Continuous solutions to a balance law

L. Caravenna, OxPDE

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G. Alberti, S. Bianchini

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Outline

Eulerian vs Lagrangian formulation for the PDE

$$u_t(t,x) + [u(t,x)^2/2]_x = g(t,x), \qquad g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$$

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Few recalls on the equation: characteristics

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PDE $\leftrightsquigarrow$ system of ODEs

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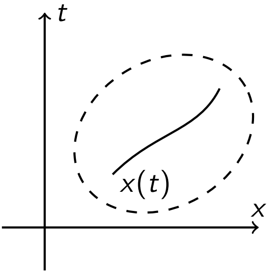
If u smooth chain rule

$u(s, x(s))$



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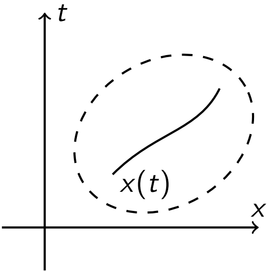
$$\frac{d}{ds} u(s,x(s)) = u_t(s,x(s)) + u_x(s,x(s)) \cdot \dot{x}(s)$$



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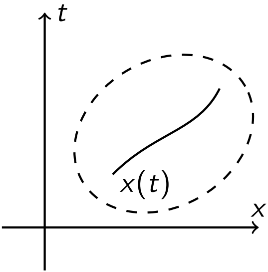
$$\frac{d}{ds}u(s,x(s)) = u_t(s,x(s)) + u_x(s,x(s)) \cdot \dot{x}(s)$$

If $\boxed{\dot{x}(s) = f'(u(s,x(s)))}$

$$= u_t(\Psi(s)) + f'(u(s,x(s))) \cdot u_x(s,x(s))$$
$$= g(\Psi(s))$$



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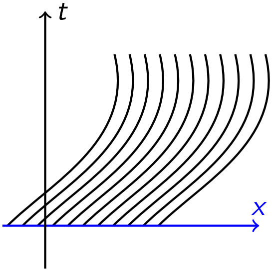
If u smooth

$$\begin{cases} \frac{d}{ds}\chi(s;x_0) = f'(u(s,\chi(s;x_0))) \\ \frac{d}{ds}u(s,x(s)) = g(s,x(s)) \\ \chi(t=0;x_0) = x_0 \qquad u(0,x) = u_0(x) \end{cases}$$



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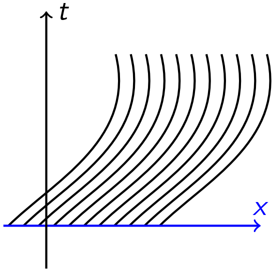

Straight lines when $g = 0$

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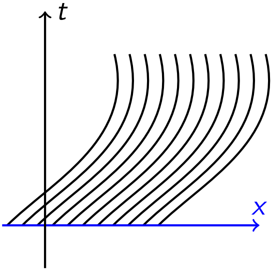
PDE $\leftrightsquigarrow$ system of ODEs

If u smooth

$$\mathbf{z}(s;x_0) = \begin{pmatrix} \chi(s;x_0) \\ u(s,\chi(s;x_0)) \end{pmatrix}$$

$$\begin{cases} \dot{z} = \begin{pmatrix} f'(\mathbf{z}) \\ g(\mathbf{z}) \end{pmatrix} \\ \mathbf{z}(s=0;x) = \begin{pmatrix} x \\ u_0(x) \end{pmatrix} \end{cases}$$



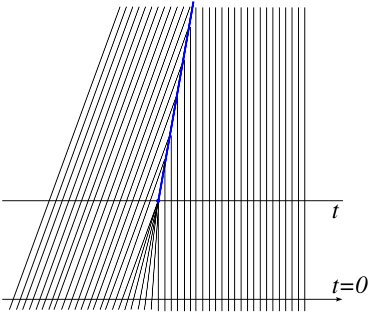
Few recalls on the equation

u may develop jumps

$$\text{e.g. } u_t + \left[\frac{u^2}{2}\right]_x = 0$$

$\rightsquigarrow$ generalized characteristics, def. by differential inclusions

Not the issue here



C. M. Dafermos.
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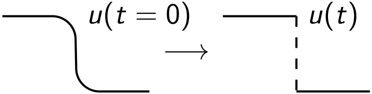
Springer, 2000.



