Time Asymptotic High Order Schemes for Dissipative BGK Hyperbolic Systems

Maya Briani

Istituto per le Applicazioni del Calcolo "M. Picone" (CNR)

m.briani@iac.cnr.it
http://www.iac.rm.cnr.it/~briani

Roberto Natalini – IAC-CNR
Denise Aregba-Driollet – Mathématiques Appliquées de Bordeaux

Hyperbolic Problems: Theory, Numerics, Applications

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Main topic of the Talk: Numerical approximation for hyperbolic equations with source term

accurate analytical time-decay properties of the solution to design **schemes** based on standard upwinding schemes, which are **increasingly accurate for large times** when computing small perturbations of constants asymptotic states.

Numerical schemes using the analytical behavior of solutions

... schemes that use some knowledge of the behavior of the solutions to improve their numerical approximation, at least in some specific regimes

- upwinding of the source term
- well-balanced schemes (finite volume, exact on non constant stationary solution)
- Asymptotic Preserving schemes

► Time Asymptotic High Order Schemes ...
⟶ the asymptotic states are constant (all the consistent schemes are exact on these states and have no special problem to keep them)
⟶ the goal is to design schemes which are able to improve their performance for large times by using accurate analytical time-decay properties of the solutions

The simplest example: the linear wave equation with damping

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = -v. \end{cases}$$

- Decay properties of the solution in norm L^∞ , $t \to +\infty$:

$$u \sim t^{-1/2} \qquad \partial_x^k u \sim t^{-1/2-k/2}$$

$$v \sim t^{-1} \qquad \partial_x^k v \sim t^{-1-k/2}$$

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The Asymptotic behavior

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = -v \end{cases}$$

Heat equation!

$$\partial_t u - \partial_{xx} u = 0$$

- Rigorous framework!

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Main features:

- symmetric system
- the source term is simple (only $-v$)

All dissipative systems have this structure !!

All dissipative systems can be reduced to
the CONSERVATIVE-DISSIPATIVE form

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One Dimensional BGK systems

for $f^i \in \mathbb{R}^k$, $i = 1, \ldots, m$,

$$\begin{cases} \partial_t f^i(x) + \lambda_i \partial_x f^i = M_i(u) - f^i, \\ \\ u := \sum_{i=1}^m f^i. \end{cases}$$

Here $x \in \mathbb{R}$ and $t > 0$, and $M_i = M_i(u) \in \mathbb{R}^k$ are smooth functions of u such that:

$$\sum_{i=1}^m M_i(u) = u.$$

- $\eta(u)$ convex entropy function; the Jacobian matrices M_i' are symmetrized by η and have real and positive eigenvalues

Conservative-Dissipative variables

for $Z = (w_c, w_d)^T$, $w_c = u$ there exists an invertible matrix D such that

$$Z = D\ f$$

$$\begin{cases} \partial_t w_c + A_{11}\partial_x w_c + A_{12}\partial_x w_d = 0, \\ \\ \partial_t w_d + A_{21}\partial_x w_c + A_{22}\partial_x w_d = \tilde{Q}(w_c) - w_d, \end{cases} \tag{1}$$

where $\tilde{Q}(w_c)$ is quadratic in w_c and $A = D\Lambda D^{-1}$ is symmetric

Asymptotic results - S. Bianchini, B. Hanouzet, R. Natalini CPAM 2007

Let $(w_c(t), w_d(t))$ be a solution to problem (1), then:

$$\|D^\beta(w_c(t), w_d(t))\|_{L^\infty} \leq C \min\Big\{1, t^{-\frac{1}{2}-|\beta|/2}\Big\} E_{|\beta|+2}$$

For w_d we have extra decay

$$\|D^\beta w_d(t)\|_{L^\infty} \leq C \min\Big\{1, t^{-\frac{1}{2}-|\beta|/2-1/2}\Big\} E_{|\beta|+2}$$

for time-derivatives

$$\|D^\beta (w_c)_t(t)\|_{L^\infty} \leq C \min\{1, t^{-1-\frac{|\beta|}{2}}\} E_{|\beta|+[\frac{1}{2}]+2}$$

$$\|D^\beta (w_d)_t(t)\|_{L^\infty} \leq C \min\{1, t^{-\frac{3}{2}-\frac{|\beta|}{2}}\} E_{|\beta|+[\frac{1}{2}]+3}$$

Time Asymptotic High Order Schemes

A scheme is Time - AHO if

- it verifies consistency and convergence properties
- it is increasingly accurate for large times

HOW ?

