A non singular Vlasov equation for magnetic plasmas

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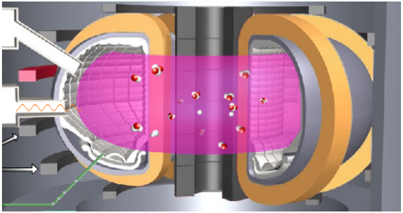
Outline

Context : fusion plasmas



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- Weakly collisional plasmas ⇒ kinetic mean field models more adapted.
- Mesoscopic scale for ions and/or electrons.

$$f_i(t,x,v) \;:\; \text{density functions in ions}$$
$$f_e(t,x,v) \;:\; \text{density functions in electrons}$$

- Full Vlasov-Maxwell systems : computationally too expensive !

Mean field kinetic models

Vlasov-Maxwell equations

$$\begin{cases} \partial_t f_s + v \cdot \nabla_x f_s + \dfrac{q_s}{m_s} \nabla_v \cdot [(\mathbf{E} + v \wedge \mathbf{B}) f_s] = 0, \quad s \in \{i, e\} \\ -\dfrac{1}{c^2} \partial_t \mathbf{E} + \mathrm{curl}\mathbf{B} = \mu_0 \displaystyle\sum_s q_s \int_{\mathbb{R}^3} f_s(v) v dv, \quad \partial_t \mathbf{B} + \mathrm{curl}\mathbf{E} = 0 \\ \mathrm{div}\mathbf{E} = \dfrac{1}{\epsilon_0} \displaystyle\sum_s q_s \int_{\mathbb{R}^3} f_s(v) dv, \quad \mathrm{div}\mathbf{B} = 0. \end{cases}$$

Mean field kinetic models

Scaled Vlasov-Maxwell equations

$$\begin{cases} \partial_t f_s + \hat{v}\cdot\nabla_x f_s + \dfrac{q_s}{m_s}\nabla_v \cdot \left[(\hat{\mathbf{E}} + \dfrac{1}{c}\hat{v}\wedge\hat{\mathbf{B}})f\right] = 0, \quad s\in\{i,e\} \\ -\dfrac{1}{c}\partial_t\hat{\mathbf{E}} + \mathrm{curl}\hat{\mathbf{B}} = \dfrac{1}{c}\sum_s q_s\int_{\mathbb{R}^3} f_s(\hat{v})\hat{v}d\hat{v}, \quad \dfrac{1}{c}\partial_t\hat{\mathbf{B}} + \mathrm{curl}\hat{\mathbf{E}} = 0 \\ \mathrm{div}\hat{\mathbf{E}} = \dfrac{1}{\epsilon_0}\sum_s q_s\int_{\mathbb{R}^3} f_s(\hat{v})d\hat{v}, \quad \mathrm{div}\hat{\mathbf{B}} = 0. \end{cases}$$

$\downarrow c\to\infty$ [Degond ; Asano and Ukai (1986) for one specie]

Vlasov-Poisson equations [relevant if $L^\circ \ll t^\circ c$]

$$\begin{cases} \partial_t f_s + v\cdot\nabla_x f_s + \dfrac{q_s}{m_s}\nabla_v\cdot[\mathbf{E}f_s] = 0, \quad s\in\{i,e\} \\ \mathrm{div}\mathbf{E} = \dfrac{1}{\epsilon_0}\sum_s q_s\int_{\mathbb{R}^3} f_s(v)dv, \end{cases}$$

Mean field kinetic models

Weak mass approximation in Vlasov-Poisson equation ($m_e \ll m_i$)

- At ions's time scale, we can assume that electrons reach immediatly their thermal equilibrium. $\Rightarrow$ electrons described by hydrodynamic equations
- Equation of conservation of momentum, with assumptions T_e =cste ($q := q_i$).

$$\partial_t(n_e \mathbf{u}_e) + \text{div} \cdot (n_e \mathbf{u}_e \otimes \mathbf{u}_e) + \frac{k_B T_e}{m_e} \nabla_x (n_e) + \frac{q}{m_e} n_e \mathbf{E} = 0$$