Non uniqueness of distributional solutions $\rightsquigarrow$ **entropy solutions**
Existence, stability, uniqueness clear in the scalar, $1D$-case
[Oleinik, Kruzhkov,. . .]

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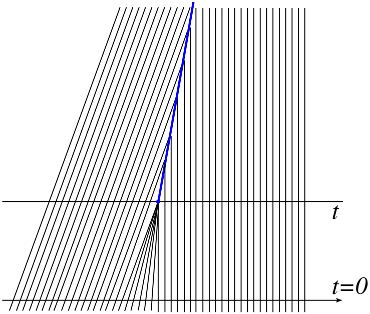

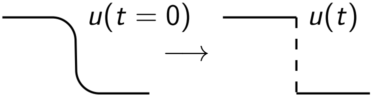

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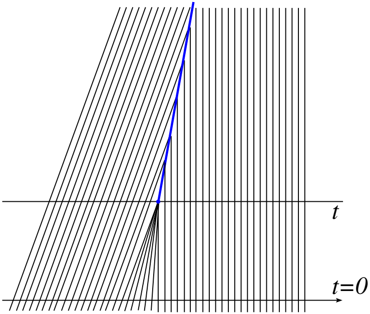

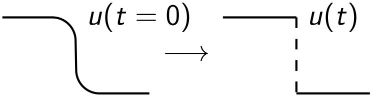

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An example: Hunter-Saxton equation

$$u_t + \left[\frac{u^2}{2}\right]_x = g, \qquad g(t,x) = \frac{1}{2}\int_{-\infty}^{x} u_x^2(y,t)dy$$

g may be interpreted as a control device preserving continuity

C. M. Dafermos
Continuous solutions for balance laws
Ricerche di Matematica 55 (2006), 79–91.

Many authors

An example: Hunter-Saxton equation

$$u_t(t,x) + \left[\frac{u^2(t,x)}{2}\right]_x = g(t,x)$$

We restrict from now on to the particular case

u continuous, g bounded

u continuous is a low regularity for this equation
Not enough for the theory on ODEs with rough coefficients

Motivation

Sub-Riemannian Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1} = \{(\underline{x}, \underline{y}, z)\}$

$$\phi : \mathbb{W} = \{x_1 = 0\} \subset \mathbb{H}^n \to \mathbb{R} \text{ continuous} \quad \rightsquigarrow \quad \mathrm{Graph}_{\mathbb{H}}\phi$$

It induces a graph quasidistance d_ϕ and an intrinsic differentiable structure on $\mathbb{W}$. One has notions of intrinsic cones, intrinsic differentiability, intrinsic gradient, ...

The graph is **intrinsic regular** if

- ϕ is uniformly intrinsic differentiable
- $(n = 1)$ ϕ satisfies $\phi_y + \left[\frac{\phi^2}{2}\right]_z = g$ with g continuous

Ambrosio-Serra Cassano-Vittone

Franchi-Serapioni-Serra Cassano

Bigolin-Serra Cassano

Very irregular from the Euclidean point of view

Kirchheim-Serra Cassano

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$n = 1$ for simplicity of presentation

The graph is **intrinsic Lipschitz** if ϕ satisfies

$$\phi_y + \left[\frac{\phi^2}{2}\right]_z = g \qquad g \text{ bounded}$$

Citti-Manfredini-Pinamonti-Serra Cassano

Franchi-Serapioni-Serra Cassano

Bigolin-C-Serra Cassano

Example

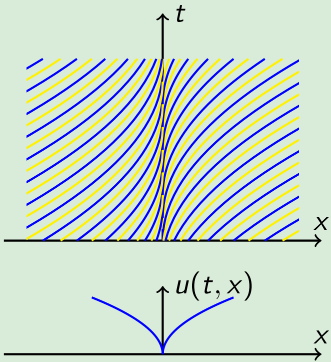
$$u_t + \left[\frac{u^2}{2}\right]_x = \begin{cases} 1/2 & x \geq 0 \\ -1/2 & x < 0 \end{cases}$$

