} = a \qquad \text{on} \quad \Gamma_T.$$

Navier slip boundary conditions in terms of the vorticity

$$\omega = b(\mathbf{v}) \quad \text{on} \quad \Gamma_T$$

with

$$b(\mathbf{v}) := (\alpha - 2k)\,\mathbf{v} \cdot \mathrm{s} + b, \qquad b = \beta - 2\,a_s'.$$

The Euler equations are equivalent to the system

$$\begin{cases} \partial_t \omega + \mathbf{v} \cdot \nabla \omega = 0 \quad \text{in} \quad \Omega_T, \\ \omega(\mathbf{x}, 0) = \omega_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \quad \text{with} \quad \omega_0 := \mathrm{rot}\,(\mathbf{v}_0)\,, \\ \mathbf{v} \cdot \mathrm{n} = a \quad \text{on} \quad \Gamma_T. \end{cases}$$

The trajectories for the transport equation start at $t = 0$ and on the inflow region Γ^-, where $a < 0$.

The Navier-slip boundary condition is given just on Γ^-

$$\omega = b(\mathbf{v}).$$

Homogeneous case

Navier-Stokes equations with

$$\mathbf{v}\cdot \mathrm{n} = 0, \qquad \omega = (\alpha - 2k)\mathbf{v}\cdot \mathrm{s} \quad \text{on} \quad \Gamma$$

Euler equations with

$$\mathbf{v}\cdot \mathrm{n} = 0 \qquad \text{on} \qquad \Gamma$$

For $\alpha = 2k$:

- Lions P.-L., Math. Topics in Fluid Mechanics (1996)
- Beirão da Veiga H., Crispo F., *J. Math. Fluid Mech.*, **13,** (2011)

For $\alpha \neq 2k$:

- Clopeau T., Mikelic A., Robert R.,*Nonlinearity* **11**, (1998)
- Lopes Filho M.C., Nussenzveig Lopes H.J., Planas G., *SIAM Math. Anal.*, **36**, 4, (2005)
- Kelliher J., *SIAM J. Math. Anal.* **38**, 1, (2006)

Non-Homogeneous case

We consider the boundary conditions

$$\begin{cases} \mathbf{v} \cdot \mathrm{n} = a, \\ \omega = b(\mathbf{v}) \end{cases} \quad \text{on} \quad \Gamma$$

for the Navier-Stokes equations

and

$$\begin{cases} \mathbf{v} \cdot \mathrm{n} = a & \text{on} \quad \Gamma, \\ \omega = b(\mathbf{v}) & \text{on} \quad \Gamma^{-} \end{cases}$$

for the Euler equations.

Solvability results of the Euler equations

In L_p for any $p \in (1, \infty]$:

- Chemetov N.V., Antontsev S.N., *Physica D: Nonlinear Phenomena*, **237**, 1, (2008).
- Chemetov N.V., Cipriano F.; Gavrilyuk, S., *Math. Methods Appl. Sci.* **33**, 6, (2010).

First main result

Theorem 1

Let $a \in L_\infty(0,T;W_p^1(\Gamma)) \cap W_2^1(0,T;W_2^{-\frac{1}{2}}(\Gamma))$,

$$b \in W_p^{2,1}(\Gamma_T), \qquad \omega_0 \in W_p^2(\Omega), \qquad \forall p \in (2,+\infty].$$

$\exists$ *a subsequence of* $\{\omega_\nu, \mathbf{v}_\nu\}$, *solutions of the N-S:*

$$\begin{cases} \omega_\nu \rightharpoonup \omega & \text{weakly-* in} \quad L_\infty(0,T;L_p(\Omega)), \\ \mathbf{v}_\nu \to \mathbf{v} & \text{strongly in} \quad C(\overline{\Omega}_T), \end{cases}$$

$$\int_{\Omega_T} \omega(\psi_t + \mathbf{v} \cdot \nabla\psi)\, d\mathbf{x}dt + \int_\Omega \omega_0\, \psi(\mathbf{x},0)\, d\mathbf{x} = \int_{\Gamma_T^-} a\, b(\mathbf{v})\, \psi\, d\mathbf{x}dt$$

for every $\psi \in C^{1,1}(\overline{\Omega}_T)$ *with* $\psi = 0$ *at* $t = 0$ *and on* Γ^+.