$\downarrow m_e \to 0$

$$\left.\begin{aligned} &\mathbf{E} = -\frac{k_B T_e}{q} \nabla_x \ln(n_e) \\ &+ \text{ Gauss Law :} \\ &\text{div}\mathbf{E} = \frac{q}{\epsilon_0}\left(\int_{\mathbb{R}^3} f_i(v)dv - n_e\right) \end{aligned}\right\}$$

Non-linear Poisson equation :

$$-\lambda^2 \Delta \ln(n_e) = \int_{\mathbb{R}^3} f_i(v)dv - n_e$$

$$\lambda^2 = \frac{\epsilon_0 k_B T_e}{n^\circ q^2} \text{ : Debye length}$$

Outline

The model

- Vlasov equation for density in ions $f(t,x,v)$ $(=f_i)$:

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m_i} \nabla_v \cdot [\mathbf{F}\, f] = 0$$

  We denote $n_I = \int f(v)dv$, $n_I \mathbf{u}_I = \int f(v) v dv$.

- $n_e(t,x)$ is governed by the non-linear Poisson equation :

$$-\lambda^2 \Delta \ln n_e = \int_{\mathbb{R}^3} f_i(v)dv - n_e$$

- Get rid of u_e :

$$\text{Maxwell-Ampère : } \mathrm{curl}\mathbf{B} = \mu_0 q (n_I \mathbf{u}_I - n_e \mathbf{u}_e) + \underbrace{\mu_0 \varepsilon_0}_{=1/c^2 \ll 1} \partial_t \mathbf{E}$$

$$\Rightarrow \mathbf{u}_e \approx \frac{1}{n_e} \left[n_I \mathbf{u}_I - \frac{\mathrm{curl}\mathbf{B}}{\mu_0 q} \right]$$

The model

- Equation of conservation of momentum for electrons + Approximation $m_e = 0$: $\Rightarrow$ Generalized Ohm's law :

$$n_e\mathbf{E} = -\frac{k_B T_e}{e}\nabla(n_e) + \underbrace{\left[\frac{\text{curl}\mathbf{B}}{\mu_0 q} - n_I \mathbf{u}_I\right] \wedge \mathbf{B}}_{\text{comes from } -u_e \wedge \mathbf{B}} + n_e \underbrace{\eta \text{curl}\mathbf{B}}_{\text{Joule effect}}$$

$$\mathbf{E} = \mathbf{E}_0 + \eta\text{curl}\mathbf{B}$$

- Force in Vlasov equation : $\mathbf{F} = \mathbf{E}_0 + v \wedge \mathbf{B}$
- Maxwell-faraday equation :

$$\partial_t \mathbf{B} + \text{curl}\mathbf{E} = 0$$

$\Rightarrow$ PDE on $\mathbf{B}$.

The model

For $x \in \Omega \subset \mathbb{R}^3$ a bounded and regular domain, $v \in \mathbb{R}^3$

$$\begin{cases} -\lambda^2 \Delta \ln n_e = n_I - n_e, \\ \dfrac{\partial \mathbf{B}}{\partial t} - \text{curl}\left(\dfrac{n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B}\right) + \overbrace{\text{curl}\left(\dfrac{\text{curl}\mathbf{B}}{n_e} \wedge \mathbf{B}\right)}^{\text{Hall effect}} + \overbrace{\text{curl}\,(\eta \text{curl}\mathbf{B})}^{\text{Joule effect}} = 0, \\ \dfrac{\partial f}{\partial t} + v \cdot \nabla f + \nabla_v \cdot \left[\left(-\nabla \ln n_e + \dfrac{\text{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B} + v \wedge \mathbf{B}\right) f\right] = 0, \\ \text{div}\mathbf{B} = 0. \end{cases}$$

Boundary conditions

$$\begin{cases} \mathbf{n}_x \cdot n_e(t,x) = 0 & x \in \partial\Omega \\ \mathbf{n}_x \wedge \mathbf{B}(t,x) = \mathbf{n}_x \wedge \mathbf{B}_{imp} & x \in \partial\Omega \\ f(t,x,v - 2(v \cdot \mathbf{n}_x)\mathbf{n}_x) = f(t,x,v) & x \in \partial\Omega \end{cases}$$

