Entropy satisfying and shock capturing approximate Riemann solver based on the Suliciu Relaxation approach

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$\longrightarrow$ We consider the following PDE model

$$\partial_t \mathbf{v} + \partial_x \mathbf{f}(\mathbf{v}) = 0 \quad (x,t) \in \mathbb{R} \times \mathbb{R}^{+*} \tag{1}$$

assumed to be strictly hyperbolic with eigenvalues $(\lambda^k)_{k=1,\dots,N}$

$\longrightarrow$ The model is supplemented with an entropy inequality

$$\partial_t \mathcal{U}(\mathbf{v}) + \partial_x \mathcal{F}(\mathbf{v}) \leq 0 \tag{2}$$

$\longrightarrow$ We focus on a Riemann initial data

$$\mathbf{v}_0(x) = \begin{cases} \mathbf{v}_L & \text{if} \quad x < 0 \\ \mathbf{v}_R & \text{if} \quad x > 0 \end{cases} \tag{3}$$

⟶ The Riemann solutions are self-similar and generally made of the juxtaposition of simple λ_k-waves : rarefaction waves, contact/shock discontinuities

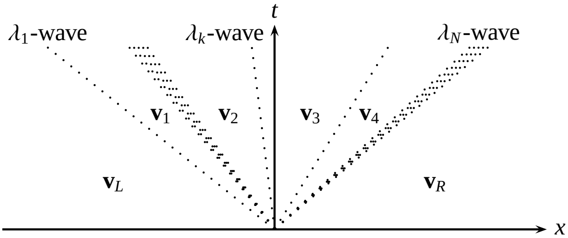


FIG.: Typical wave structure of a Riemann solution of (1)-(2)-(3)

⟶ The Riemann solutions are generally difficult to calculate and we generally consider Approximate Riemann Solutions

⟶ We only consider discontinuities with a given number of intermediate states (not necessarily the same as in the exact solution !)

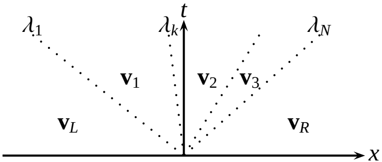


Fig.: Typical structure of a Approximate Riemann solution of (1)-(2)-(3)

⟶ Consistency with the integral form of the conservation laws (1)

$$\mathbf{f}(\mathbf{v}_R) - \mathbf{f}(\mathbf{v}_L) = \sum_{k=1}^{l} \lambda_k(\mathbf{v}_{k+1} - \mathbf{v}_k)$$

⟶ Consistency with the integral form of the entropy inequality (2)

$$\mathcal{F}(\mathbf{v}_R) - \mathcal{F}(\mathbf{v}_L) \leq \sum_{k=1}^{l} \lambda_k(\mathcal{U}(\mathbf{v}_{k+1}) - \mathcal{U}(\mathbf{v}_k))$$

HLL Approximate Riemann Solver

⟶ HLL Approximate Riemann Solver : two discontinuities and one intermediate state

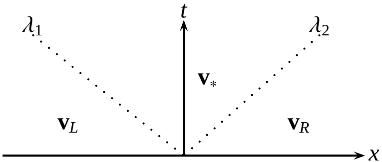


Fig.: Typical structure of a HLL Approximate Riemann solution of (1)-(2)-(3)

⟶ Consistency with the integral form of the conservation laws (1)

$$\mathbf{v}_* = \frac{\lambda_2\,\mathbf{v}_R - \lambda_1\,\mathbf{v}_L}{\lambda_2 - \lambda_1} - \frac{\mathbf{f}(\mathbf{v}_R) - \mathbf{f}(\mathbf{v}_L)}{\lambda_2 - \lambda_1}$$

⟶ Consistency with the integral form of the entropy inequality (2) under the sub-characteristic condition

$$\lambda_1 < \lambda^{(k)} < \lambda_2 \quad \text{for all} \quad k$$

Our requirements

⟶ Consistency with the integral form of the conservation laws (1)

$$\mathbf{f}(\mathbf{v}_R) - \mathbf{f}(\mathbf{v}_L) = \sum_{k=1}^{l} \lambda_k (\mathbf{v}_{k+1} - \mathbf{v}_k)$$

⟶ Consistency with the integral form of the entropy inequality (2)

$$\mathcal{F}(\mathbf{v}_R) - \mathcal{F}(\mathbf{v}_L) \leq \sum_{k=1}^{l} \lambda_k (\mathcal{U}(\mathbf{v}_{k+1}) - \mathcal{U}(\mathbf{v}_k))$$

⟶ Exact capture of isolated entropy shock solutions

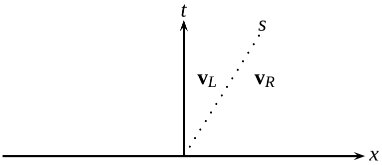


FIG.: Expected Approximate Riemann solution for an isolated entropy shock solution

The model

$\longrightarrow$ We consider the p-system in Lagrangian coordinates

$$\begin{cases} \partial_t \tau - \partial_x u = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^{+*} \\ \partial_t u + \partial_x p(\tau) = 0 \end{cases}$$