Distributional solution:

$$u(t,x) = \sqrt{|x|}$$

Characteristics:

$$\dot{x}(t) = u(t,x)$$



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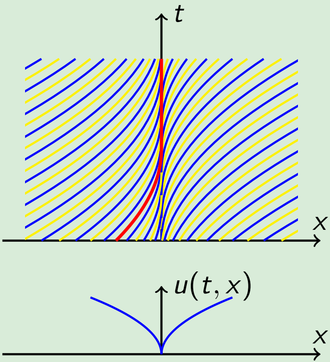

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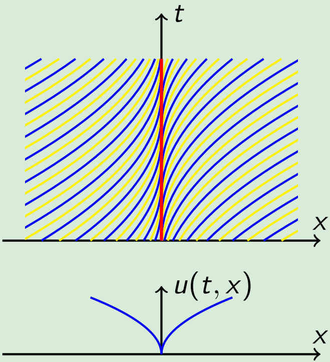

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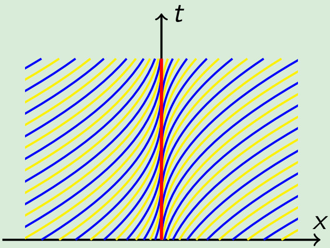

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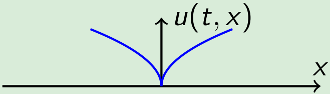


Characteristics:

$$\dot{x}(t) = u(t,x)$$

$$\frac{d}{dt}u(t,x(t)) = g(t,x(t))$$

The right representative is $\hat{g}(t,x) = \begin{cases} 1/2 & x > 0 \\ 0 & x = 0 \\ -1/2 & x < 0 \end{cases}$

Two notions of continuous solutions

Let $f \in C^2(\mathbb{R}^+ \times \mathbb{R})$ and $g \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$. Let $u \in C(\mathbb{R}^+ \times \mathbb{R})$

Distributional solution:

$$u_t + f(u)_x = g \qquad \text{in } \mathcal{D}(\mathbb{R}^+ \times \mathbb{R})$$

Broad solution: $\exists\ \hat{g} \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ s.t. $\|g - \hat{g}\|_{\mathcal{L}^2} = 0$ and

$$\frac{d}{dt} u(t, \gamma(t)) = \hat{g}(t, \gamma(t)) \qquad \forall \gamma(t) \, : \, \dot{\gamma} = u \circ \gamma.$$

Theorem [Bigolin-C.-Serra Cassano, available on arxiv]

Let u be a continuous distributional solution of

$$u_t(t,x) + \left[\frac{u^2(t,x)}{2}\right]_x = g(t,x), \qquad g \in L^\infty(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}).$$

Choosing suitably $\hat{g} \in \mathfrak{L}^\infty(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$ representative of g

$$\frac{d}{dt} u(t,\gamma(t)) = \hat{g}(t,\gamma(t)) \qquad \forall \gamma(t) \,:\, \dot{\gamma} = u \circ \gamma.$$

u is moreoveover $1/2$-Hölder continuous

For the converse implication

CITTI-MANFREDINI-PINAMONTI-SERRA CASSANO

Let $f \in C^2(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$

Theorem [Alberti-Bianchini-C, forthcoming]

Let u be a continuous distributional solution of

$$u_t(t,x) + f(u(t,x))_x = g(t,x), \qquad g \in L^\infty(\mathbb{R}^+ \times \mathbb{R}).$$

If $\overline{\mathrm{Infl}(f)} = 0$ is negligible, then $\exists \hat{g} \in \mathfrak{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$:

$$\frac{d}{dt} u(t,\gamma(t)) = \hat{g}(t,\gamma(t)) \qquad \forall \gamma(t) \, : \, \dot{\gamma} = f'(u) \circ \gamma.$$