The model

For $t \in [0,T]$, $x \in \Omega \subset \mathbb{R}^3$ a bounded and regular domain, $v \in \mathbb{R}^3$

$$\begin{cases} -\lambda^2 \Delta \ln n_e = n_I - n_e, \\ \dfrac{\partial \mathbf{B}}{\partial t} - \mathrm{curl}\left(\dfrac{n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B}\right) + \overbrace{\mathrm{curl}\left(\dfrac{\mathrm{curl}\mathbf{B}}{n_e} \wedge \mathbf{B}\right)}^{\text{Hall effect}} + \overbrace{\mathrm{curl}\left(\eta \mathrm{curl}\mathbf{B}\right)}^{\text{Joule effect}} = 0 \\ \dfrac{\partial f}{\partial t} + v \cdot \nabla f + \nabla_v \cdot \left[\left(-\nabla \ln n_e + \dfrac{\mathrm{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B} + v \wedge \mathbf{B}\right) f\right] = 0, \\ \mathrm{div}\mathbf{B} = 0. \end{cases}$$

Boundary conditions

$$\begin{cases} \mathbf{n}_x \cdot n_e(t,x) = 0 & x \in \partial\Omega \\ \mathbf{n}_x \wedge \mathbf{B}(t,x) = 0 & x \in \partial\Omega \\ f(t,x,v - 2(v \cdot \mathbf{n}_x)\mathbf{n}_x) = f(t,x,v) & x \in \partial\Omega \end{cases}$$

The energy balance

$$\mathcal{E}_I(t) = \frac{1}{2}\int_\Omega \int_{\mathbb{R}^3} f(t,x,v)|v|^2 dv dx, \qquad \mathcal{E}_m(t) = \frac{1}{2}\int_\Omega |\mathbf{B}(t,x)|^2 dx,$$

Proposition

Classical solutions to the previous system satisfy the energy dissipation relation

$$\frac{d}{dt}\left[\overbrace{\mathcal{E}_I + \mathcal{E}_m + \frac{\lambda^2}{2}\int_\Omega |\nabla(\ln n_e)|^2 dx + \int_\Omega (n_e \ln n_e - n_e + 1)\, dx}^{\mathcal{E}_{\text{tot}}(t)}\right]$$
$$= -\int_\Omega \eta \left|\text{curl}\mathbf{B}\right|^2 dx.$$

Outline

Theorem (weak stability)

We consider a sequence of strong solutions f^ε, $\mathbf{B}^\varepsilon$, and n_e^ε of the model with

$$f^\varepsilon(0,x,v) = f^{\mathrm{ini},\varepsilon}(x,v) \geq 0, \qquad f^{\mathrm{ini},\varepsilon} \text{ is bounded in } \mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega \times \mathbb{R}^3),$$

$$\mathbf{B}^\varepsilon(0,x) = \mathbf{B}^{\mathrm{ini},\varepsilon}(x), \qquad \mathrm{div}\mathbf{B}^{\mathrm{ini},\varepsilon} = 0.$$

$$\sup_{0<\varepsilon\leq 1} [\mathcal{E}_I(0) + \mathcal{E}_m(0)] < \infty, \qquad 0 < \eta_{\min} \leq \eta \in \mathrm{L}^\infty(\Omega).$$

Then we can extract a subsequence that converges to a weak solution with finite energy. Moreover, we have for all $T > 0$,

$$\mathbf{B}^\varepsilon \to \mathbf{B} \in \mathrm{L}^\infty\big(0,T;\mathrm{L}^2(\Omega)\big) \cap \mathrm{L}^2\big(0,T;\mathrm{L}^q(\Omega)\big), \qquad \text{for } q < 6,$$

$$f^\varepsilon \rightharpoonup f \in \mathrm{L}^\infty\big(0,T;\mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega \times \mathbb{R}^3)\big)\big),$$

$$\mathrm{curl}\mathbf{B}^\varepsilon \rightharpoonup \mathrm{curl}\mathbf{B} \quad \text{in } \mathrm{L}^2((0,T)\times\Omega),$$

$$n_e^\varepsilon \to n_e \quad \text{in } \mathrm{L}^q((0,T)\times\Omega), \text{ and } \quad 0 < K_- \leq n_e \leq K_+,$$

$$n_I^\varepsilon \mathbf{u}_I^\varepsilon \to n_I \mathbf{u}_I \in (\mathrm{L}^q(0,T\times\Omega))^3, \qquad \text{for } 1 \leq q < 5/4.$$

