Two-waves PVM-WAF method for non-conservative systems

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Outline

Model problem

Let us consider the system

$$w_t + F(w)_x + B(w) \cdot w_x = G(w)H_x, \tag{1}$$

where

- $w(x,t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$,
- F is a regular function from $\mathcal{O}$ to $\mathbb{R}^N$,
- B is a regular matrix function from $\mathcal{O}$ to $\mathcal{M}_{N\times N}(\mathbb{R})$,
- G is a function from $\mathcal{O}$ to $\mathbb{R}^N$, and
- H is a function from $\mathbb{R}$ to $\mathbb{R}$.

By adding to (1) the equation $H_t = 0$, the system (1) can be rewritten under the form

$$W_t + \mathcal{A}(W) \cdot W_x = 0, \tag{2}$$

where

W is the augmented vector

$$W = \begin{bmatrix} w \\ H \end{bmatrix} \in \Omega = \mathcal{O} \times \mathbb{R} \subset \mathbb{R}^{N+1}$$

and

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By adding to (1) the equation $H_t = 0$, the system (1) can be rewritten under the form

$$W_t + \mathcal{A}(W) \cdot W_x = 0, \qquad (2)$$

where

$\mathcal{A}(W)$ is the matrix whose block structure is given by:

$$\mathcal{A}(W) = \left[\begin{array}{c|c} A(w) & -G(w) \\ \hline 0 & 0 \end{array}\right],$$

where

$$A(w) = J(w) + B(w), \quad \text{being } J(w) = \frac{\partial F}{\partial w}(w).$$

Difficulties

Main difficulties

- Non conservative products $\mathcal{A}(W) \cdot W_x$. Solutions may develop discontinuities and the concept of weak solution in the sense of distributions cannot be used. The theory introduced by DLM 1995 is used here to define the weak solutions of the system. This theory allows one to give a sense to the non conservative terms of the system as Borel measures provided a prescribed family of paths in the space of states.
- Derivation of numerical schemes for non-conservative systems: path-conservative numerical schemes (Parés 2006).
- The eigenstructure of systems like bilayer Shallow-Water system or two-phase flow model of Pitman Le are not explicitly known: PVM and/or WAF schemes.

PC-Roe-based schemes I

Let us consider path-conservative numerical schemes that can be written as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(D_{i-1/2}^{+} + D_{i+1/2}^{-}\right), \tag{3}$$

where

- Δx and Δt are, for simplicity, assumed to be constant;
- w_i^n is the approximation provided by the numerical scheme of the cell average of the exact solution at the i-th cell, $I_i = [x_{i-1/2}, x_{i+1/2}]$ at the n-th time level $t^n = n\Delta t$. H_i is the cell average of the function $H(x)$.

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$$\begin{aligned} D_{i+1/2}^{\pm} = \quad & \frac{1}{2}\Big(F(w_{i+1}) - F(w_i) + B_{i+1/2}\cdot(w_{i+1}-w_i) - G_{i+1/2}(H_{i+1}-H_i) \\ & \pm Q_{i+1/2}(w_{i+1} - w_i - A_{i+1/2}^{-1} G_{i+1/2}(H_{i+1}-H_i))\Big), \end{aligned} \tag{4}$$

where

- $A_{i+1/2} = J_{i+1/2} + B_{i+1/2}$. Here, $J_{i+1/2}$ is a Roe matrix of the Jacobian of the flux F in the usual sense:

$$J_{i+1/2}\cdot(w_{i+1}-w_i) = F(w_{i+1}) - F(w_i);$$

- $B_{i+1/2}\cdot(w_{i+1}-w_i) = \displaystyle\int_0^1 B(\Phi_w(s;W_i,W_{i+1}))\frac{\partial\Phi_w}{\partial s}(s;W_i,W_{i+1})\,ds;$
- $G_{i+1/2}(H_{i+1}-H_i) = \displaystyle\int_0^1 G(\Phi_w(s;W_i,W_{i+1}))\frac{\partial\Phi_H}{\partial s}(s;W_i,W_{i+1})\,ds;$
- $Q_{i+1/2}$ is a numerical viscosity matrix.

PC-Roe-based schemes I

Let us consider path-conservative numerical schemes that can be written as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(D^+_{i-1/2} + D^-_{i+1/2}\right), \tag{3}$$

$$\begin{aligned} D^{\pm}_{i+1/2} = \quad & \frac{1}{2}\Big(F(w_{i+1}) - F(w_i) + B_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i) \\ & \pm Q_{i+1/2}(w_{i+1} - w_i - A^{-1}_{i+1/2}G_{i+1/2}(H_{i+1} - H_i))\Big), \end{aligned} \tag{4}$$

Conservative systems

If the system is conservative and

$$\mathcal{F}_{i+1/2} = \frac{F(w_i) + F(w_{i+1})}{2} - \frac{1}{2}Q_{i+1/2}(w_{i+1} - w_i)$$

is a conservative flux, where $Q_{i+1/2}$ is defined in terms of $J_{i+1/2}$, then

$$D^-_{i+1/2} = \mathcal{F}_{i+1/2} - F(w_i) \quad D^+_{i+1/2} = F(w_{i+1}) - \mathcal{F}_{i+1/2}.$$