$t \mapsto u(t,\gamma(t))$ not generally Lipschitz without assumptions on f

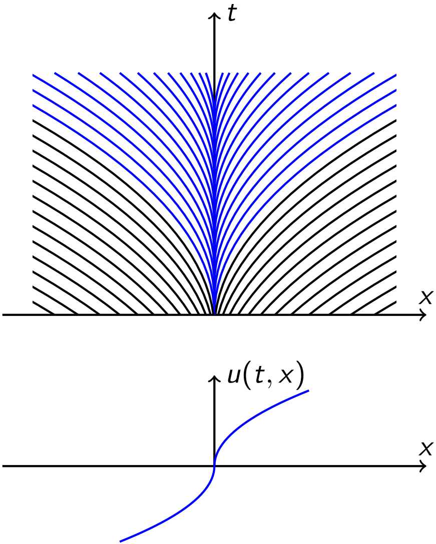
Outline

Some examples

Characteristics split

e.g. $\dot{x}(t) = \quad \operatorname{sgn} x(t)\sqrt{|x(t)|},$

$$u_t + \left[\frac{u^2}{2}\right]_x = \begin{cases} 1/2 & x \geq 0 \\ -1/2 & x < 0 \end{cases}$$



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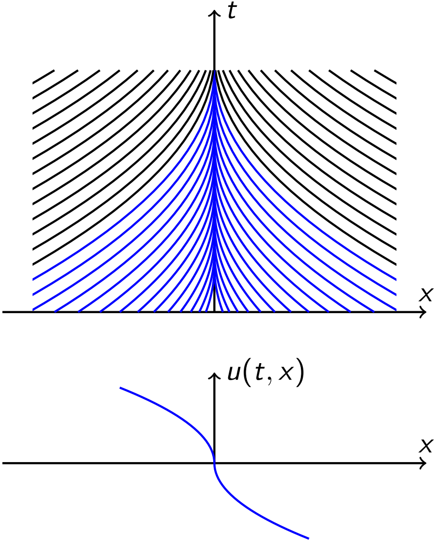
They may also concentrate



Some examples

Sometimes not

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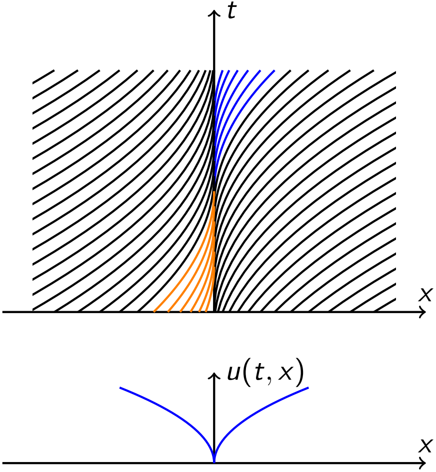

We are not in this case

G. Crippa.
Lagrangian flows and the one dimensional Peano phenomenon for ODEs.
JDE, to appear.

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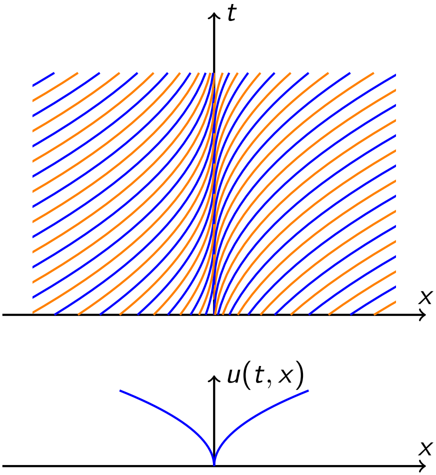

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Some hints of the proof

1. Dafermos' computation

Given any characteristic γ, compute the balance in the region

$$\Gamma^\varepsilon = \{(t,x) : \ t_1 \le t \le t_2, \ \gamma(t) \le x \le \gamma(t) + \varepsilon\}$$

$$\begin{aligned}
&\int_{\gamma(t_2)}^{\gamma(t_2)+\varepsilon} u(t,x)dx - \int_{\gamma(t_1)}^{\gamma(t_1)+\varepsilon} u(t,x)dx \\
&\quad = \int_{t_1}^{t_2} \int_{\gamma(t)}^{\gamma(t)+\varepsilon} g(t,x)dtdx - \frac{1}{2} \int_{t_1}^{t_2} \left[u(t,\gamma(t)+\varepsilon) - u(t,\gamma(t))\right]^2 dt \\
&\quad \ge \int_{t_1}^{t_2} \int_{\gamma(t)}^{\gamma(t)+\varepsilon} g(t,x)dtdx
\end{aligned}$$

