

# Optima and Equilibria for Traffic Flow on a Network

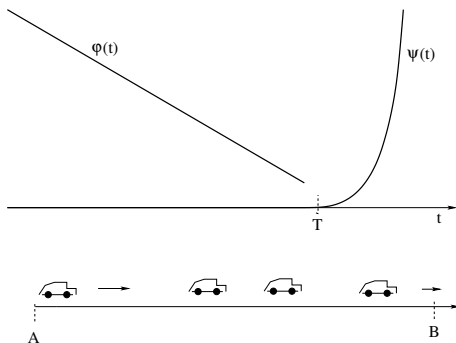
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# A Traffic Flow Problem

- Car drivers starting from a location  $A$  (a residential neighborhood) need to reach a destination  $B$  (a working place) at a given time  $T$ .
- There is a cost  $\varphi(\tau_d)$  for departing early and a cost  $\psi(\tau_a)$  for arriving late.



# Elementary solution

$L$  = length of the road,       $v$  = speed of cars

$$\tau_a = \tau_d + \frac{L}{v}$$

Optimal departure time:

$$\tau_d^{\text{opt}} = \operatorname{argmin}_t \left\{ \varphi(t) + \psi\left(t + \frac{L}{v}\right) \right\}.$$

If everyone departs exactly at the same optimal time,  
a traffic jam is created and this strategy is not optimal anymore.

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Problem: choose the departure rate  $\bar{u}(t)$  so that the solution of the conservation law

$$\begin{cases} \rho_t + [\rho v(\rho)]_x = 0 & x \in [0, L] \\ \rho(t, 0)v(\rho(t, 0)) = \bar{u}(t) \end{cases}$$

minimizes the sum of the costs to all drivers.

$$u(t, x) \doteq \rho(t, x) v(\rho(t, x)) = \text{flux of cars}$$

$$\text{minimize: } J(\bar{u}) \doteq \int \varphi(t) \cdot u(t, 0) dt + \int \psi(t) u(t, L) dt$$

Choose the optimal departure rate  $\bar{u}(t)$ , subject to the constraint

$$\bar{u}(t) \geq 0, \quad \int \bar{u}(t) dt = \kappa = [\text{total number of drivers}]$$

# Existence of a globally optimal solution

**(A1)** The flux function  $\rho \mapsto f(\rho) = \rho v(\rho)$  is strictly concave down.

$$f(0) = f(\rho_{max}) = 0, \quad f'' < 0.$$

**(A2)** The cost functions  $\varphi, \psi$  satisfy  $\varphi' < 0$ ,  $\psi, \psi' \geq 0$ ,

$$\lim_{t \rightarrow -\infty} \varphi(t) = +\infty, \quad \lim_{t \rightarrow +\infty} (\varphi(t) + \psi(t)) = +\infty$$

**Theorem (A.B. and K. Han, *SIAM J. Math. Anal.*, 2012).**

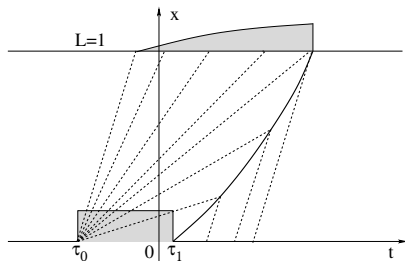
Let **(A1)**-**(A2)** hold. Then, for any  $\kappa > 0$ , there exists a unique admissible initial data  $\bar{u}$  minimizing the total cost  $J(\cdot)$ .

# An Example

Cost functions:  $\varphi(t) = -t$ ,  $\psi(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ t^2, & \text{if } t > 0 \end{cases}$

$$L = 1, \quad u = \rho(2 - \rho), \quad M = 1, \quad \kappa = 3.80758$$

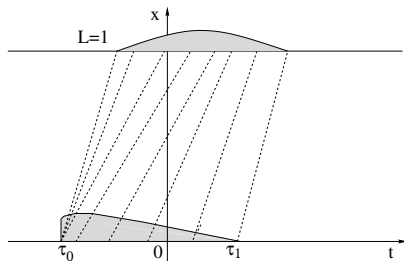
**Bang-bang solution**



$$\tau_0 = -2.78836, \quad \tau_1 = 1.01924$$

total cost = 5.86767

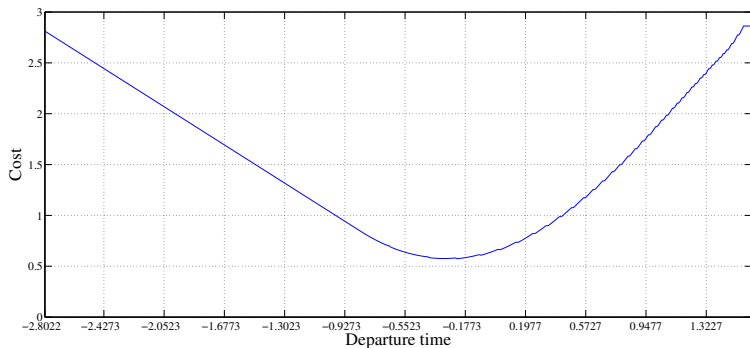
**Optimal solution**



$$\tau_0 = -2.8023, \quad \tau_1 = 1.5976$$

total cost = 5.5714

# Does everyone pay the same cost?



Departure time vs. cost in the Pareto optimal solution



# The Nash equilibrium solution

A solution  $u = u(t, x)$  is a **Nash equilibrium** if no driver can reduce his/her own cost by choosing a different departure time.