PC-Roe-based schemes I

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$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(D_{i-1/2}^{+} + D_{i+1/2}^{-}\right), \quad (3)$$

$$\begin{aligned} D_{i+1/2}^{\pm} = \; & \frac{1}{2}\Big(F(w_{i+1}) - F(w_i) + B_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i) \\ & \pm Q_{i+1/2}(w_{i+1} - w_i - A_{i+1/2}^{-1} G_{i+1/2}(H_{i+1} - H_i))\Big), \end{aligned} \quad (4)$$

Different numerical schemes can be obtained for different definitions of $Q_{i+1/2}$

PC-Roe-based schemes II

- Roe scheme corresponds to the choice

$$Q_{i+1/2} = |A_{i+1/2}|,$$

- Lax-Friedrichs scheme:

$$Q_{i+1/2} = \frac{\Delta x}{\Delta t} Id,$$

being *Id* the identity matrix.

- Lax-Wendroff scheme:

$$Q_{i+1/2} = \frac{\Delta t}{\Delta x} A^2_{i+1/2},$$

- FORCE and GFORCE schemes:

$$Q_{i+1/2} = (1-\omega)\frac{\Delta x}{\Delta t} Id + \omega \frac{\Delta t}{\Delta x} A^2_{i+1/2},$$

with $\omega = 0.5$ and $\omega = \frac{1}{1+\alpha}$, respectively, being α the CFL parameter.

PVM methods

We propose a class of finite volume methods defined by

$$Q_{i+1/2} = P_l(A_{i+1/2}),$$

being $P_l(x)$ a polinomial of degree l,

$$P_l(x) = \sum_{j=0}^{l} \alpha_j^{i+1/2} x^j,$$

and $A_{i+1/2}$ a Roe matrix. That is, $Q_{i+1/2}$ can be seen as a Polynomial Viscosity Matrix (PVM).

See also: P. Degond, P.F. Peyrard, G. Russo, Ph. Villedieu. *Polynomial upwind schemes for hyperbolic systems.* C. R. Acad. Sci. Paris 1 328, 479-483, 1999.

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$Q_{i+1/2}$ has the same eigenvectors than $A_{i+1/2}$ and if $\lambda_{i+1/2}$ is an eigenvalue of $A_{i+1/2}$, then $P_l(\lambda_{i+1/2})$ is an eigenvalue of $Q_{i+1/2}$.

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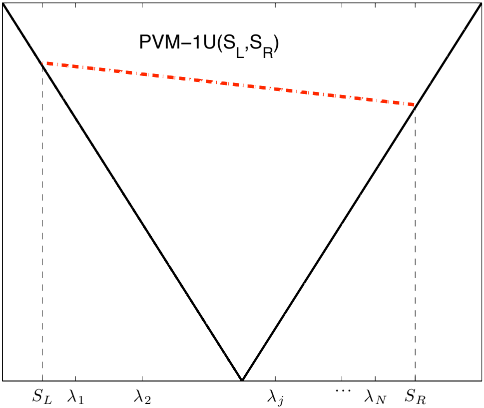
Some well-known solvers as Lax-Friedrichs, Rusanov, FORCE/GFORCE, HLL, Roe, Lax-Wendroff, ... can be recovered as PVM methods

PVM-1U(S_L, S_R) or HLL method

$$P_1(x) = \alpha_0 + \alpha_1 x \quad \text{such as } P_1(S_L) = |S_L|,\ P_1(S_R) = |S_R|.$$

$$Q_{i+1/2} = \alpha_0 Id + \alpha_1 A_{i+1/2}$$



PVM-1U(S_L, S_R) or HLL method

The usual HLL scheme coincides with PVM-1U(S_L, S_R) in the case of conservative systems.
Let us suppose that the system is conservative. Then, the conservative flux associated to PVM-1U(S_L,S_R) is $\mathcal{F}_{i+1/2} = D^-_{i+1/2} + F(w_i)$. Taking into account that

$$\alpha_0 = \frac{S_R|S_L| - S_L|S_R|}{S_R - S_L}, \quad \alpha_1 = \frac{|S_R| - |S_L|}{S_R - S_L},$$

then

$$\begin{aligned} \mathcal{F}_{i+1/2} &= \frac{F(w_i)(S_R + |S_R| - S_L - |S_L|) + F(w_{i+1})(S_R - |S_R| - S_L + |S_L|)}{2S_R - 2S_L} \\ &\quad - \frac{(S_R|S_L| - S_L|S_R|)(w_{i+1} - w_i)}{2S_R - 2S_L} \\ &= \frac{S_R^+ F(w_i) - S_L^- F(w_{i+1}) + (S_R^+ S_L^-)(w_{i+1} - w_i)}{S_R^+ - S_L^-} \end{aligned}$$

which is a compact definition of the HLL flux, being $S_R^+ = \max(S_R, 0)$ and $S_L^- = \min(S_L, 0)$.