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In the analogous region $\Gamma^{-\varepsilon}$ one has the opposite inequality:

$$\int_{t_1}^{t_2} \int_{\gamma(t)-\varepsilon}^{\gamma(t)} g(t,x) dt dx \le \int_{\gamma(t_2)}^{\gamma(t_2)+\varepsilon} u(t,x) dx - \int_{\gamma(t_1)}^{\gamma(t_1)+\varepsilon} u(t,x) dx$$
$$\le \int_{t_1}^{t_2} \int_{\gamma(t)}^{\gamma(t)+\varepsilon} g(t,x) dt dx$$

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When g continuous in x one can take the limit as $\varepsilon \downarrow 0$:

$$u(t_1, \gamma(t_2)) - u(t_1, \gamma(t_1)) = \int_{t_1}^{t_2} g(t, \gamma(t))dt$$

If g only bounded

$$|u(t_1, \gamma(t_2)) - u(t_1, \gamma(t_1))| \le \|g\|_\infty (t_2 - t_1)$$

Therefore u Lipschitz along characteristics

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This can be done under the hypothesis on the inflection points of f

Otherwise there is a counterexample

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2. Monotone change of variable

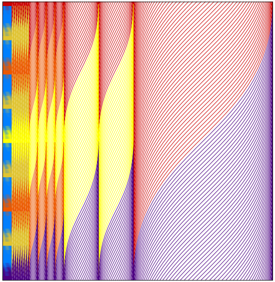
Focus on $\mathbb{R}^+ \times \mathbb{R}$, all is local by the finite speed of propagation

Define a change of variables $(t, y) \mapsto (t, x) = (t, \chi(t, y))$ where

- χ is surjective
- for all t, $y \mapsto \chi(t, y)$ is nondecreasing
- for all t, $\frac{d}{dt}\chi(t, \chi(t, y)) = u(t, \chi(t, y))$

Some hints of the proof

This change of variable is BV not Lipschitz in general



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This is possible by the continuity of u, applying Peano's theorem

Let

$$\tilde{g} : \ \frac{d}{dt}u(t, \gamma(t)) = \frac{d^2}{dt^2}\chi(t, y) = \tilde{g}(t, \chi(t, y)) \text{ in } \mathcal{D}'$$

Still, it is not fine.

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- for all t, $\frac{d}{dt}\chi(t, \chi(t, y)) = u(t, \chi(t, y))$

This is possible by the continuity of u, applying Peano's theorem

Let

$$\tilde{g} : \ \frac{d}{dt}u(t, \gamma(t)) = \frac{d^2}{dt^2}\chi(t, y) = \tilde{g}(t, \chi(t, y)) \text{ in } \mathcal{D}'$$

Still, it is not fine.

Some hints of the proof

2. Monotone change of variable

Focus on $\mathbb{R}^+ \times \mathbb{R}$, all is local by the finite speed of propagation

Define a change of variables $(t, y) \mapsto (t, x) = (t, \chi(t, y))$ where

- χ is surjective
- for all t, $y \mapsto \chi(t, y)$ is nondecreasing
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Some hints of the proof

3. Selection

Consider the subset of $\{t > \delta\} \times \mathbb{R} \times C((-\delta,\delta)) \times [-\|g\|_\infty, \|g\|_\infty]$

$$\{(t,x,\gamma,\xi) : \ \gamma(0) = x, \ \dot\gamma(0) = u(t,x), \ \exists \ddot\gamma(0) = \xi\}$$

One can define a Borel function $\hat g(t,x)$ by a selection theorem

- given a characteristic, it touches characteristics with different second derivative only at most at countably many times
- at (t,x)-a.e. point, every characteristic through that point is differentiable with derivative $g(t,x)$

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Some hints of the proof

4. Approximation
Let $u \in C(\mathbb{R}^+ \times \mathbb{R})$
Let $(t, \chi(t, y))$ be a a monotone change of variables as above: for $\tilde{g} \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$

$$\partial_t \chi(t, y) = u(t, \chi(t, y)), \qquad \partial_t^2 \chi(t, y) = \tilde{g}(t, \chi(t, y))$$