This implies that **all drivers pay the same cost**.

To find a Nash equilibrium, introduce the integrated variable

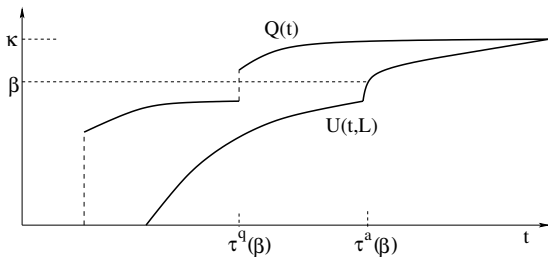
$$U(t, x) \doteq \int_{-\infty}^t \rho(s, x) v(\rho(s, x)) ds = [\text{number of drivers}$$

that have crossed the point  $x$  along the road within time  $t]$

This solves a Hamilton-Jacobi equation

$$U_x + F(U_t) = 0 \qquad U(t, 0) = Q(t)$$

Note: a **queue** can form at the entrance of the highway

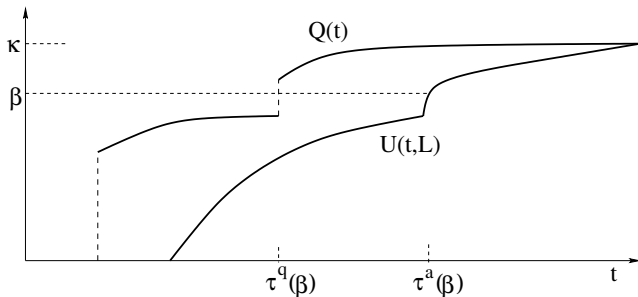


$Q(t)$  = number of drivers who have started their journey before time  $t$   
(possibly joining the queue)

$L$  = length of the road

$U(t, L)$  = number of drivers who have reached destination before time  $t$

# Characterization of a Nash equilibrium



$\beta \in [0, \kappa]$  = Lagrangian variable labeling one particular driver

$\tau^q(\beta)$  = time when driver  $\beta$  joins the queue

$\tau^a(\beta)$  = time when driver  $\beta$  arrives at destination

$$\text{Nash equilibrium} \quad \implies \quad \varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) = c$$

# Existence and Uniqueness of Nash equilibrium

## Theorem (A.B. - K. Han).

Let the flux  $f$  and cost functions  $\varphi, \psi$  satisfy the assumptions (A1)-(A2). Then, for every  $\kappa > 0$ , the Hamilton-Jacobi equation

$$U_x + F(U_t) = 0$$

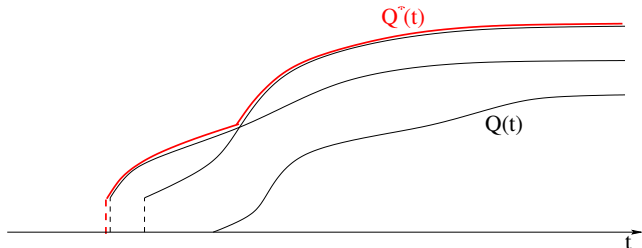
admits a unique Nash equilibrium solution with total mass  $\kappa$

# Sketch of the proof

1. For a given cost  $c$ , let  $\mathcal{Q}_c^-$  be the set of all initial data  $Q(\cdot)$  for which every driver has a cost  $\leq c$ :

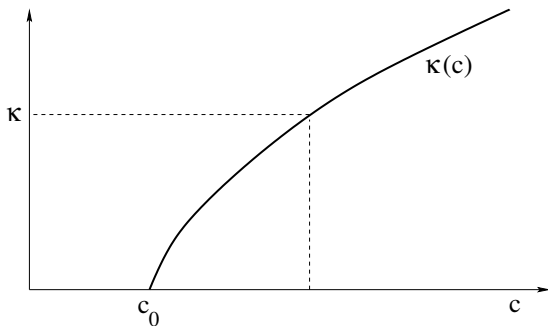
$$\varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) \leq c \quad \text{for a.e. } \beta \in [0, Q(+\infty)].$$

2. Claim:  $Q^*(t) \doteq \sup \{ Q(t); Q \in \mathcal{Q}_c^- \}$   
is the initial data for a Nash equilibrium with common cost  $c$ .