$\tau > 0$ is the inverse of a density, u is the velocity and $p > 0$ such that $p'(\tau) < 0$ is the non linear pressure

$\longrightarrow$ It is strictly hyperbolic with characteristic speeds

$$\lambda_-(\tau) = -\sqrt{-p'(\tau)}, \quad \lambda_+(\tau) = \sqrt{-p'(\tau)}$$

$\longrightarrow$ It satisfies the entropy inequality

$$\partial_t \mathcal{U}(\mathbf{v}) + \partial_x \mathcal{F}(\mathbf{v}) \leq 0$$

with

$$\mathcal{U} = \frac{u^2}{2} + e(\tau), \quad e(\tau) = -\int^{\tau} p(t)dt, \quad \mathcal{F} = pu$$

⟶ We consider the following relaxation system

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \lambda(\tau - \mathcal{T}) \end{cases}$$

with

$$\pi = \pi(\mathcal{T}) = p(\mathcal{T}) + a^2(\mathcal{T} - \tau)$$

⟶ It is strictly hyperbolic with characteristic speeds

$$\lambda^r_- = -a, \quad \lambda^r_0 = 0, \quad \lambda^r_+ = a$$

⟶ Under the sub-characteristic condition $a^2 > \max_\tau -p'(\tau)$
it satisfies the entropy inequality

$$\partial_t \Sigma + \partial_x \pi u = -\lambda(a^2 + p'(\mathcal{T}))(\mathcal{T} - \tau)^2 \leq 0$$

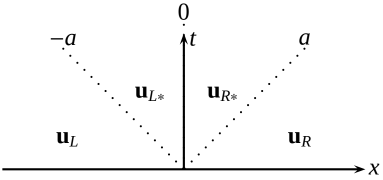
with

$$\Sigma = \frac{u^2}{2} + e(\mathcal{T}) + \frac{\pi^2 - p^2(\mathcal{T})}{2\,a^2}$$

⟶ The Approximate Riemann Solver is the Exact Riemann Solver of

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = 0, \end{cases}$$

with initial data at equilibrium : $\mathcal{T}_L = \tau_L$ and $\mathcal{T}_R = \tau_R$



$$\begin{cases} u_{L*} = u_{R*} = u_* = \dfrac{1}{2}(u_L + u_R) - \dfrac{1}{2a}(p_R - p_L), \quad \tau_{L*} = \tau_L + \dfrac{u_* - u_L}{a} \\ \pi_{L*} = \pi_{R*} = \pi_* = \dfrac{1}{2}(p_L + p_R) - \dfrac{a}{2}(u_R - u_L), \quad \tau_{R*} = \tau_R - \dfrac{u_* - u_R}{a} \end{cases}$$

$\longrightarrow$ Suliciu Relaxation Approximate Riemann Solver : three discontinuities and two intermediate states $\mathbf{v}_{L*} = \mathbf{v}(\mathbf{u}_{L*}) = (\tau_{L*}, u_*)$ and $\mathbf{v}_{R*} = \mathbf{v}(\mathbf{u}_{R*}) = (\tau_{R*}, u_*)$

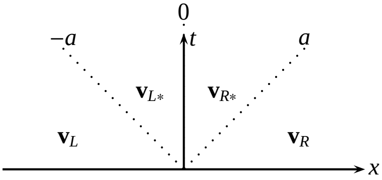


Fig.: Typical structure of a Suliciu Relaxation Approximate Riemann Solver

$\longrightarrow$ Consistency with the integral form of the conservation laws provided that $\begin{cases} \mathcal{T}_L = \tau_L \\ \mathcal{T}_R = \tau_R \end{cases}$

$\longrightarrow$ Consistency with the integral form of the entropy inequality under the sub-characteristic condition

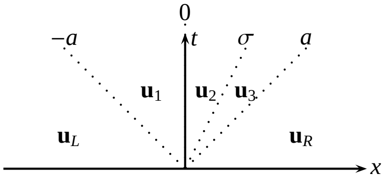
$$a^2 > \max_{\tau} -p'(\tau)$$

$\longrightarrow$ Note that under the sub-characteristic condition there is no chance to be exact for an admissible isolated shock discontinuity (note also that $\mathcal{T}_1 = \tau_L$ and $\mathcal{T}_2 = \tau_R$!)

A new Suliciu-like Approximate Riemann Solver

⟶ The Approximate Riemann Solver is now the Exact Riemann Solver of

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \mathcal{M}(\mathbf{u}_L, \mathbf{u}_R, \theta)\, \delta_{x=\sigma t} \end{cases}$$



⟶ The shock speed σ **has to be defined** ($\sigma = s$ for an isolated shock !)