Then choosing a convolution kernel $\rho^\varepsilon(t, y)$ define

$$\chi^\varepsilon = \chi * \rho^\varepsilon$$

smooth change of variables. Then at $x = \chi^\varepsilon(t, y)$ define

$$u^\varepsilon(t, x) = \partial_t \chi^\varepsilon(t, y) \quad g^\varepsilon(t, x) = \partial_t^2 \chi^\varepsilon(t, y)$$

They are smooth approximations satisfying

$$u_t^\varepsilon(t, x) + \frac{[(u^\varepsilon(t, x))^2]_x}{2} = g^\varepsilon(t, x)$$

Some hints of the proof

5. A counterexample

Let $f \in C^2(\mathbb{R})$ positive, strictly increasing such that

$$N = \{v : \ f'(v) = f''(v) = 0\} \text{ satisfies } \mathcal{L}^1(N) > 0$$

Then

$$\tilde{f}(v + L^1(N \cap [0, v])) = f(v) \quad \tilde{f}'(v) = f'(f^{-1}(\tilde{f}(v))$$

The function $u(t, x) = f^{-1}(x)$ satisfies $u_t + f(u)_x = 1$, but on the characteristic $\gamma(t) = \tilde{f}(t)$

$$\frac{d}{dt} f^{-1}(\tilde{f}(t)) = \mathcal{L}^1 + f_\sharp \mathcal{L}^1 \restriction_N$$

Thanks for your attention!

Outline

3. Introduction to intrinsic Lipschitz graphs in $\mathbb{H}^n$
    - Carnot-Carathéodory spaces
    - Graphs in the Heisenberg groups
    - Main statement

$X_1, \dots, X_m$ smooth ('orthonormal') horizontal vector fileds in $\mathbb{R}^n$.
A curve $\gamma \in \mathrm{AC}([0,1]; \mathbb{R}^N)$ is horizontal if $\dot{\gamma} \in \langle X_1, \dots, X_m \rangle$ a.e.

The Carnot-Carathéodory distance d_C between $P, Q \in \mathbb{R}^N$ is

$$d_C(P, B) = \inf \{ \mathrm{length}_X(\gamma) : \gamma \text{ horizontal curve joining } P \text{ to } Q \}$$

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Sub-Riemannian Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1} = \{(\underline{x}, \underline{y}, z)\}$
Noncommutative Lie group: $P \cdot Q = (\underline{x}, \underline{y}, z) \cdot (\underline{x}', \underline{y}', z')$ is

$$\left(\underline{x} + \underline{x}',\ \underline{y} + \underline{y}',\ z + z' - \frac{1}{2}\Im(\langle \underline{x} + i\underline{y}, \underline{y}' - i\underline{y}' \rangle)\right)$$

Horizontal vector fileds: $X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial z}$, $Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial z}$
Vertical vector field: $Z = \frac{\partial}{\partial z} \equiv [X_j, Y_j]$, $j = 1, \ldots, n$

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Vertical vector field: $Z = \frac{\partial}{\partial z} \equiv [X_j, Y_j]$, $j = 1, \dots, n$

$$d_C \sim d_\infty(P,Q) := \max\left\{ |\underline{x}'' + i\underline{y}''|, |z''|^{1/2} \right\}, \quad (\underline{x}'', \underline{y}'', z'') = P^{-1} \cdot Q$$

X_1-graphs in $\mathbb{H}^n$

Let $\phi : \mathbb{W} = \{x_1 = 0\} \subset \mathbb{H}^n \to \mathbb{R}$ continuous **$n = 1$ for shortness**

$$\text{Graph}_{\mathbb{H}}\phi := \{A \cdot \phi(A)e_1 : A \in \mathbb{W}\}$$
$$= \left\{ \left(\phi(A), y^A, z^A - \frac{y^A}{2}\phi(A) \right) : \ A \in \mathbb{W} \right\}$$

It induces the graph quasidistance d_ϕ on $\mathbb{W}$

$$d_\phi(A, B) = \left| y^B - y^A \right| + \left| z^B - z^A - \frac{1}{2}(\phi(A) + \phi(B))(y^B - y^A) \right|^{1/2}$$

[This formula for d_ϕ is from Ambrosio-Serra Cassano-Vittone]

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It induces an intrinsic differentiable structure. One has notions of intrinsic cones, intrinsic differentiability, intrinsic gradient, ...