PVM-2U(S_L, S_R) method

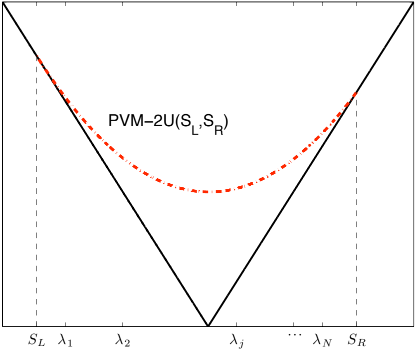
$$P_2(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2,$$

such as

$$P_2(S_m) = |S_m|, \; P_2(S_M) = |S_M|, \; P_2'(S_M) = \mathrm{sgn}(S_M),$$

where

$$S_M = \begin{cases} S_L & \text{if } |S_L| \geq |S_R|, \\ S_R & \text{if } |S_L| < |S_R|. \end{cases} \quad S_m = \begin{cases} S_R & \text{if } |S_L| \geq |S_R|, \\ S_L & \text{if } |S_L| < |S_R|. \end{cases}$$

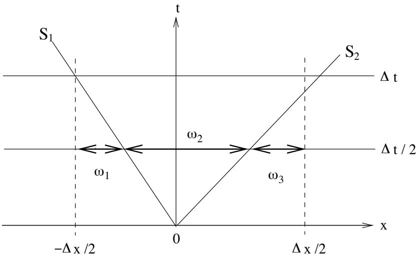


WAF method

Let us consider the following Riemann problem:

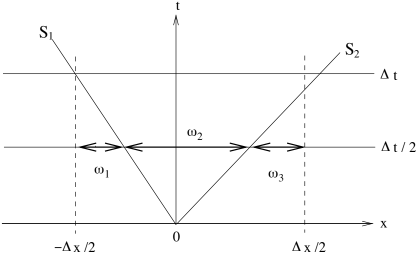
$$\begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{\partial F(w)}{\partial x} = 0; \\ w(x,0) = \begin{cases} w_i & x < 0; \\ w_{i+1} & x > 0. \end{cases} \end{cases}$$

We denote by S_i for $i = 1, \cdots, N$ some approximation of the characteristic velocities.



$$\mathcal{F}^{\text{WAF}}_{i+1/2} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} F(w(x, \frac{\Delta t}{2}))dx.$$

WAF method (Toro [1989])



- $\omega_k = \frac{1}{2}(c_k - c_{k-1})$,
- $c_0 = -1$, $c_{N+1} = 1$ and $c_l = \dfrac{\Delta t}{\Delta x} S_l$, for $1 \leq l \leq N$,
- N is the number of waves.

So, we can write the numerical flux as follows:

$$\mathcal{F}^{\text{WAF}}_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2}\sum_{k=1}^{N} c_k\, \Delta F^{(k)}_{i+1/2},$$

where

- $F^{(k)}_{i+1/2}$ is the value of the flux function in the interval k,

WAF method

TVD WAF method (Toro [1989])

We denote $\chi(v)$ a flux limiter function, and

$$\Psi(v, c) = 1 - (1 - |c|)\chi(v).$$

So the TVD-WAF flux function is defined by:

$$\mathcal{F}^{\text{WAF}}_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2}\sum_{k=1}^{N} sign(S_k)\Psi_k\, \Delta F^{(k)}_{i+1/2},$$

where

$$\Psi_k = \Psi(v^{(k)}, c_k) = 1 - (1 - |c_k|)\chi(v^{(k)}).$$

Some suitable choices for χ can be found in [Toro]. Let us consider here the Van Albada's limiter:

$$\chi(v^{(k)}) = \begin{cases} 0 & \text{if } v^{(k)} \le 0 \\ \dfrac{v^{(k)}(1 + v^{(k)})}{1 + {v^{(k)}}^2} & \text{if } v^{(k)} \ge 0 \end{cases}, \quad \text{where} \quad v^{(k)} = \begin{cases} \dfrac{p_i^{(k)} - p_{i-1}^{(k)}}{p_{i+1}^{(k)} - p_i^{(k)}} & \text{if } S_k > 0 \\ \dfrac{p_{i+2}^{(k)} - p_{i+1}^{(k)}}{p_{i+1}^{(k)} - p_i^{(k)}} & \text{if } S_k < 0 \end{cases}$$

being $p^{(k)}$ a scalar value.

HLL-WAF method

We consider now $N = 2$, $S_1 = S_L$, $S_2 = S_R$, $F^1_{i+1/2} = F(w_i)$, $F^3_{i+1/2} = F(w_{i+1})$ and

$$F^{(2)}_{i+1/2} = \frac{S_R F_i - S_L F_{i+1} + S_R S_L (w_{i+1} - w_i)}{S_R - S_L}$$

$$\begin{aligned}\mathcal{F}^{\text{HLL-WAF}}_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) \quad & - \quad \frac{1}{2}\left(\nu_1(\chi_L, \chi_R)(w_{i+1} - w_i) + \nu_2(\chi_L, \chi_R)(F_{i+1} - F_i)\right) \\ & - \quad \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\mu_1(\chi_L, \chi_R)(w_{i+1} - w_i) + \mu_2(\chi_L, \chi_R)(F_{i+1} - F_i)\right),\end{aligned}$$

where

$$\begin{aligned}\nu_1(\chi_L, \chi_R) &= \frac{S_L S_R((1 - \chi_L)sgn(S_L) - (1 - \chi_R)sgn(S_R))}{S_R - S_L} \\ \nu_2(\chi_L, \chi_R) &= \frac{(1 - \chi_R)|S_R| - (1 - \chi_L)|S_L|}{S_R - S_L} \\ \mu_1(\chi_L, \chi_R) &= \frac{S_L S_R(S_L \chi_L - S_R \chi_R)}{S_R - S_L} \\ \mu_2(\chi_L, \chi_R) &= \frac{S_R^2 \chi_R - S_L^2 \chi_L}{S_R - S_L}.\end{aligned}$$