⟶ Note that we will have $\mathcal{T}_1 = \mathcal{T}_L$ and $\mathcal{T}_3 = \mathcal{T}_R$ but now

$$\mathcal{T}_2 \in \lfloor \mathcal{T}_L, \mathcal{T}_R \rceil \Longleftrightarrow \mathcal{T}_2 = \mathcal{T}_2^{\theta} = (1-\theta)\mathcal{T}_R + \theta \mathcal{T}_L$$

⟶ **Importantly**, θ is used to switch continuously from the **classical solver** solver to an **isolated discontinuity**

$$\begin{cases} \theta = 0 & \text{classical relaxation solver} \\ \theta = 1 & \text{isolated discontinuity propagating with velocity } \sigma \end{cases}$$

Definition of σ

$\longrightarrow$ An entropy shock wave $(\mathbf{u}_-, \mathbf{u}_+, s)$ satisfies the Rankine-Hugoniot relations

$$s = -\frac{u_+ - u_-}{\tau_+ - \tau_-} \quad s = \frac{p_+ - p_-}{u_+ - u_-}$$

and

$$\begin{cases} \tau_+ < \tau_- & \text{if} \quad s < 0 \quad (\lambda_-\text{-shock wave}) \\ \tau_+ > \tau_- & \text{if} \quad s > 0 \quad (\lambda_+\text{-shock wave}) \end{cases}$$

$\longrightarrow$ We also have the following relation involving only the variable τ :

$$s^2 = -\frac{p_+ - p_-}{\tau_+ - \tau_-}$$

$\longrightarrow$ As a natural and generic prediction of s we propose

$$\sigma = \sigma(\mathbf{u}_L, \mathbf{u}_R) = \sigma(\tau_L, \tau_R) = \begin{cases} -\sqrt{-\dfrac{p_R - p_L}{\tau_R - \tau_L}} & \text{if} \quad \tau_R < \tau_L \\ \sqrt{-\dfrac{p_R - p_L}{\tau_R - \tau_L}} & \text{if} \quad \tau_R > \tau_L \end{cases}$$

Definition of the three intermediate states (9 unknowns)

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \mathcal{M}(\mathbf{u}_L, \mathbf{u}_R, \theta)\, \delta_{x=\sigma t} \end{cases}$$

⟶ Consistency relations with the integral form of the model : 2 equations

$$\begin{cases} u_L - u_R = -a(\tau_1 - \tau_L) + \sigma(\tau_3 - \tau_2) + a(\tau_R - \tau_3) \\ \pi_R - \pi_L = -a(u_1 - u_L) + \sigma(u_3 - u_2) + a(u_R - u_3) \end{cases}$$

⟶ Mass conservation across each wave : 3 equations

$$\begin{cases} u_L - a\tau_L = u_1 - a\tau_1 \\ u_1 = u_2 \end{cases} \qquad \begin{cases} u_2 + \sigma\tau_2 = u_3 + \sigma\tau_3 \\ u_R + a\tau_R = u_3 + a\tau_3 \end{cases}$$

⟶ Momentum conservation across each wave : 3 equations

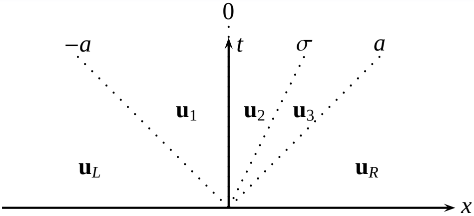
$$\begin{cases} au_L + \pi_L = au_1 + \pi_1 \\ \pi_1 = \pi_2 \end{cases} \qquad \begin{cases} \sigma u_2 - \pi_2 = \sigma u_3 - \pi_3 \\ au_R - \pi_R = au_3 - \pi_3 \end{cases}$$

⟶ Generalized jump relation at the σ-wave : 1 equation

$$\mathcal{I}_2 = (1-\theta)\mathcal{I}_R + \theta \mathcal{I}_L, \quad \theta \in [0,1], \quad \text{with} \quad \mathcal{I}(\mathcal{T}) = p(\mathcal{T}) + a^2 \mathcal{T}$$

Definition of the three intermediate states (9 unknowns)

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \mathcal{M}(\mathbf{u}_L, \mathbf{u}_R, \theta)\, \delta_{x=\sigma t} \end{cases}$$



$$\begin{cases} u_1 = u_2 = u_* + \dfrac{\sigma\theta(\mathcal{I}_R - \mathcal{I}_L)}{2a(a+\sigma)} \\ u_3 = u_* - \dfrac{\sigma\theta(\mathcal{I}_R - \mathcal{I}_L)}{2a(a-\sigma)} \end{cases} \qquad \begin{cases} \tau_1 = \tau_L + \dfrac{1}{a}(u_- - u_L) \\ \tau_3 = \tau_R + \dfrac{1}{a}(u_R - u_+) \end{cases}$$

$$\tau_2 = \begin{cases} \tau_1 + \dfrac{\sigma\theta(\mathcal{I}_R - \mathcal{I}_L)}{\sigma(a^2-\sigma^2)}, & \sigma < 0 \\ \tau_3 - \dfrac{\sigma\theta(\mathcal{I}_R - \mathcal{I}_L)}{\sigma(a^2-\sigma^2)}, & \sigma > 0 \end{cases} \qquad \begin{cases} \pi_1 = \pi_2 = \pi_* - \dfrac{\sigma\theta(\mathcal{I}_R - \mathcal{I}_L)}{2(a+\sigma)} \\ \pi_3 = \pi_* - \dfrac{\sigma\theta(\mathcal{I}_R - \mathcal{I}_L)}{2(a-\sigma)} \end{cases}$$