Controls on moments of f

- f_ε strong solution of Vlasov equation with $\nabla_v F = 0$

$$\Rightarrow \|f^\varepsilon(t,\cdot,\cdot)\|_p = \|f^{\mathrm{ini},\varepsilon}(\cdot,\cdot)\|_p \quad \forall t \geq 0,\ \forall p \in [1,\infty]$$

- Moments of order 0 and 1 :

$$\|n_I^\varepsilon(t,\cdot)\|_{5/3} \leq C\|f^{\mathrm{ini},\varepsilon}\|_\infty^{2/5} \left(\int_\Omega \int_{\mathbb{R}^3} f(t,x,v)|v|^2 dv dx\right)^{3/5}$$

$$\|n_I^\varepsilon \mathbf{u}_I^\varepsilon(t,\cdot)\|_{5/4} \leq C\|f^{\mathrm{ini},\varepsilon}\|_\infty^{1/5} \left(\int_\Omega \int_{\mathbb{R}^3} f(t,x,v)|v|^2 d\mathbf{v} dx\right)^{4/5}.$$

Uniform lower bound on n_e^ε

If n_e^ε is a strong solution of

$$\begin{cases} -\lambda^2 \Delta \ln n_e^\varepsilon + n_e^\varepsilon = n_I^\varepsilon, & x \in \Omega, \\ \mathbf{n}_x \cdot \nabla n_e^\varepsilon = 0, & x \in \partial\Omega, \end{cases}$$

with $n_I^\varepsilon \geq 0$, $n_I^\varepsilon \in \mathrm{L}^\infty(0, T, \mathrm{L}^{5/3}(\Omega))$, then we have the two-sided control

$$0 < K_-(\|n_I^\varepsilon\|_{5/3}) \leq n_e^\varepsilon \leq K_+(\|n_I^\varepsilon\|_{5/3}),$$

for some continuous positive functions $K_\pm(\cdot)$ with $K_+ > 1$ increasing, K_- decreasing.

Consequence

$$\sup_{0<\varepsilon\leq 1} [\mathcal{E}_I(0) + \mathcal{E}_m(0)] < +\infty \quad \text{and} \quad 0 < \eta_{\min} < \eta(x) < \eta_{\max}$$

$$\Rightarrow \int_\Omega |\nabla_x (\ln n_e^\varepsilon(0,x))|^2 \, dx + \int_\Omega n_e^\varepsilon(0,x) \ln n_e^\varepsilon(0,x) dx < +\infty$$

$$\Rightarrow \mathcal{E}_{\text{tot}} < +\infty$$

Estimate on the magnetic field

- Control on energy dissipation

$$\Rightarrow \sup_{0<\varepsilon\leq 1} \int_0^T \int_\Omega \Big|\mathrm{curl}\mathbf{B}^\varepsilon\Big|^2 dxdt \leq C,$$

- If Ω is of class $\mathcal{C}^{1,1}$, then

$$X_N(\Omega) := \{\mathbf{b} \in \mathrm{H}_{\mathrm{curl}}(\Omega) \cap \mathrm{H}_{\mathrm{div}}(\Omega) \,/\quad \mathbf{b}\wedge\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\}$$

  is continuously imbedded in $\mathrm{H}^1(\Omega)^3$.
- Thanks to Sobolev injections, we get

$$\sup_{0<\varepsilon\leq 1} \|\mathbf{B}^\varepsilon\|_{L^2\left(0,\infty;L^q(\Omega)^3\right)} \leq C \quad \text{for} \quad q < 6.$$

Estimate on the electric field

Every term of $\mathbf{E}^\varepsilon$ is uniformely bounded in a Lebesgue space :

$$\mathbf{E}^\varepsilon = \underbrace{-\nabla \ln n_e^\varepsilon}_{\in \mathrm{L}_t^\infty(\mathrm{L}_x^2)} - \underbrace{\frac{n_I^\varepsilon \mathbf{u}_I^\varepsilon}{n_e^\varepsilon} \wedge \mathbf{B}^\varepsilon}_{\in \mathrm{L}_t^2(\mathrm{L}_x^{30/29})} + \underbrace{\frac{\mathrm{curl}\mathbf{B}^\varepsilon \wedge \mathbf{B}^\varepsilon}{n_e^\varepsilon}}_{\in \mathrm{L}_t^{20/11}(\mathrm{L}_x^{30/29})} + \underbrace{\eta \nabla \wedge \mathbf{B}^\varepsilon}_{\in \mathrm{L}_t^2(\mathrm{L}_x^2)}$$