HLL-WAF method as a PVM-type method

Then, we can rewrite the HLL-WAF method as follows:

$$\mathcal{F}_{i+1/2}^{\text{HLL-WAF}} = \frac{1}{2}(F_{i+1} + F_i) - \frac{1}{2}Q_{i+1/2}^{HLL-WAF}(w_{i+1} - w_i),$$

$$Q_{i+1/2}^{HLL-WAF}(\chi_L, \chi_R) = Q_{o1,i+1/2}^{HLL-WAF}(\chi_L, \chi_R) + \frac{\Delta t}{\Delta x} Q_{o2,i+1/2}^{HLL-WAF}(\chi_L, \chi_R)$$

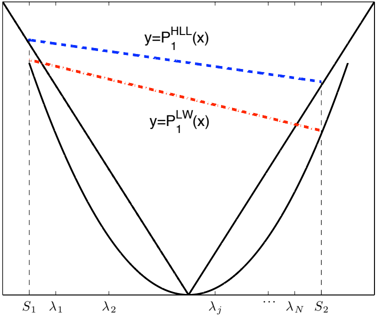
with

$$\begin{aligned} Q_{o1,i+1/2}^{HLL-WAF}(\chi_L, \chi_R) &= \nu_1(\chi_L, \chi_R)I + \nu_2(\chi_L, \chi_R)A_{i+1/2} \\ Q_{o2,i+1/2}^{HLL-WAF}(\chi_L, \chi_R) &= \mu_1(\chi_L, \chi_R)I + \mu_2(\chi_L, \chi_R)A_{i+1/2}. \end{aligned}$$

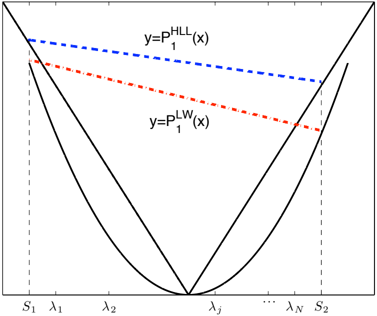
That is, the usual two-waves HLL-WAF method can be seen as a non-linear combination of two PVM schemes associated to the first order polynomials:

$$P_1^{o1}(x) = \nu_1(\chi_L, \chi_R) + \nu_2(\chi_L, \chi_R)x \quad \text{and} \quad P_1^{o2}(x) = \mu_1(\chi_L, \chi_R) + \mu_2(\chi_L, \chi_R)x.$$

HLL-WAF method as a PVM-type method



HLL-WAF method as a PVM-type method

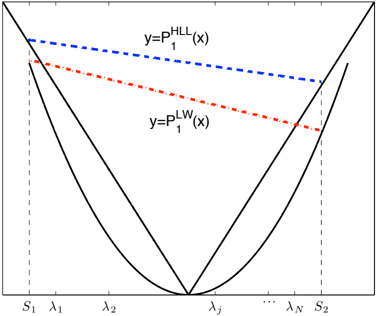


Remarks

For systems with N=2. If $S_1 = \lambda_{1,i+1/2}$ and $S_2 = \lambda_{2,i+1/2}$ being $\lambda_{j,i+1/2}$ the eigenvalues of Roe matrix, then

- HLL-WAF method coincides with Lax-Wendroff method if $(\chi_L = \chi_R = 1)$. For $N > 2$ it is not true!
- HLL-WAF method coincides with HLL if $(\chi_L = \chi_R = 0)$ $(N \geq 2)$.

HLL-WAF method as a PVM-type method



Objective

We want to define a new two-wave WAF method so that coincides with Lax-Wendroff for $N > 2$ if $\chi_L = \chi_R = 1$

Two-waves PVM2U-WAF methods

We consider

$$\mathcal{F}^{\text{2U-FL}}_{i+1/2} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2} Q^{2U-FL}_{i+1/2}(\chi_L, \chi_R)(w_{i+1} - w_i),$$

where $Q^{2U-FL}_{i+1/2}(\chi_L, \chi_R)$ is defined as follows:

$$Q^{2U-FL}_{i+1/2}(\chi_L, \chi_R) = Q^{2U-FL}_{o1,i+1/2}(\chi_L, \chi_R) + \frac{\Delta t}{\Delta x} Q^{2U-FL}_{o2,i+1/2}(\chi_L, \chi_R),$$

with

$$\begin{aligned} Q^{2U-FL}_{o1,i+1/2}(\chi_L, \chi_R) &= \frac{\text{sgn}(S_L)(1-\chi_L) + \text{sgn}(S_R)(1-\chi_R)}{2} A_{i+1/2} \\ &+ \frac{\text{sgn}(S_R)(1-\chi_R)}{2} P_{2,\alpha_R}(A_{i+1/2}) - \frac{\text{sgn}(S_L)(1-\chi_L)}{2} P_{2,\alpha_L}(A_{i+1/2}), \end{aligned}$$