$$\begin{cases} \mathcal{I}_1 = \mathcal{I}_L, \\ \mathcal{I}_2(\theta) = \begin{cases} (1-\theta)\mathcal{I}_L + \theta\mathcal{I}_R = \mathcal{I}_L + \theta(\mathcal{I}_R - \mathcal{I}_L), & \sigma < 0 \\ (1-\theta)\mathcal{I}_R + \theta\mathcal{I}_L = \mathcal{I}_R - \theta(\mathcal{I}_R - \mathcal{I}_L), & \sigma > 0 \end{cases} \\ \mathcal{I}_3 = \mathcal{I}_R \end{cases}$$

Theorem 1. *If* $\theta = 0$, *then*

$$\mathcal{T}_2 = \mathcal{T}_R$$

and the classical relaxation solver is recovered

Theorem 2. *If* $\theta = 1$ *and* $\mathbf{v}_L$ *and* $\mathbf{v}_R$ *can be joined by an admissible shock discontinuity propagating with velocity* s, *then*

$$\mathcal{T}_2 = \mathcal{T}_L$$

and

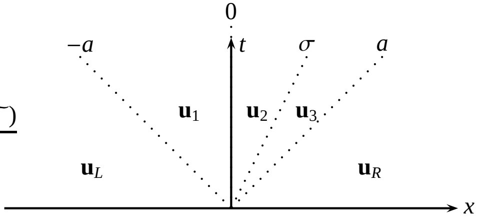
$$\sigma = s$$

and by uniqueness the approximate solution gives the isolated discontinuity between $\mathbf{v}_L$ *and* $\mathbf{v}_R$

Definition of θ and consistency with the entropy inequality

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \mathcal{M}(\mathbf{u}_L, \mathbf{u}_R, \theta)\, \delta_{x=\sigma t} \end{cases}$$

$$\begin{cases} \Sigma(\mathbf{u}) = \dfrac{u^2}{2} + e(\mathcal{T}) + \dfrac{\pi^2 - p^2(\mathcal{T})}{2a^2} \\ \mathcal{F}^r(\mathbf{u}) = \pi u \end{cases}$$



$\longrightarrow$ We want $\theta = 1$ in the case of an isolated shock wave

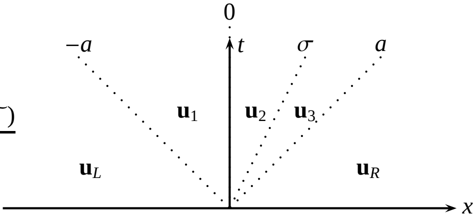
$\longrightarrow$ We want the consistency with the integral form of the entropy inequality

$$\mathcal{F}(\mathbf{v}_R) - \mathcal{F}(\mathbf{v}_L) \leq$$

$$-a(\mathcal{U}(\mathbf{v}_1) - \mathcal{U}(\mathbf{v}_L)) + \sigma(\mathcal{U}(\mathbf{v}_3) - \mathcal{U}(\mathbf{v}_2)) + a(\mathcal{U}(\mathbf{v}_R) - \mathcal{U}(\mathbf{v}_3))$$

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \mathcal{M}(\mathbf{u}_L, \mathbf{u}_R, \theta)\, \delta_{x=\sigma t} \end{cases}$$

$$\begin{cases} \Sigma(\mathbf{u}) = \dfrac{u^2}{2} + e(\mathcal{T}) + \dfrac{\pi^2 - p^2(\mathcal{T})}{2a^2} \\ \mathcal{F}^r(\mathbf{u}) = \pi u \end{cases}$$

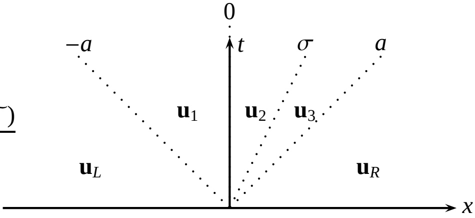


$\longrightarrow$ It is sufficient to show the following set of equalities and inequalities

$$\begin{cases} \mathcal{F}(\mathbf{v}_R) - \mathcal{F}^r(\mathbf{u}_3) & = & a(\mathcal{U}(\mathbf{v}_R) - \Sigma(\mathbf{u}_3)), \\ 0 & \leq & (a-\sigma)(\Sigma(\mathbf{u}_3) - \mathcal{U}(\mathbf{v}_3)) \\ \mathcal{F}^r(\mathbf{u}_3) - \mathcal{F}^r(\mathbf{u}_2) & \leq & \sigma(\Sigma(\mathbf{u}_3) - \Sigma(\mathbf{u}_2)) \\ 0 & \leq & \sigma(\Sigma(\mathbf{u}_2) - \mathcal{U}(\mathbf{v}_2)) \\ \mathcal{F}^r(\mathbf{u}_2) - \mathcal{F}^r(\mathbf{u}_1) & = & 0 \\ 0 & \leq & \sigma(\Sigma(\mathbf{u}_1) - \mathcal{U}(\mathbf{v}_1)) \\ \mathcal{F}^r(\mathbf{u}_1) - \mathcal{F}(\mathbf{v}_L) & = & -a(\Sigma(\mathbf{u}_1) - \mathcal{U}(\mathbf{v}_L)) \\ 0 & \leq & (a+\sigma)(\Sigma(\mathbf{u}_1) - \mathcal{U}(\mathbf{v}_1)) \end{cases}$$