Using accurate analytical time-decay properties of the local truncation error

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One Dimensional BGK systems: A general Finite Difference Method

for $i = 1, ..., m$, q_i artificial diffusion terms, $j \in \mathbb{Z}$, $n \in \mathbb{N}$, $f^i_{j,0} = \mathrm{f}^i_0(x_j)$

$$\frac{f^i_{j,n+1} - f^i_{j,n}}{\Delta t} + \frac{\lambda_i}{2\Delta x}\left(f^i_{j+1,n} - f^i_{j-1,n}\right) - \frac{q_i}{2\Delta x}\delta^2_x f^i_{j,n} = \sum_{l=-1,0,1}\left(\mathscr{B}^i_l(u_{j+l,n}) - \beta^i_l f^i_{j+l,n}\right)$$

for $l = -1, 0, 1$, $\beta^i_l \in \mathbb{R}^{k\times k}$ diagonal matrix, $\mathscr{B}^i_l(\cdot) \in \mathbb{R}^k$ vectors of functions define the source term approximation

Consistency

$$\beta^i_{-1} + \beta^i_0 + \beta^i_1 = I_{m_1} + \Delta x C^i, \qquad \mathscr{B}^i_{-1}(u) + \mathscr{B}^i_0(u) + \mathscr{B}^i_1(u) = M_i(u) + \Delta x \mathscr{C}_i(u),$$

where $C^i = diag(c^i_1, ..., c^i_k)$ and $\mathscr{C}_i(u)$ are k functions to be defined

Finite Difference Method in conservative-dissipative variables

$$Z = D\ f$$

$$\frac{Z_{j,n+1} - Z_{j,n}}{\Delta t} + \frac{A}{2\Delta x}\left(Z_{j+1,n} - Z_{j-1,n}\right) - \frac{\bar{Q}}{2\Delta x}\delta_x^2 Z_j^n$$

$$= \sum_{l=-1,0,1} \left(\bar{\mathscr{B}}_l(u_{j+l,n}) - \bar{b}_l Z_{j,n}\right),$$

for $l = -1, 0, 1$

$$\begin{array}{ll} b_l = (\beta_l^1, ..., \beta_l^k) & \bar{b}_l = D b_l D^{-1} \\ \mathscr{B}_l(u) = (\mathscr{B}_l^1(u), ..., \mathscr{B}^k(u)) & \bar{\mathscr{B}}_l = D\mathscr{B}_l(u) \\ Q = diag(q_1, ..., q_m) & \bar{Q} = DQD^{-1} \end{array}$$

A general Finite Difference Method

for the **conservative-dissipative** variables

$$\begin{cases} \dfrac{w_c^{j,n+1} - w_c^{j,n}}{\Delta t} = \mathscr{F}_c(w_c^{j,n}), \\[2ex] \dfrac{w_d^{j,n+1} - w_d^{j,n}}{\Delta t} = \mathscr{F}_d(w_d^{j,n}) \end{cases}$$

$w_{c,d}^{j,n} = w_{c,d}(x_j, t^n),\ (x_j, t^n) \in \mathbb{R} \times \mathbb{R}^+, \quad j \in \mathbb{Z}, n \in \mathbb{N}$

- local truncation error

$$\mathscr{T}_j^n(Z) := \frac{Z_j^{n+1} - Z_j^n}{\Delta t} - \mathscr{F}(Z_j^n), \quad Z \text{ exact solution}$$