and

$$Q^{2U-FL}_{o2,i+1/2}(\chi_L, \chi_R) = \frac{S_L\chi_L + S_R\chi_R}{2} A_{i+1/2} + \frac{S_R\chi_R}{2} P_{2,\alpha_R}(A_{i+1/2}) - \frac{S_L\chi_L}{2} P_{2,\alpha_L}(A_{i+1/2}),$$

where

$$\alpha_K = 1 - (1-\chi_K)(1-\alpha), \quad K = L, R,$$

$$\alpha = \frac{(S_R - S_L)sgn(S_M) - (S_R + S_L)}{4S_M - 2(S_L + S_R)},$$

Two-waves PVM2U-WAF methods

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and

$$Q^{2U-FL}_{o2,i+1/2}(\chi_L, \chi_R) = \frac{S_L\chi_L + S_R\chi_R}{2} A_{i+1/2} + \frac{S_R\chi_R}{2} P_{2,\alpha_R}(A_{i+1/2}) - \frac{S_L\chi_L}{2} P_{2,\alpha_L}(A_{i+1/2}),$$

$$P_{2,\alpha}(x) = \alpha P_{2M}(x) + (1-\alpha) P_{1M}(x),$$

and

$$P_{1M}(x) = \frac{-2S_R S_L}{S_R - S_L} + \frac{S_R + S_L}{S_R - S_L} x.$$

Two-wave PVM-2U-WAF method

Properties

- Only uses the information of the two fastest waves.
- If $N = 2$ it coindices with the usual HLL-WAF scheme.
- If $\chi_L = \chi_R = 0$, then we recover the PVM-2U first order scheme for $N \geq 2$.
- If $\chi_L = \chi_R = 1$, then scheme reduces to Lax-Wendroff scheme for $N \geq 2$.

Two-wave PVM-2U-WAF method

Remark

A natural extension to balance laws and non-conservative system is straightforward:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(D_{i-1/2}^{+} + D_{i+1/2}^{-}\right),$$

with

$$\begin{aligned} D_{i+1/2}^{\pm} = \; & \frac{1}{2}\Big(F(w_{i+1}) - F(w_i) + B_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i) \\ & \pm Q_{i+1/2}(w_{i+1} - w_i - A_{i+1/2}^{-1} G_{i+1/2}(H_{i+1} - H_i))\Big), \end{aligned}$$

being

$$Q_{i+1/2} = Q_{i+1/2}^{2U-FL}(\chi_L, \chi_R)$$

But it is not second order.

Two-wave PVM-2U-WAF method

To recover the second order, new terms appear in the Lax-Wendroff scheme due to the non-conservative products and source terms (see Castro, Pares & Toro Math. Comp. 2010):

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(D^+_{i-1/2} + D^-_{i+1/2}\right) + \frac{\Delta t^2}{4\Delta x^2}\left(\mathcal{R}(\chi_L,\chi_R)^n_{i-1/2} + \mathcal{R}(\chi_L,\chi_R)^n_{i+1/2}\right)$$

with

$$\begin{aligned}
\mathcal{R}(\chi_L,\chi_R)^n_{i+1/2} = \quad & \frac{1}{2}\Big(\quad \chi_L DA(W_i)[A_{i+1/2}(w_{i+1}-w_i) - G_{i+1/2}(H_{i+1}-H_i), w_{i+1}-w_i] \\
& + \quad \chi_R DA(W_{i+1})[A_{i+1/2}(w_{i+1}-w_i) - G_{i+1/2}(H_{i+1}-H_i), w_{i+1}-w_i] \\
& - \quad \chi_L DA(W_i)[w_{i+1}-w_i, A_{i+1/2}(w_{i+1}-w_i) - G_{i+1/2}(H_{i+1}-H_i)] \\
& - \quad \chi_R DA(W_{i+1})[w_{i+1}-w_i, A_{i+1/2}(w_{i+1}-w_i) - G_{i+1/2}(H_{i+1}-H_i)] \\
& - \quad \chi_L G_w(w_i)(A_{i+1/2}(w_{i+1}-w_i) - G_{i+1/2}(H_{i+1}-H_i))(H_{i+1}-H_i) \\
& - \quad \chi_R G_w(w_{i+1})(A_{i+1/2}(w_{i+1}-w_i) - G_{i+1/2}(H_{i+1}-H_i))(H_{i+1}-H_i) \Big)
\end{aligned}$$

being $DA(W)[U,V] = \left(\sum_{l=1}^N u_l \partial_{w_l} A(W)\right) V$ and $\partial_{w_l} A(W)$ is the $N \times N$ matrix whose (i,j) element is $\partial_{w_l} a_{ij}(W)$. $G_w(w)$ denotes the Jacobian matrix of $G(w)$.

Multilayer shallow water system

$$\begin{cases} \partial_t h_j + \partial_x q_j = 0, \\ \partial_t q_j + \partial_x \left(\frac{q_j^2}{h_j} + \frac{1}{2} g h_j^2\right) + g h_j \partial_x \left(z_b + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k\right) = 0. \end{cases} \qquad j = 1, \dots, m,$$

where m is the number of layers, $h_j, j = 1, \dots, m$ are the fluid depths, $q_j = h_j u_j$ are the discharges, u_j are the velocites and $z_b(x)$ is the topography. g is the gravity constant and ρ_j the densisites of the stratified fluid layers, with

$$0 < \rho_1 < \cdots < \rho_m.$$

We use the Van Albada's limiter with a smooth indicator of the fluid interfaces. Total energy of the system provides also good results.