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \mathcal{M}(\mathbf{u}_L, \mathbf{u}_R, \theta)\, \delta_{x=\sigma t} \end{cases}$$

$$\begin{cases} \Sigma(\mathbf{u}) = \dfrac{u^2}{2} + e(\mathcal{T}) + \dfrac{\pi^2 - p^2(\mathcal{T})}{2a^2} \\ \mathcal{F}^r(\mathbf{u}) = \pi u \end{cases}$$



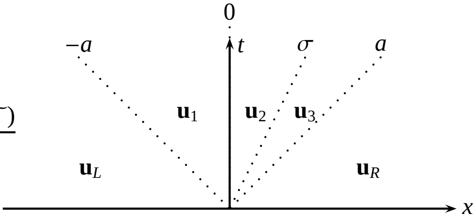
⟶ The 3 equalities are valid by the jump relations imposed to define the intermediate states

$$\begin{cases} \mathcal{F}(\mathbf{v}_R) - \mathcal{F}^r(\mathbf{u}_3) & = & a(\mathcal{U}(\mathbf{v}_R) - \Sigma(\mathbf{u}_3)) \\ \mathcal{F}^r(\mathbf{u}_2) - \mathcal{F}^r(\mathbf{u}_1) & = & 0 \\ \mathcal{F}^r(\mathbf{u}_1) - \mathcal{F}(\mathbf{v}_L) & = & -a(\Sigma(\mathbf{u}_1) - \mathcal{U}(\mathbf{v}_L)) \end{cases}$$

Consistency with the entropy inequality

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \mathcal{M}(\mathbf{u}_L, \mathbf{u}_R, \theta)\, \delta_{x=\sigma t} \end{cases}$$

$$\begin{cases} \Sigma(\mathbf{u}) = \dfrac{u^2}{2} + e(\mathcal{T}) + \dfrac{\pi^2 - p^2(\mathcal{T})}{2a^2} \\ \mathcal{F}^r(\mathbf{u}) = \pi u \end{cases}$$



⟶ The 3 inequalities

$$\begin{cases} 0 & \leq & (a-\sigma)(\Sigma(\mathbf{u}_3) - \mathcal{U}(\mathbf{v}_3)) \\ 0 & \leq & \sigma(\Sigma(\mathbf{u}_2) - \mathcal{U}(\mathbf{v}_2)) \\ 0 & \leq & (a+\sigma)(\Sigma(\mathbf{u}_1) - \mathcal{U}(\mathbf{v}_1)) \end{cases} \iff \begin{cases} \Sigma(\mathbf{u}_3) & \geq & \mathcal{U}(\mathbf{v}_3) \\ \Sigma(\mathbf{u}_2) & \geq & \mathcal{U}(\mathbf{v}_2) \\ \Sigma(\mathbf{u}_1) & \geq & \mathcal{U}(\mathbf{v}_1) \end{cases}$$

are valid under the sub-characteristic condition

$$a^2 > \min_{\tau \in (\tau_m, \tau_M)} -p'(\tau) \quad \text{with} \quad \begin{cases} \tau_m = \min(\tau_L, \tau_1, \tau_2, \tau_3, \tau_R) \\ \tau_M = \max(\tau_L, \tau_1, \tau_2, \tau_3, \tau_R) \end{cases}$$

Consistency with the entropy inequality

Indeed, by definition of $\mathcal{U}$ and Σ, $\begin{cases} \Sigma(\mathbf{u}) = \dfrac{u^2}{2} + e(\mathcal{T}) + \dfrac{\pi^2 - p^2(\mathcal{T})}{2a^2} \\ \mathcal{F}^r(\mathbf{u}) = \pi u \end{cases}$

$$\begin{cases} \Sigma(\mathbf{u}_3) & \geq & \mathcal{U}(\mathbf{v}_3) \\ \Sigma(\mathbf{u}_2) & \geq & \mathcal{U}(\mathbf{v}_2) \\ \Sigma(\mathbf{u}_1) & \geq & \mathcal{U}(\mathbf{v}_1) \end{cases} \iff \begin{cases} 2a^2(e(\tau_3) - e(\mathcal{T}_3)) & \leq & \pi_3^2 - p^2(\mathcal{T}_3) \\ 2a^2(e(\tau_2) - e(\mathcal{T}_2)) & \leq & \pi_2^2 - p^2(\mathcal{T}_2) \\ 2a^2(e(\tau_1) - e(\mathcal{T}_1)) & \leq & \pi_1^2 - p^2(\mathcal{T}_1) \end{cases}$$