A general Finite Difference Method

$$\mathscr{T}(Z_j^n) = \frac{Z_j^{n+1} - Z_j^n}{\Delta t} - \mathscr{F}(Z_j^n) = \text{by Taylor expansion}$$

$$= \mathscr{S}_1(Z_j^n)\Delta x + \mathscr{S}_2(Z_j^n)\Delta t + \mathscr{S}_3(Z_j^n)\Delta x^2 + \mathscr{S}_4(Z_j^n)\Delta t^2 + \ldots$$

where $\mathscr{S}_{\cdot}(\cdot)$ depend on Z derivatives and on scheme parameters

- High Order Schemes

$$\mathscr{T}_j^n(Z) = O(\Delta x^{\alpha}) + O(\Delta t^{\beta}), \quad \alpha, \beta > 1$$

Assume $\Delta t = \rho \Delta x, \ \rho > 0$

$$\mathcal{T}_j^n(Z) = \frac{Z_j^{n+1} - Z_j^n}{\Delta t} - \mathscr{F}(Z_j^n) = \text{by Taylor expansion}$$

$$= \mathscr{S}(Z(t))\Delta x + O(\Delta x^2),$$

The aim is to select the scheme parameters in $\mathscr{S}(\cdot)$ such that

$$\mathcal{T} \sim O(t^{-\gamma}\Delta x) + O(\Delta x^2)$$

for γ as big as possible ...

going into details, to build a Time AHO scheme ...

local truncation error in conservative-dissipative variables

$$\begin{aligned}
\mathcal{T}_{\mathrm{w}_c} &= \Big[T_0^1\,\mathrm{w}_c + T_1^1\,\partial_x \mathrm{w}_c + T_2^1\,\partial_{xx}\mathrm{w}_c \\
&\quad + S_0^1\,\mathrm{w}_d - S_1^1\,\partial_{tx}\mathrm{w}_d + S_2^1\,\partial_{xx}\mathrm{w}_d \Big] \Delta x + O(\Delta x^2), \\
\mathcal{T}_{\mathrm{w}_d} &= \Big[T_0^2\,\mathrm{w}_c + T_1^2\,\partial_x \mathrm{w}_c + T_2^2\,\partial_{xx}\mathrm{w}_c \\
&\quad + S_0^2\,\mathrm{w}_d + S_2^2\,\partial_{xx}\mathrm{w}_d \Big] \Delta x + O(\Delta x^2),
\end{aligned}$$

going into details, to build a Time AHO scheme ...

check the decay rates

$$
\begin{aligned}
\mathcal{T}_{\mathrm{w}_c} &= \Big[T_0^1 \overbrace{\mathrm{w}_c}^{t^{-1/2}} + T_1^1 \overbrace{\partial_x \mathrm{w}_c}^{t^{-1}} + T_2^1 \overbrace{\partial_{xx} \mathrm{w}_c}^{t^{-3/2}} \\
&\quad + S_0^1 \overbrace{\mathrm{w}_d}^{t^{-1}} - S_1^1 \overbrace{\partial_{tx} \mathrm{w}_d}^{t^{-2}} + S_2^1 \overbrace{\partial_{xx} \mathrm{w}_d}^{t^{-2}} \Big] \Delta x + O(\Delta x^2 \;\; t^{-3/2}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{\mathrm{w}_d} &= \Big[T_0^2 \overbrace{\mathrm{w}_c}^{t^{-1/2}} + T_1^2 \overbrace{\partial_x \mathrm{w}_c}^{t^{-1}} + T_2^2 \overbrace{\partial_{xx} \mathrm{w}_c}^{t^{-3/2}} \\
&\quad + S_0^2 \overbrace{\mathrm{w}_d}^{t^{-1}} + S_2^2 \overbrace{\partial_{xx} \mathrm{w}_d}^{t^{-2}} \Big] \Delta x + O(\Delta x^2 \;\; t^{-3/2}),
\end{aligned}
$$

going into details, to build a Time AHO scheme ...

delete terms that decay more slowly!