Sketch of the proof

1st step : Estimates, independent of the viscosity

Lemma

$\exists$ *a weak solution* $\{\omega_\nu, \mathbf{v}_\nu\}$ *of the Navier-Stokes equations:*

$$||\omega_\nu||_{L_\infty(0,T;L_p(\Omega))} \leqslant C,$$
$$||\mathbf{v}_\nu||_{L_\infty(0,T;\, W_p^1(\Omega))} \leqslant C, \qquad ||\partial_t \mathbf{v}_\nu||_{L_2(\Omega_T)} \leqslant C.$$

2 important estimates:

$$\|\mathbf{v}_\nu(t)\|^2_{L_2(\Omega)} \leqslant C\Big(\|\mathbf{v}_0\|^2_{L_2(\Omega)} + \int_0^t f(s)\|\omega_\nu(s)\|^2_{L_p(\Omega)}\, ds + 1\Big)$$

and

$$\|\omega_\nu(t)\|^2_{L_p(\Omega)} \leqslant C(\|\mathbf{v}_\nu(t)\|^2_{L_2(\Omega)} + \int_0^t f(s)\|\omega_\nu(s)\|^2_{L_p(\Omega)}\, ds + 1),$$

where the constants C and $f(t) \in L_1(0,T)$ are independent of ν.

2nd step : Trace value of the vorticity

Let B be an extension in Ω_T of the initial condition $B\big|_{t=0} = \omega_0$ and boundary condition $B\big|_{\Gamma_T} = b(\mathbf{v}_\nu)$;

$d(\mathbf{x})$-the distance function from $\mathbf{x} \in \Omega$ to Γ.

Lemma

$$\lim_{\sigma\to 0^+} \left(\overline{\lim_{\nu\to 0^+}} \frac{1}{\sigma} \int_{\Omega_T \cap [0<d<\sigma]} |\omega_\nu - B| \, |(\mathbf{v}_\nu \bigtriangledown d)|^- \, d\mathbf{x}dt \right) = 0.$$

3d step : Weak convergence

Taking $\nu \to 0$:

$$\begin{cases} \omega_\nu \rightharpoonup \omega & \text{weakly-* in } L_\infty(0,T;L_p(\Omega)), \\ \mathbf{v}_\nu \to \mathbf{v} & \text{strongly in } C(\overline{\Omega}_T), \end{cases}$$

we will obtain the Euler equation

$$\int_{\Omega_T} \omega(\psi_t + \mathbf{v}\cdot\nabla\psi)\, d\mathbf{x}dt + \int_\Omega \omega_0\, \psi(\mathbf{x},0)\, d\mathbf{x} = \int_{\Gamma_T^-} a\, b(\mathbf{v})\, \psi\, d\mathbf{x}dt$$

Second main result

Theorem 2

Let $\omega_0 \in L_p(\Omega)$, $\qquad a \in L_\infty(0,T; W_p^1(\Gamma))$,

$$b \in L_2(0,T; W_p^1(\Gamma) \cap W_2^1(0,T; W_p^{-\frac{1}{p}}(\Gamma)), \qquad \forall p \in (2,+\infty].$$

$\exists$ a subsequence of $\{\omega_\nu, \mathbf{v}_\nu\}$, solutions of N-S:

$$\begin{cases} \omega_\nu \to \omega & \text{strongly in} \quad L_r(0,T; L_p(\Omega)),\ \forall r < \infty, \\ \mathbf{v}_\nu \to \mathbf{v} & \text{strongly in} \quad L_r(0,T; W_p^1(\Omega)), \end{cases}$$

$$\int_{\Omega_T} \beta(\omega)\,(\psi_t + \mathbf{v}\cdot\nabla\psi)\,d\mathbf{x}dt + \int_\Omega \beta(\omega_0)\,\psi(\mathbf{x},0)\,d\mathbf{x} = \int_{\Gamma_T^-} a\beta(b(\mathbf{v}))\psi\,d\mathbf{x}dt$$

holds for any $\beta \in C(\mathbb{R})$ and any test function ψ.

Sketch of the proof

Uniform estimates

Lemma

$\exists$ *a weak solution* $\{\omega_\nu, \mathbf{v}_\nu\}$ *of N-S:*

$$\begin{aligned} ||\omega_\nu||_{L_\infty(0,T;L_p(\Omega))} &\leqslant C, \\ ||\mathbf{v}_\nu||_{L_\infty(0,T;\,W^1_p(\Omega))} &\leqslant C, \qquad ||\partial_t(\mathbf{v}_\nu - \nabla A)||_{L_2(\Omega_T)} \leqslant C, \end{aligned}$$

where A *is the solution of the system*

$$\begin{cases} -\Delta A = 0 & \text{in} \quad \Omega, \\ \frac{\partial A}{\partial \mathbf{n}} = a & \text{on} \quad \Gamma \end{cases}$$

Strong convergence

- Malek J., Necas J., Rokyta M., Ruzicka M., Weak and measure-valued solutions to evolutionary PDEs. (1996).

We denote $h(\mathbf{x}) = \min\{\delta,\ d(\mathbf{x})\}$ for any $\mathbf{x} \in \Omega$ with a fixed $\delta > 0$.