Using that $\pi = p(\mathcal{T}) + a^2(\mathcal{T} - \tau)$, it equivalently recast as

$$\begin{cases} 2(e(\tau_3) - e(\mathcal{T}_3)) & \leq & a^2(\mathcal{T}_3 - \tau_3)^2 + 2p(\mathcal{T}_3)(\mathcal{T}_3 - \tau_3) \\ 2(e(\tau_2) - e(\mathcal{T}_2)) & \leq & a^2(\mathcal{T}_2 - \tau_2)^2 + 2p(\mathcal{T}_2)(\mathcal{T}_2 - \tau_2) \\ 2(e(\tau_1) - e(\mathcal{T}_1)) & \leq & a^2(\mathcal{T}_1 - \tau_1)^2 + 2p(\mathcal{T}_1)(\mathcal{T}_1 - \tau_1) \end{cases}$$

so that a second-order Taylor expansion around $\mathcal{T}$ easily gives for some ξ_i

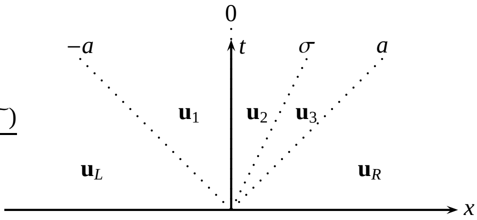
$$\begin{cases} -p'(\xi_3) & \leq & a^2 \\ -p'(\xi_2) & \leq & a^2 \\ -p'(\xi_1) & \leq & a^2 \end{cases}$$

This holds true under the sub-characteristic condition

$$a^2 > \min_{\tau \in (\tau_m, \tau_M)} -p'(\tau) \quad \text{with} \quad \begin{cases} \tau_m = \min(\tau_L, \tau_1, \tau_2, \tau_3, \tau_R) \\ \tau_M = \max(\tau_L, \tau_1, \tau_2, \tau_3, \tau_R) \end{cases}$$

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \pi = 0 \\ \partial_t \mathcal{T} = \mathcal{M}(\mathbf{u}_L, \mathbf{u}_R, \theta)\, \delta_{x=\sigma t} \end{cases}$$

$$\begin{cases} \Sigma(\mathbf{u}) = \dfrac{u^2}{2} + e(\mathcal{T}) + \dfrac{\pi^2 - p^2(\mathcal{T})}{2a^2} \\ \mathcal{F}^r(\mathbf{u}) = \pi u \end{cases}$$



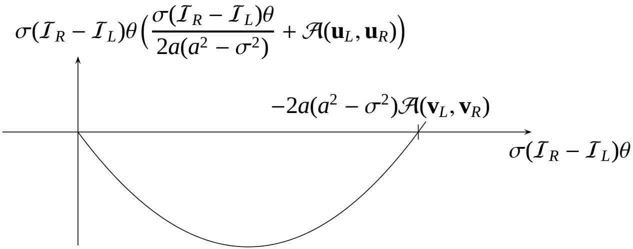
$\longrightarrow$ The last inequality $\mathcal{F}^r(\mathbf{u}_3) - \mathcal{F}^r(\mathbf{u}_2) \le \sigma(\Sigma(\mathbf{u}_3) - \Sigma(\mathbf{u}_2))$ is valid if

$$\sigma(\mathcal{I}_R - \mathcal{I}_L)\theta\left(\frac{\sigma(\mathcal{I}_R - \mathcal{I}_L)\theta}{2a(a^2 - \sigma^2)} + \mathcal{A}(\mathbf{u}_L, \mathbf{u}_R)\right) \le 0$$

with $\mathcal{A}(\mathbf{u}_L, \mathbf{u}_R)$ explicitly defined from $\mathbf{u}_L$ and $\mathbf{u}_R$ by

$$\mathcal{A}(\mathbf{v}_L, \mathbf{v}_R) = -\frac{\left((e - \frac{p^2}{2a^2})_R - (e - \frac{p^2}{2a^2})_L\right)}{(\mathcal{I}_R - \mathcal{I}_L)} - \frac{\pi_*}{a^2}.$$

Consistency with the entropy inequality



$\longrightarrow$ Then we set

$$\sigma(\mathcal{I}_R - \mathcal{I}_L)\theta = \max\Big(0, \min\left(\sigma(\mathcal{I}_R - \mathcal{I}_L), -2a(a^2 - \sigma^2)\mathcal{A}(\mathbf{v}_L, \mathbf{v}_R)\right)\Big)$$

$\longrightarrow$ $\theta \geq 0$, $\theta \leq 1$
$\longrightarrow$ If $\mathbf{v}_L$ and $\mathbf{v}_R$ can be joined by an admissible shock discontinuity propagating with velocity s, then $\theta = 1$

$$\sigma(\mathcal{I}_R - \mathcal{I}_L) \leq -2a(a^2 - \sigma^2)\mathcal{A}(\mathbf{v}_L, \mathbf{v}_R)$$