1LSW: Stationary subcritical solution over bump

- $I = [0, 20]$.
- $z_b(x) = 0.2e^{-0.16(x-10)^2}$
- Boundary conditions: $q(0,t) = 4.42$ and $h(20,t) = 2.0$
- $\Delta x = 1/20$.
- $cfl = 0.9$.

Nodes	L^1 err h	L^1 order h	L^1 err q	L^1 order q
20	1.58×10^{-3}	-	5.02×10^{-3}	-
40	5.07×10^{-4}	1.646	1.21×10^{-3}	2.0513
80	1.3×10^{-4}	1.967	3.04×10^{-4}	1.9965
160	3.2×10^{-5}	1.995	7.6×10^{-5}	1.9985
320	9×10^{-6}	1.904	1.9×10^{-5}	1.9987

Table: Errors and order. Subcritical stationary solution.

1LSW: Order of accuracy

- $I = [0, 1]$, $T = 0.1$.
- $z_b(x) = \text{sen}^2(\pi x)$.
- Initial condition: $h(x, 0) = 5 + e^{\cos(2\pi x)}, \quad q(x, 0) = \sin(\cos(2\pi x))$,
- $CFL = 0.8$.
- Reference solution computed with ROE scheme with $\Delta x = 1/12800$.

Nodes	L^1 err h	L^1 order h	L^1 err q	L^1 order q
25	2.802×10^{-2}	-	3.14×10^{-1}	-
50	1.021×10^{-2}	1.45	9.702×10^{-2}	1.69
100	3.228×10^{-3}	1.66	2.677×10^{-2}	1.85
200	9.15×10^{-4}	1.81	6.594×10^{-3}	2.02
400	2.53×10^{-4}	1.85	1.553×10^{-3}	2.08
800	6.45×10^{-5}	1.97	3.78×10^{-4}	2.02

Table: Errors and order.

1LSW stationary transcritical flow with a shock

- $I = [0, 20]$
- bottom topography:

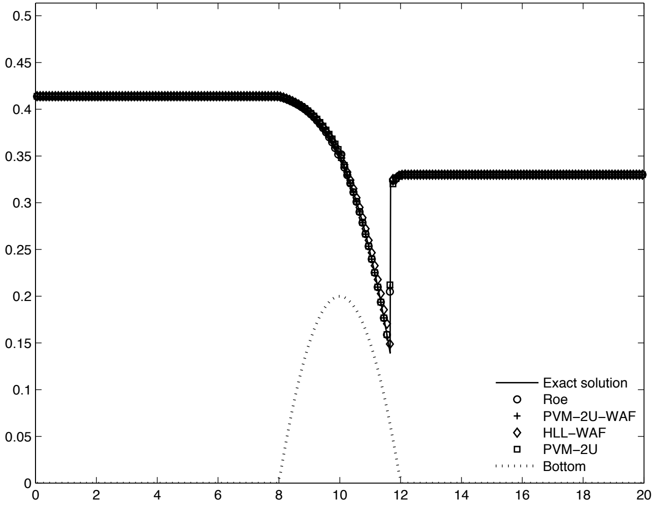
$$z_b(x) = \begin{cases} 0.2 - 0.05(x-10)^2 & 8 < x < 12 \\ 0 & otherwise \end{cases} .$$

- Initial condition

$$h(x,0) = 0.33 - z_b, q(x,0) = 0.$$

- Boundary conditions: $q = 0.18$ at x=0 and $h = 0.33$ at $x = 20$
- $\Delta x = 1/10$ and $CFL = 0.9$.
- Concerning CPU time similar results are obtained for Roe solver and PVM-2U WAF method.

1LSW stationary transcritical flow with a shock

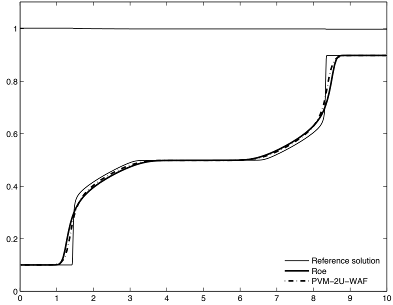


(a) Free surface an bottom topography

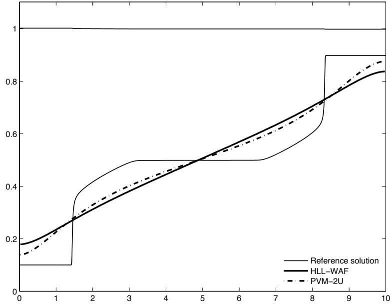
2LSW: Internal dam break problem.

- $I = [0, 10]$.
- $zb(x) = 0$
- Initial condition:
  $q_1(x,0) = q_2(x,0) = 0, \quad \forall x \in [0,10],$
  $h_1(x,0) = \begin{cases} 0.9 & \text{if } x < 5, \\ 0.1 & \text{if } x \geq 5, \end{cases}$
  $h_2(x,0) = 1.0 - h_1(x,0) \quad \forall x \in [0,10].$
- Open boundary conditions
- $\Delta x = 1/20$.
- A reference solution is computed Roe scheme with $\Delta x = 1/200$.
- $r = 0.99$,
- $cfl = 0.9$.
- Concerning CPU time PVM-2U-WAF method is 2.8 times faster than Roe and similar to original HLL-WAF scheme.