We consider the cut-off function

$$\xi_\nu(\mathbf{x}) := 1 - \exp(-\frac{M + \nu L}{\nu}\, h(\mathbf{x})), \qquad \forall \nu > 0,$$

with $\|\mathbf{v}_\nu\|_{L_\infty(\Omega_T)} \leqslant M$, Multiplying the Navier-Stokes equations by $\eta'(\omega_\nu)\xi_\nu\psi$, with $\eta \in C^2(\mathbb{R})$ convex and $\psi \in C_0^\infty(\mathbb{R}^{2+1})$ *non-negative*, we obtain

$$\int_{\Omega_T} \eta(\omega_\nu)\,[\psi_t + \mathbf{v}_\nu \cdot \nabla\psi + \nu\,\Delta\psi]\,\xi_\nu + 2\,\nu\,\eta(\omega_\nu)\,(\nabla\psi \cdot \nabla\xi_\nu)\;d\mathbf{x}dt$$

$$+ \int_\Omega \eta(\omega_0)\,\psi(\mathbf{x},0)\,\xi_\nu\;d\mathbf{x} - \int_\Omega \eta(\omega(\mathbf{x},T))\,\psi(\mathbf{x},T)\,\xi_\nu\;d\mathbf{x}$$

$$+ \int_{\Gamma_T} (M + \nu L)\eta(b)\,\psi\;d\mathbf{x}dt \geqslant 0$$

$$\mathbf{v}_\nu \to \mathbf{v} \qquad \text{strongly in} \qquad L_\infty(0, T; L_2(\Omega)),$$

$$|\omega_\nu - c|^+ \rightharpoonup z^1, \qquad |\omega_\nu - c|^- \rightharpoonup z^2,$$

$$(\omega_\nu - c) \rightharpoonup \omega - c = z^1 - z^2, \qquad \text{weakly-* in} \qquad L_\infty(0, T; L_p(\Omega)),$$

Hence

$$\int_{\Omega_T} z^1 \left(\psi_t + \mathbf{v}\cdot\nabla\psi\right)\, d\mathbf{x}\, dt$$

$$+M\int_{\Gamma_T} |b-c|^+\psi\, d\mathbf{x}\, dt + \int_{\Omega} |\omega_0 - c|^+\psi(\mathbf{x},0)\, d\mathbf{x} \geqslant 0,$$

and

$$\int_{\Omega_T} z^2 \left(\psi_t + \mathbf{v}\cdot\nabla\psi\right)\, d\mathbf{x}\, dt$$

$$+M\int_{\Gamma_T} |b-c|^-\psi\, d\mathbf{x}\, dt + \int_{\Omega} |\omega_0 - c|^-\psi(\mathbf{x},0)\, d\mathbf{x} \geqslant 0.$$

The strong convergence of ω_ν to ω follows from the result:

If $\quad z^1 z^2 = 0 \quad$ in $\quad \Omega_T$, then

$$z^1 = |\omega - c|^+, \qquad z^2 = |\omega - c|^-.$$

Hence

$$|\omega_\nu - c|^{\pm} \rightharpoonup |\omega - c|^{\pm} \qquad \text{weakly-* in} \quad L_\infty(0,T;L_p(\Omega)), \quad \forall c \in \mathbb{R},$$

that implies the strong convergence

$$\omega_\nu \to \omega \quad \text{in} \quad L_r(0,T;L_p(\Omega)), \quad \mathbf{v}_\nu \to \mathbf{v} \quad \text{in} \quad L_r(0,T;W_p^1(\Omega)), \ \forall r,$$

and the limit $\{\omega, \mathbf{v}\}$ satisfies

$$\int_{\Omega_T} |\omega - c| \left(\psi_t + \mathbf{v}\cdot\nabla\psi\right) \, d\mathbf{x}\, dt$$

$$+M\int_{\Gamma_T} |b-c| \ \psi \ d\mathbf{x}\, dt + \int_\Omega |\omega_0 - c| \ \psi(0,x) \ d\mathbf{x} \geqslant 0.$$

Renormalization property

Applying

- Boyer F., Trace theorems ..., *Differential and integral equations*, **18,** 8 (2005)

we derive

$$\int_{\Omega_T} \beta(\omega)\ (\psi_t + \mathbf{v} \cdot \nabla\psi)\ d\mathbf{x}\, dt + \int_{\Omega} \beta(\omega_0)\ \psi(\mathbf{x}, 0)\ d\mathbf{x}$$

$$= \int_{\Gamma_T} a\beta(b(\mathbf{v}))\psi\ d\mathbf{x}\, dt$$

for any $\beta \in C(\mathbb{R})$ and arbitrary $\psi \in C^{1,1}(\overline{\Omega}_T)$ with $\psi = 0$ at $t = T$.

Conclusion

-The sharp change of the velocity inside the boundary layer could be avoided if the mathematical model allows fluid motion along the boundary.

-By this reason the vanishing viscosity convergence results were obtained due to Navier slip type conditions.