$\longrightarrow$ Let $\mathbf{v}_L$ and $\mathbf{v}_R$ be such that

$$\begin{cases} -s(\tau_R - \tau_L) - (u_R - u_L) = 0 \\ -s(u_R - u_L) + (p_R - p_L) = 0 \end{cases}$$

and

$$-s(\mathcal{U}_R - \mathcal{U}_L) + (\mathcal{F}_R - \mathcal{F}_L) \leq 0$$

or equivalently

$$s(\tau_R - \tau_L) > 0$$

$\longrightarrow$ In this case σ clearly coincides with s and satisfies in particular

$$s^2 = \sigma^2 = -\frac{p_R - p_L}{\tau_R - \tau_L}$$

$\longrightarrow$ We have by easy manipulations $\sigma(\mathcal{I}_R - \mathcal{I}_L) \leq -2a(a^2 - \sigma^2)\mathcal{A}(\mathbf{v}_L, \mathbf{v}_R)$ so that

$$\sigma(\mathcal{I}_R - \mathcal{I}_L)\theta = \max\Big(0, \min\left(\sigma(\mathcal{I}_R - \mathcal{I}_L), -2a(a^2 - \sigma^2)\mathcal{A}(\mathbf{v}_L, \mathbf{v}_R)\right)\Big) = \sigma(\mathcal{I}_R - \mathcal{I}_L)$$

or equivalently

$$\theta = 1$$

$\longrightarrow$ Simply replacing in the formulas gives

$$\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_L \quad \text{and} \quad \mathbf{v}_3 = \mathbf{v}_R$$

⟶ Recall that σ is discontinuous in the asymptotic regime $\tau_L = \tau_R$ since

$$\lim_{\tau_R \to \tau_L^{\pm}} \sigma = \pm\sqrt{-p'(\tau_L)}$$

⟶ But σ is always multiplied by $\theta(\mathcal{I}_R - \mathcal{I}_L)$ and one can easily prove that

$$\sigma\theta(\mathcal{I}_R - \mathcal{I}_L) \sim \sigma(\mathcal{I}_R - \mathcal{I}_L) \quad (\theta = 1)$$

in this asymptotic regime $\tau_L = \tau_R$

⟶ This quantity is Lipschitz-continuous in this asymptotic regime since

$$\lim_{\tau_R \to \tau_L^{\pm}} \sigma(\mathcal{I}_R - \mathcal{I}_L) = \lim_{\tau_R \to \tau_L^{\pm}} \sigma\Big((p_R - p_L) + a^2(\tau_R - \tau_L)\Big) = \pm\sqrt{-p'(\tau_L)} \times 0 = 0$$

and

$$\lim_{\tau_R \to \tau_L^{\pm}} \sigma\frac{(\mathcal{I}_R - \mathcal{I}_L)}{\tau_R - \tau_L} = \lim_{\tau_R \to \tau_L^{\pm}} \sigma\frac{\Big((p_R - p_L) + a^2(\tau_R - \tau_L)\Big)}{\tau_R - \tau_L} = \pm\sqrt{-p'(\tau_L)} \times \Big(p'(\tau_L) + a^2\Big)$$

Theorem. *Under the sub-characteristic condition*

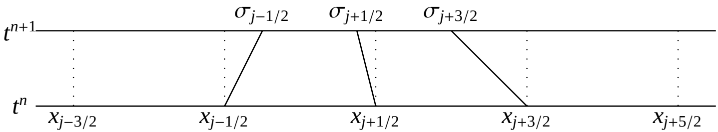
$$a^2 > \min_{\tau \in (\tau_m, \tau_M)} -p'(\tau) \quad \textit{with} \quad \begin{cases} \tau_m = \min(\tau_L, \tau_1, \tau_2, \tau_3, \tau_R) \\ \tau_M = \max(\tau_L, \tau_1, \tau_2, \tau_3, \tau_R) \end{cases}$$

the proposed approximate Riemann solver
(i) is exact for isolated admissible shock waves
(ii) is entropy satisfying
(iii) is Lipschitz-continuous with respect to the initial states $\mathbf{v_L}$ *and* $\mathbf{v_R}$
(iv) is L^1 *stable in the sense that the intermediate states belong to the phase space*

Remark. Under a suitable modification of the definition of θ, the sub-characteristic condition can be given by

$$a^2 > \min_{\tau \in (\tau_m, \tau_M)} -p'(\tau) \quad \text{with} \quad \begin{cases} \tau_m = \min(\tau_L, \tau_{L*}, \tau_{R*}, \tau_R) \\ \tau_M = \max(\tau_L, \tau_{L*}, \tau_{R*}, \tau_R) \end{cases}$$

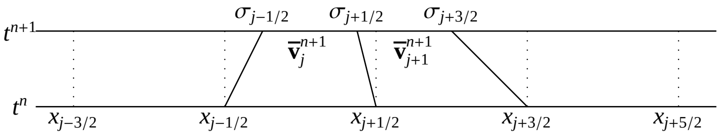
which is nothing but the usual sub-characteristic condition associated with the classical Suliciu relaxation scheme



$\longrightarrow$ We average the proposed Approximate Riemann Solutions at time $t = \Delta t$ on the cells $[x_{j-1/2}, x_{j+1/2}]$ as in the classical Godunov scheme

$$\mathbf{v}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(x, \Delta t) dt$$