2LSW: Internal dam break problem.



(b) PVM-2U-WAF and Roe



(c) HLL-WAF and PVM-2U

Figure: Internal dam break: free surface and interface at $t = 20$ seg.

2LSW: Stationary transcritical flow with an internal shock

- $I = [0, 10]$
- Bottom topography:

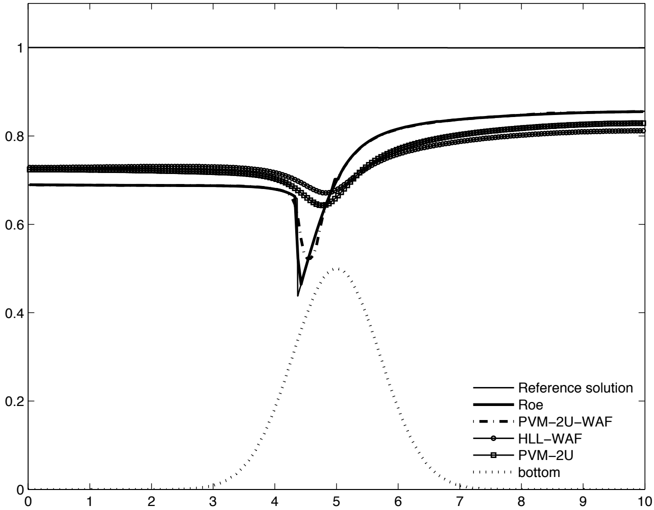
$$z_b(x) = 0.5e^{(x-5)^2}.$$

- Initial condition: $q_1(x,0) = q_2(x,0) = 0.$ and

$$h_1(x,0) = \begin{cases} 0.48 & \text{if } x < 5, \\ 0.5 & \text{if } x \geq 5, \end{cases} \qquad h_2(x,0) = 1 - h_1(x,0) - z_b(x),$$

- $\rho_1/\rho_2 = 0.99.$
- Free boundary conditions.
- $CFL = 0.9$, $\Delta x = 1/20$. Reference solution computed with $\Delta x = 1/200$.

2LSW: Stationary transcritical flow with an internal shock



(a) Free surface, bottom topography and interface

4LSW: Internal dam breaks.

- $I = [0, 10]$
- Bottom topography:

$$z_b(x) = 0.0$$

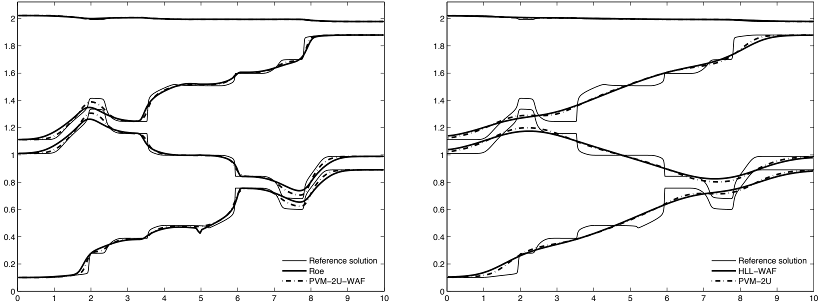
- Initial condition: $q_i(x,0) = 0., i = 1, \ldots, 4$ and

$$h_1(x,0) = \begin{cases} 0.9 & \text{if } x < 5, \\ 0.1 & \text{if } x \geq 5, \end{cases}$$

$$h_2(x,0) = 1 - h_1(x,0), \quad h_3(x,0) = h_1(x,0), \quad h_4(x,0) = h_2(x,0)$$

- $\rho_1/\rho_4 = 0.85$, $\rho_2/\rho_4 = 0.9$, $\rho_3/\rho_4 = 0.95$.
- Free boundary conditions.
- $CFL = 0.9$, $\Delta x = 1/20$. Reference solution computed with $\Delta x = 1/200$.
- Concerning CPU time PVM-2U-WAF method is 9.8 times faster than Roe and similar to original HLL-WAF scheme.

4LSW: Internal dam breaks.



(b) PVM-2U-WAF and Roe (c) HLL-WAF and PVM-2U

Figure: Internal dam breaks: free surface and interfaces at $t = 5$ seg.

2D 1LSW: Circular dam break

- $D = [-2, 2] \times [-2, 2]$
- Bottom topography:

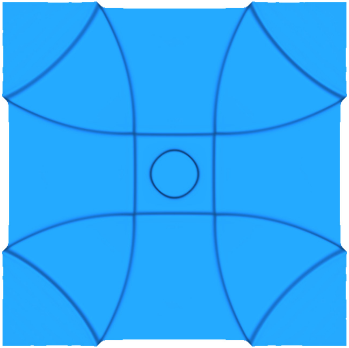
$$zb(x, y) = 0.8\, e^{-x^2 - y^2}$$

- Initial condition: $q_x(x, y, 0) = q_y(x, y, 0) = 0$ and

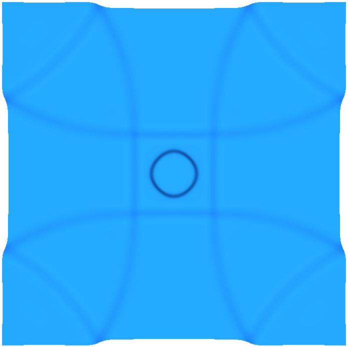
$$h(x, y, 0) = \begin{cases} 1 - zb(x, y) + 0.5 & \text{if } \sqrt{x^2 + y^2} < 0.5 \\ 1 - zb(x, y) & \text{otherwise} \end{cases}$$