Space-time compactness

- $\{B^\varepsilon(t,\cdot)\}$ is relatively compact in $\mathrm{L}^q(\Omega)^3, \quad 1 \leq q < 6.$
- Simon-Aubins-Lions Theorem and

$$\partial_t B^\varepsilon = \mathrm{curl} E^\varepsilon$$

$\Rightarrow$ Space-time compactness.

- Up to an extraction,

$$B^\varepsilon \to B \quad \text{in} \quad \mathrm{L}^2(0,T,\mathrm{L}^q(\Omega)^3), \quad 1 \leq q < 6$$

- Thanks to a kinetic averaging Lemma [Perthame & Souganidis, 1998]

$$n_I^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} n_I, \text{ strongly in } \mathrm{L}^q(0,T;\Omega),$$

$$n_I \mathbf{u}_I^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} n_I \mathbf{u}_I \quad \text{strongly in } \mathrm{L}^r(0,T;\Omega)^3$$

- Then $n_e^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} n_e$ a.e. and strongly in $\mathrm{L}^q(0,T;\Omega)^3$.

Passing to the Limit

$$\mathbf{E}^\varepsilon = -\nabla \ln n_e^\varepsilon - \frac{n_I^\varepsilon \mathbf{u}_I^\varepsilon}{n_e^\varepsilon} \wedge \mathbf{B}^\varepsilon + \frac{\mathrm{curl}\mathbf{B}^\varepsilon \wedge \mathbf{B}^\varepsilon}{n_e^\varepsilon} + \eta \nabla \wedge \mathbf{B}^\varepsilon$$

Magnetic equation

$$-\iint_{(0,T)\times\Omega} \Big[\frac{\partial \Psi(t,x)}{\partial t}\mathbf{B}^\varepsilon + \mathbf{E}^\varepsilon(t,x) \cdot \nabla \wedge \Psi(t,x)\Big] dtdx = \int_\Omega \mathbf{B}^{in,\varepsilon}(x,v)\Psi(0,x),$$

$$\downarrow \varepsilon \to 0$$

$$-\iint_{(0,T)\times\Omega} \Big[\frac{\partial \Psi(t,x)}{\partial t}\mathbf{B} + \mathbf{E}(t,x) \cdot \nabla \wedge \Psi(t,x)\Big] dtdx = \int_\Omega \mathbf{B}^{in}(x,v)\Psi(0,x),$$

Passing to the Limit

Kinetic equation

$$\iint_{\Omega\times\mathbb{R}^3} f^{in,\varepsilon}(x,v)\varphi(v)\chi(t,x)dvdx =$$
$$-\iiint \Big[\varphi(v)\frac{\partial\chi(t,x)}{\partial t} + \varphi(v)v\cdot\nabla_x\chi(t,x) + \chi(t,x)(\mathbf{B}^\varepsilon\wedge v)\cdot\nabla_v\varphi(v)\Big] f^\varepsilon(t,x,v)dt$$
$$+\iint_{(0,T)\times\Omega} \chi(t,x)\mathbf{E}^{0,\varepsilon}\cdot\langle\nabla_v\varphi f^\varepsilon\rangle dtdx$$

and thanks to an averaging Lemma (after extraction)

$$\chi\left\langle\psi_i f^\varepsilon\right\rangle(t,x)\to\chi\left\langle\psi_i f\right\rangle \quad \text{in} \quad L^q([0,T]\times\Omega) \quad \text{with } 1\le q<30/29$$

Outline

A last remark

If $\lambda \to 0$, then n_e solution of

$$-\lambda^2 \Delta \ln(n_e) = n_I - n_e$$

satisfies $n_e \to n_I$.

Why don't we take the approximation $n_e = n_I$?