$\longrightarrow$ The scheme is then conservative and entropy-satisfying



$\longrightarrow$ We first average the proposed Approximate Riemann Solutions at time $t = \Delta t$ on the modified cells $[\overline{x}_{j-1/2}, \overline{x}_{j+1/2}]$ as follows

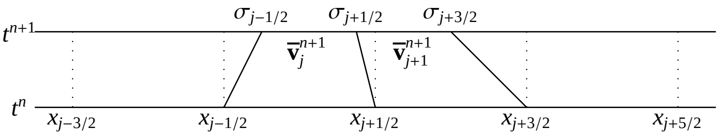
$$\overline{\mathbf{v}}_j^{n+1} = \frac{1}{\overline{\Delta x_j}} \int_{\overline{x}_{j-1/2}}^{\overline{x}_{j+1/2}} \mathbf{v}(x, \Delta t) dt$$

with

$$\overline{x}_{j+1/2} = x_{j+1/2} + \sigma_{j-1/2}\Delta t, \quad \text{and} \quad \overline{\Delta x_j} = \overline{x}_{j+1/2} - \overline{x}_{j-1/2}$$

$\longrightarrow$ We then proceed with a random sampling procedure to recover the initial mesh and define $\mathbf{v}_j^{n+1}$

Numerical approximation : a Glimm-type scheme



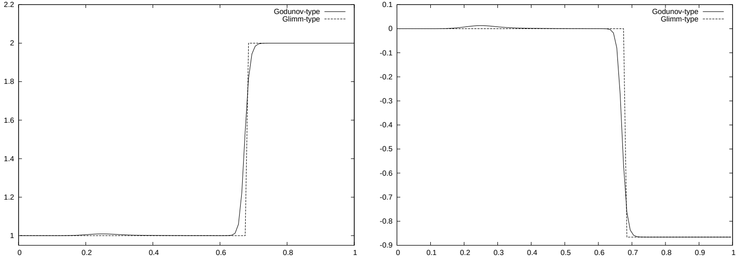
Let be given $(a_n)_n$ a well-distributed random sequence within $]0, \Delta x[$ (*e.g.* van der Corput sequence)

Random choice procedure gives

$$\mathbf{v}_j^{n+1} = \begin{cases} \overline{\mathbf{v}}_{j-1}^{n+1} & \text{if} \quad a_{n+1} \in (0, \frac{\Delta t}{\Delta x}\sigma^+_{j-1/2}) \\ \overline{\mathbf{v}}_{j}^{n+1} & \text{if} \quad a_{n+1} \in [\frac{\Delta t}{\Delta x}\sigma^+_{j-1/2}, 1 + \frac{\Delta t}{\Delta x}\sigma^-_{j+1/2}) \\ \overline{\mathbf{v}}_{j+1}^{n+1} & \text{if} \quad a_{n+1} \in [1 + \frac{\Delta t}{\Delta x}\sigma^-_{j+1/2}, 1) \end{cases}$$

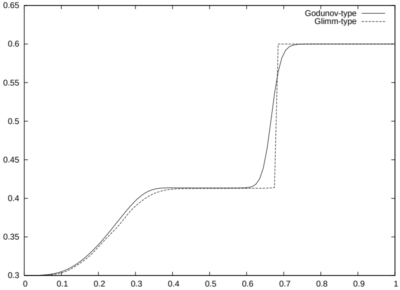
$\sigma_{j+1/2}$ is the speed of the additional wave coming from $x_{j+1/2}$
$\sigma^+_{j+1/2} = \max(\sigma_{j+1/2}, 0)$, $\sigma^-_{j+1/2} = \min(\sigma_{j+1/2}, 0)$

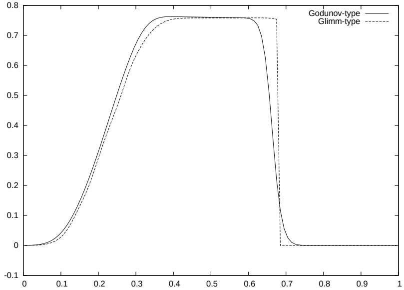
Isolated admissible shock wave (100 points)



No numerical diffusion across the shock for the Glimm-type method !

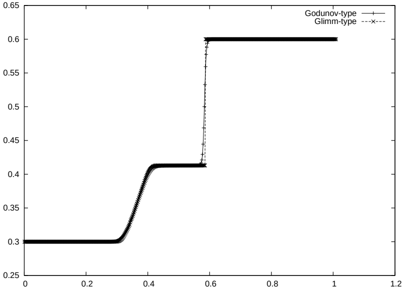
General Riemann problem (100 points)

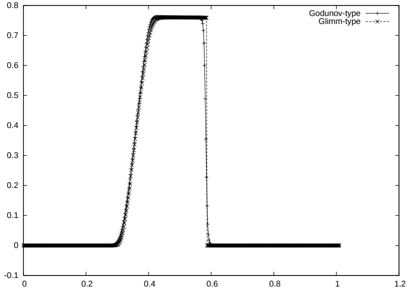




No numerical diffusion across the shock for the Glimm-type method !

General Riemann problem (500 points)





No numerical diffusion across the shock for the Glimm-type method !