$$\mathcal{T}_{w_c} = \Big[\cancel{T_0^1}\overbrace{w_c}^{t^{-1/2}} + \cancel{T_1^1}\overbrace{\partial_x w_c}^{t^{-1}} + \cancel{T_2^1}\overbrace{\partial_{xx} w_c}^{t^{-3/2}}$$

$$+\cancel{S_0^1}\overbrace{w_d}^{t^{-1}} - S_1^1\overbrace{\partial_{tx} w_d}^{t^{-2}} + S_2^1\overbrace{\partial_{xx} w_d}^{t^{-2}}\Big]\Delta x + O(\Delta x^2\ t^{-3/2}),$$

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$$+\cancel{S_0^2}\overbrace{w_d}^{t^{-1}} + S_2^2\overbrace{\partial_{xx} w_d}^{t^{-2}}\Big]\Delta x + O(\Delta x^2\ t^{-3/2}),$$

going into details, to build a Time AHO scheme ...

$$\mathscr{T}_{\mathrm{w}_c} = \left[S_1^1 \overbrace{\partial_{tx}\mathrm{w}_d}^{t^{-2}} + S_2^1 \overbrace{\partial_{xx}\mathrm{w}_d}^{t^{-2}} \right] \Delta x + O(\Delta x^2 \ t^{-3/2}),$$

$$\mathscr{T}_{\mathrm{w}_d} = \left[S_2^2 \overbrace{\partial_{xx}\mathrm{w}_d}^{t^{-2}} \right] \Delta x + O(\Delta x^2 \ t^{-3/2}),$$

- For BGK systems it is always possible to design TAHO schemes such that
$$\mathscr{T}_{\mathrm{w}_c}, \mathscr{T}_{\mathrm{w}_d} \sim O(t^{-2}\Delta x) + O(t^{-3/2}\Delta x^2)$$

usually, for standard approximations, we have that
$$\mathscr{T}_{\mathrm{w}_c}, \mathscr{T}_{\mathrm{w}_d} \sim O(t^{-3/2}\Delta x) + O(t^{-3/2}\Delta x^2)$$

going into details, to build a Time AHO scheme ...

$$\mathcal{T}_{w_c} = \Big[S_1^1 \overbrace{\partial_{tx} w_d}^{t^{-2}} + S_2^1 \overbrace{\partial_{xx} w_d}^{t^{-2}}\Big] \Delta x + O(\Delta x^2 \; t^{-3/2}),$$

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Particular case $A_{11} = 0$ (in conservative-diffusive variables)

$$\partial_{tt} w_c \sim t^{-2} \qquad \partial_{tt} w_d \sim t^{-2}$$

upwinding of the source term (Roe scheme)

- For $A_{11} = 0$, the local truncation error of the Roe scheme verifies the time asymptotic estimate $\mathcal{T}_{w_c,w_d} \sim (t^{-2}\Delta x)$

Convergence order with respect the diffusion limit

The linear case

$$\begin{cases} u_t + \alpha u_x + z_x = 0, \\ z_t + u_x - \alpha z_x = -\beta z. \end{cases}$$

for large time simulations, we can consider to be in the diffusion (Chapman-Enskog) limit

$$\begin{cases} \hat{u}_t + \alpha \hat{u}_x = \frac{1}{\beta} \hat{u}_{xx}, \\ \hat{z} = -\frac{1}{\beta} \hat{u}_x. \end{cases}$$

Convergence order with respect the diffusion limit

$$\begin{cases} \hat{u}_t + \alpha \hat{u}_x = & \left(\frac{1}{\beta} + \Delta x \mathscr{D}\right) \hat{u}_{xx} + O(\Delta x^2), \\ \\ \hat{z} = & -\frac{1}{\beta} \hat{u}_x + O(\Delta x^2), \end{cases}$$

where the constant $\mathscr{D}$ depends on the selected scheme

$$\mathscr{D}_{UP} = -\frac{1}{2}\left(\rho\alpha^2 - \xi\right) \quad \mathscr{D}_{Roe} = -\frac{1}{2}\left(\rho\alpha^2 - \frac{\xi^2 - 1}{\xi}\right),$$

$$\mathscr{D}_{TAHO} = 0$$

- as $t \to +\infty$, TAHO approximation is consistent of order $O(\Delta x^2)$ with respect the asymptotic parabolic problem