The system become

$$\begin{aligned}
&\frac{\partial \mathbf{B}}{\partial t} - \mathrm{curl}\,(\mathbf{u}_I \wedge \mathbf{B}) + \mathrm{curl}\left(\frac{1}{n_I}\mathrm{curl}\mathbf{B} \wedge \mathbf{B})\right) + \mathrm{curl}\,(\eta \mathrm{curl}\mathbf{B}) = 0,\\
&\frac{\partial f}{\partial t} + v.\nabla f + \nabla_v \left[(\mathbf{E}^{\mathrm{lim}} + v \wedge \mathbf{B})f\right] = 0,\\
&n_I(\mathbf{E}^{\mathrm{lim}} + \mathbf{u}_I \wedge \mathbf{B}) = -\nabla n_I + \mathrm{curl}\mathbf{B} \wedge \mathbf{B}.
\end{aligned}$$

$\to$ Probabely ill-posed!! (cf Bardos, Jabin & Nouri)

Thanks for your attention !

Construction of an approximated solution

Idea : splitting and freezing some parts of the equations to linearize.

First step : Vlasov Poisson

Solving

$$\begin{cases} -\lambda^2 \Delta \ln n_e = n_I - n_e, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \operatorname{curl}\left(\dfrac{\operatorname{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B}\right) + \operatorname{curl}\left(\eta \operatorname{curl}\mathbf{B}\right) = 0, \\ \dfrac{\partial f}{\partial t} + v \cdot \nabla f + \nabla_v \cdot \left[\left(-\nabla \ln n_e + \dfrac{\operatorname{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B} + v \wedge \mathbf{B}\right) f\right] = 0, \end{cases}$$

during Δt.

Construction of an approximated solution

Second step : magnetic part I

Solving

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} + \text{curl}\left(\dfrac{\text{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B}_{\text{frozen}}\right) + \text{curl}\left(\eta \text{curl}\mathbf{B}\right) = 0, \\ \dfrac{\partial f}{\partial t} + v \cdot \nabla f + \nabla_v \cdot \left[\left(-\nabla \ln n_e + \dfrac{\text{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B}_{\text{frozen}} + v \wedge \mathbf{B}\right) f\right] = 0, \end{cases}$$

during Δt.

Construction of an approximated solution

Third step : magnetic part II

Solving

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} + \text{curl}\left(\dfrac{\text{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B}\right) + \text{curl}\left(\eta \text{curl}\mathbf{B}\right) = 0, \\ \dfrac{\partial f}{\partial t} + v \cdot \nabla f + \nabla_v \cdot \left[\left(-\nabla \ln n_e + \dfrac{\text{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B} + v \wedge \mathbf{B}\right) f\right] = 0, \end{cases}$$

during Δt.

A kinetic averaging Lemma

Theorem [Perthame, Souganidis, 1998]

Let $1 < q < \infty$ and f, $\mathbf{g} = (g_1, ..., g_d)$ belong to $\mathrm{L}^q_{t,x,\mathbf{v}}(\mathbf{R}^{1+3+3})$ and satisfy

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f = \frac{\partial}{\partial \mathbf{v}} . \left((\mathbb{I} - \Delta_{t,x,\mathbf{v}})^{1/2} \mathbf{g} \right). \tag{1}$$

Then $\langle f \psi \rangle \in \mathrm{L}^q_{t,x}(\mathbf{R}^{1+3})$ and there exists $C(p, \psi)$ such that

$$\| \langle f \psi \rangle \|_q \leq C(q, \psi) \| f \|_q^{1-\alpha} \| \mathbf{g} \|_q^{\alpha}$$

for a positive exponent $\alpha \leq \frac{1}{2} \min(\frac{1}{q}, 1 - \frac{1}{q})$.

We apply this result to χf, with $g^\varepsilon = (I - \nabla^{-1/2}_{x,t,v}) \chi F^\varepsilon f^\varepsilon$ and we get

$$\| \chi \langle f^\varepsilon \psi \rangle - \chi \langle f \psi \rangle \|_q \leq C(q, \psi) \cdot \| \mathbf{g}^\varepsilon - \mathbf{g} \|_q^\alpha \; \| f^\varepsilon - f \|_q^{(1-\alpha)}.$$