Intrinsic Regular X_1-graphs in $\mathbb{H}^n$

Let $\phi : \omega \subset \mathbb{W} = \{x_1 = 0\} \subset \mathbb{H}^n \to \mathbb{R}$ continuous

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A $\mathbb{W}$-linear functional $L : \mathbb{W} \to \mathbb{R}$ is a group homeomorphism s.t.

$$L(rx_2, \dots, ry_n, r^2 z) = rL(x_2, \dots, y_n, z) \text{ for } r > 0$$

ϕ is ∇^ϕ-differentiable at $A_0 \in \mathbb{W}$ if $\exists$ a $\mathbb{W}$-linear $L : \mathbb{W} \to \mathbb{R}$ s.t.

$$\lim_{A \to A_0} \frac{\phi(A) - \phi(A_0) - L(A_0^{-1} \cdot A)}{d_\phi(A_0, A)} = 0. \tag{1}$$

Roughly, L represented as scalar product with horizontal variables, the intrinsic gradient $\nabla^\phi \phi$ given by this representation

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The graph is **intrinsic regular** if

- ϕ is uniformly intrinsic differentiable
- it is the level set of $\Phi : \mathbb{H} \to \mathbb{R}$ which is C^1 along all X_i, Y_i and s.t. $X_1\Phi \neq 0$ (Intrinsic Implicit Function Theorem)
- $(n = 1)$ ϕ satisfies $\phi_y + \left[\frac{\phi^2}{2}\right]_z = g$ with $g : \omega \to \mathbb{R}$ continuous

In this case, ϕ is uniformly intrinsic differentiable and $\nabla^\phi \phi = g$

Ambrosio-Serra Cassano-Vittone

Franchi-Serapioni-Serra Cassano

Bigolin-Serra Cassano

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They can be very irregular from the Euclidean point of view

Kirchheim-Serra Cassano

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Intrinsic Lipschitz X_1-graphs in $\mathbb{H}^n$

Let $\phi : \omega \subset \mathbb{W} = \{x_1 = 0\} \subset \mathbb{H}^n \to \mathbb{R}$ continuous

$$\mathrm{Graph}_{\mathbb{H}}\phi := \{A \cdot \phi(A)e_1 : A \in \omega\}$$

$$C(0,\alpha) := \{||(0, x_2, \dots, y_n, z)||_\infty \leq \alpha\, ||(x_1, 0, 0)||_\infty\}_{(x,y,z)\in\mathbb{H}^n}$$

The graph is **intrinsic Lipschitz** if

- 'translating' an intrinsic cone $C(0,\alpha)$ at any point of the graph, it intersects the graph only in the vertex
- $\exists L > 0$ tale che $|\phi(A) - \phi(B)| \leq L d_\phi(A, B) \qquad \forall A, B \in \omega$
- 'by approximation' with intrinsic regular graphs

In this case, ϕ is almost everywhere intrinsically differentiable

FRANCHI-SERAPIONI-SERRA CASSANO

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Local concepts

Intrinsic Lipschitz X_1-graphs in $\mathbb{H}^n$

Let $\phi : \omega \subset \mathbb{W} = \{x_1 = 0\} \subset \mathbb{H}^n \to \mathbb{R}$ continuous

$$\mathrm{Graph}_{\mathbb{H}}\phi := \{A \cdot \phi(A)e_1 : A \in \omega\}$$

Theorem (for shortness written here for $n = 1$)

The graph is intrinsic Lipschitz if and only if ϕ satisfies

$$\phi_y + \left[\frac{\phi^2}{2}\right]_z = g$$

with $g : \omega \to \mathbb{R}$ bounded. $g = \nabla^{\phi}\phi$, the intrinsic gradient of ϕ.

- Citti-Manfredini-Pinamonti-Serra Cassano
- Bigolin-C.-Serra Cassano

We also have the analogous statement for $n > 1$