Numerical tests

Let focus on

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \lambda^2 \partial_x u = A(u) - v, \end{cases} \quad (2)$$

for $\lambda > 0$, u and v scalar and $A = A(u)$ smooth, with $A(0) = 0$

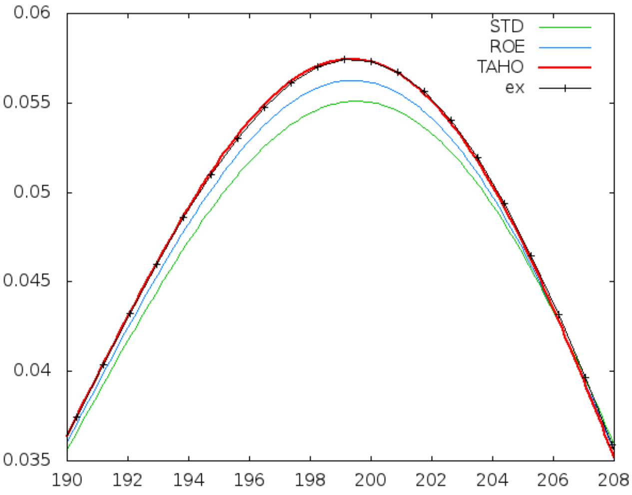
conservative-dissipative form

for $\lambda > |A'(0)|$, $a = A'(0)$, $\mu = (\lambda^2 - a^2)^{-1/2}$ and

$$w_c = u, \quad w_d = \mu(v - au).$$

$$\begin{cases} \partial_t w_c + \partial_x (a w_c + \frac{1}{\mu} w_d) = 0, \\ \partial_t w_d + \partial_x (\frac{w_c}{\mu} - a w_d) = \mu(A(w_c) - a w_c) - w_d. \end{cases} \quad (3)$$

$a \neq 0$, zoom on the conservative variable u



for Δx big!

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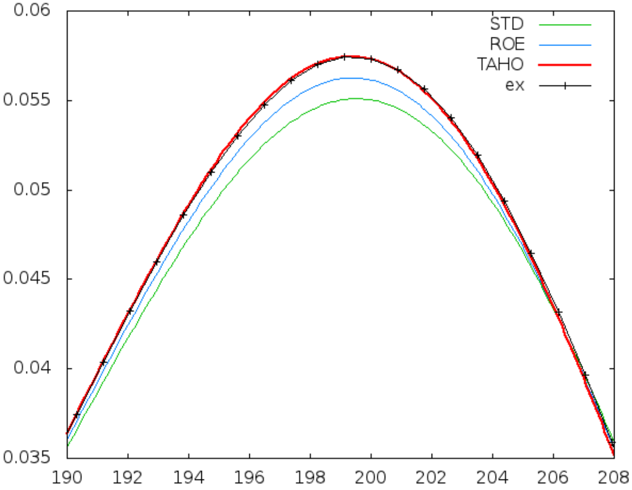
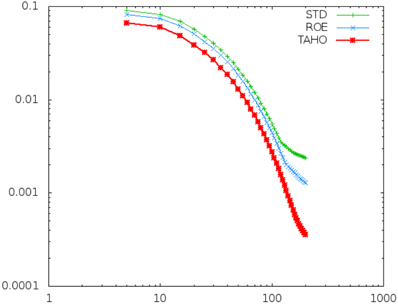
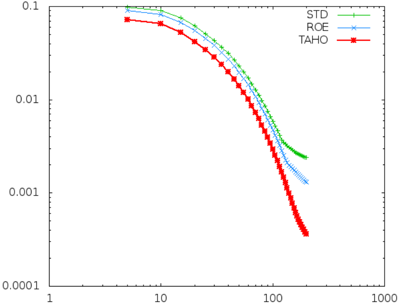


for Δx big!