- Wall boundary conditions
- $\Delta x = \Delta y = 1/100$, $CFL = 0.9$.
- Three implementations are considered: first and second order HLL and PVM-2U-WAF method.
- Algorithms implemented on GPUs: speedups of more than 200 for the three numerical schemes,
- The extension by the method of lines of 1D PVM-2U-WAF method to multidimensional problems is NOT second order accurate.

2D 1LSW: Circular dam break



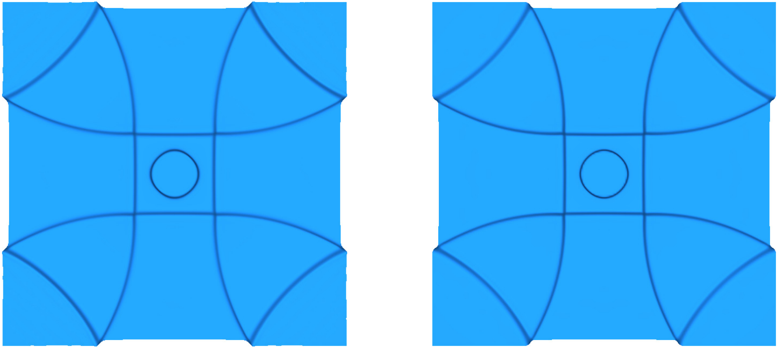
(a) PVM-2U WAF $t = 1.0$ s



(b) HLL $t = 1.0$ s

Figure: 2D circular dam break: free surface at $t = 1$ seg.

2D 1LSW: Circular dam break

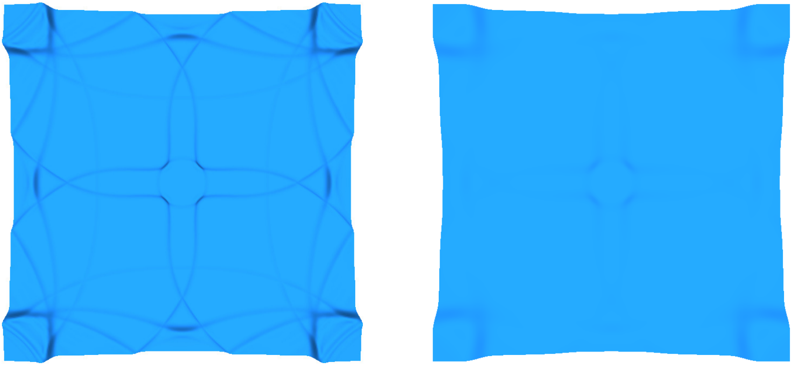


(a) PVM-2U WAF $t = 1.0$ s

(b) Second order HLL $t = 1.0$ s

Figure: 2D circular dam break: free surface at $t = 1$ seg.

2D 1LSW: Circular dam break

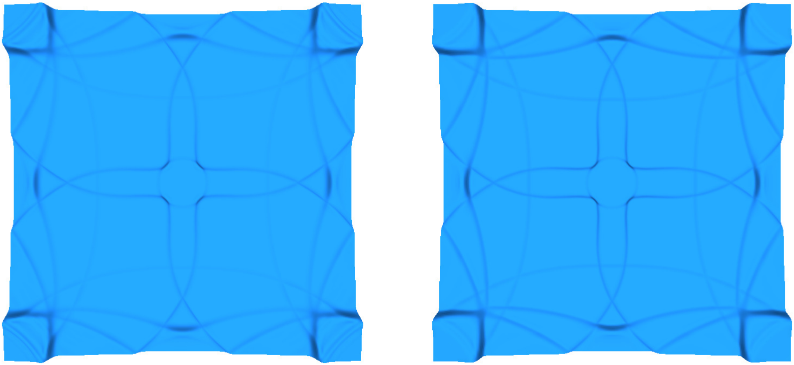


(a) PVM-2U WAF $t = 2.0$ s

(b) HLL $t = 2.0$ s

Figure: 2D circular dam break: free surface at $t = 1$ seg.

2D 1LSW: Circular dam break



(a) PVM-2U WAF $t = 2.0$ s   (b) Second order HLL $t = 2.0$ s

Figure: 2D circular dam break: free surface at $t = 2$ seg.

Conclusions

Conclusions

- The original two-wave HLL-WAF method can be seen as a PVM-based flux-limiting scheme.
- A new two-wave WAF method that ensured second order of accuracy for $N > 2$ is defined using PVM framework.
- It can be applied to conservative, balance laws and non-conservative systems.
- Its performance increases with the complexity of the system. It can be 10 times faster than Roe solver for the 1D 4LSW.
- Extension to 2D that preserves second order accuracy: comming soon, it is NOT straight forward.