Numerical Error $e(t) \sim C t^{-\gamma}$





scheme	C_u	γ_u	$C_{\mathbf{z}}$	$\gamma_{\mathbf{z}}$
STD	0.013797	0.374708	0.010744	0.341554
ROE	0.004874	0.333634	0.007850	0.439996
TAHO	0.111380	1.151517	0.495480	1.451030

Results for the 3×3 system

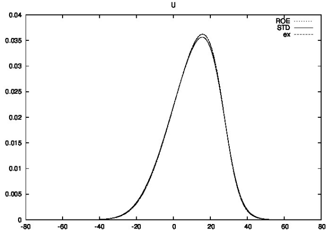
Let A be a smooth function such taht $A(0) = 0$ and γ be such that $\gamma'(u) = |A'(u)|$, $\gamma(0) = 0$

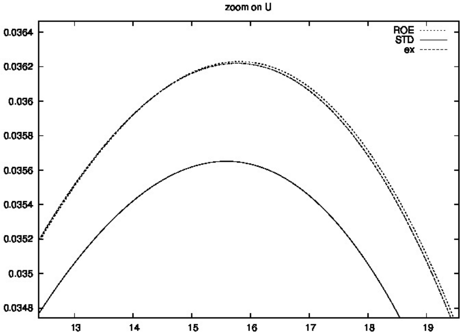
$$\begin{cases} \partial_t f_1 - \lambda \partial_x f_1 = -f_1 + M_1(u), \\ \\ \partial_t f_2 = -f_2 + M_2(u), \\ \\ \partial_t f_3 + \lambda \partial_x f_3 = -f_3 + M_3(u), \end{cases}$$

where $\beta \in +0,1[$, $\lambda > 0$, $u = \sum_i f_i$ and

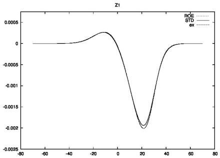
$$M_1(u) = \frac{1}{2}\left(\frac{\gamma(u) - A(u)}{\lambda} + \beta u\right),\ M_2 = (1-\beta)u - \frac{\gamma(u)}{\lambda},\ M_3(u) = \frac{1}{2}\left(\frac{\gamma(u) + A(u)}{\lambda} + \beta u\right)$$

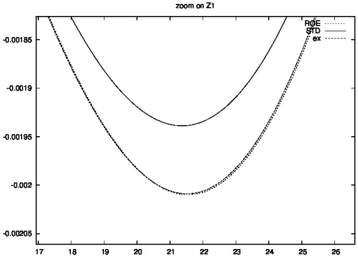
conservative variable u

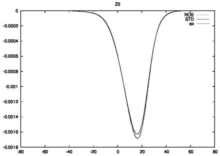


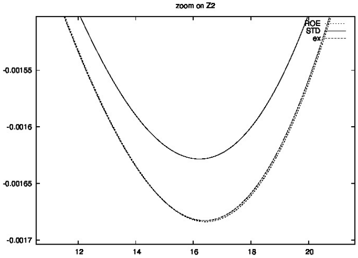


dissipative variable $\tilde{Z}$

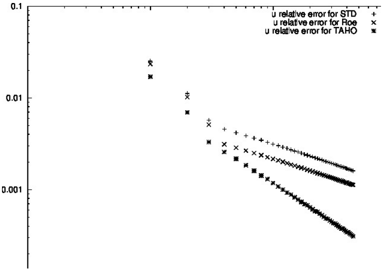


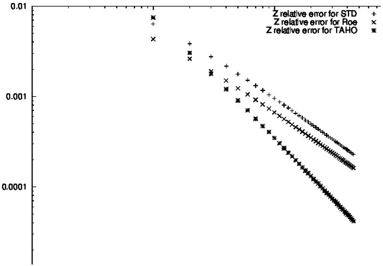






Numerical Error $e(t) \sim Ct^{-\gamma}$





scheme	C_u	γ_u	C_z	γ_z
STD	4.8E-002	0.57	6E-002	0.91
ROE	4.4E-002	0.62	4E-002	0.90
TAHO	0.1	0.96	0.19	1.38

Conclusions

- How to built asymptotic high order schemes using the analytical behavior of the solutions
- We get a class of TAHO schemes for the one dimensional BGK systems
- This idea has been used successfully in some applications, such as for hyperbolic model of chemotaxis with boundary conditions
- Future developments: multidimensional framework!

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- Future developments: multidimensional framework!

Some References

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