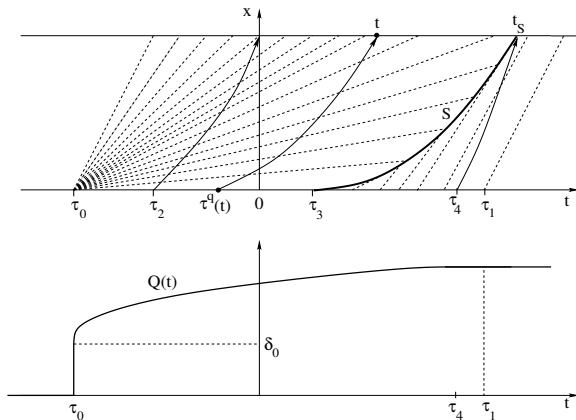


3. There exists a minimum cost  $c_0$  such that  $\kappa(c) = 0$  for  $c \leq c_0$ .

The map  $c \mapsto \kappa(c)$  is strictly increasing and continuous from  $[c_0, +\infty[$  to  $[0, +\infty[$ .



# An example of Nash equilibrium



- A queue of size  $\delta_0$  forms instantly at time  $\tau_0$
- The last driver of this queue departs at  $\tau_2$ , and arrives at exactly 0.
- The queue is depleted at time  $\tau_3$ . A shock is formed.
- The last driver departs at  $\tau_1$ .

# A comparison

Total cost of the Pareto optimal solution:  $J^{opt} = 5.5714$

Total cost of the Nash equilibrium solution:  $J^{Nash} = 10.286$

Price of anarchy:  $J^{Nash} - J^{opt} \approx 4.715$

Can one eliminate this inefficiency,  
yet allowing freedom of choice to each driver ?

(goal of non-cooperative game theory: devise incentives)



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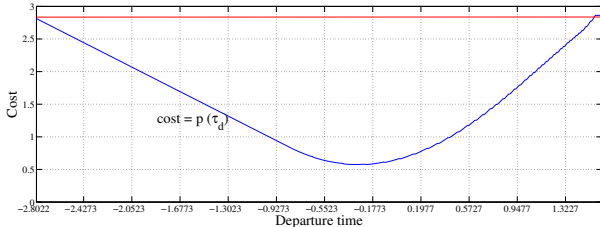
Can one eliminate this inefficiency,  
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(goal of non-cooperative game theory: devise incentives)

Suppose a fee  $b(t)$  is collected at a toll booth at the entrance of the highway, depending on the departure time.

$$\text{New departure cost: } \tilde{\varphi}(t) = \varphi(t) + b(t)$$

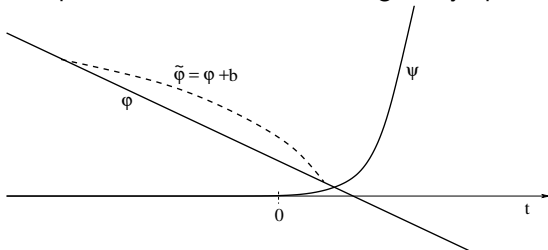
Is there an optimal choice of  $b(t)$  ?



$p(t)$  = cost to a driver starting at time  $t$ , in a globally optimal solution

Choose additional fee:  $b(t) = p_{max} - p(t) + \text{constant}$

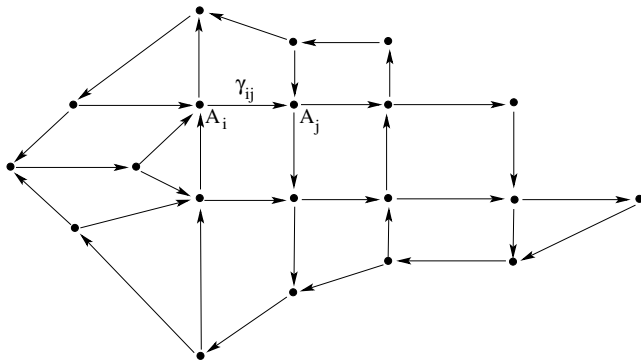
$\implies$  Nash equilibrium coincides with the globally optimal solution



# Traffic Flow on a Network

Nodes:  $A_1, \dots, A_m$       arcs:  $\gamma_{ij}$

$n$  groups of drivers with different origins and destinations, and different costs

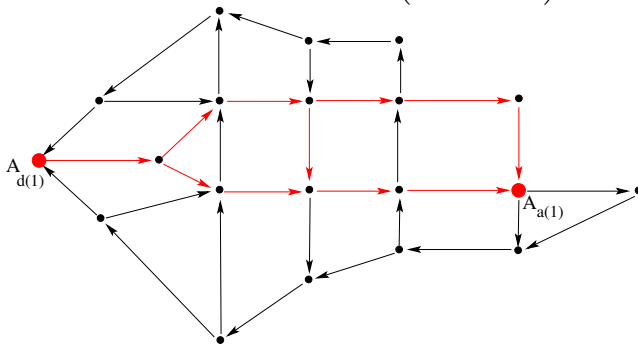


# Traffic Flow on a Network

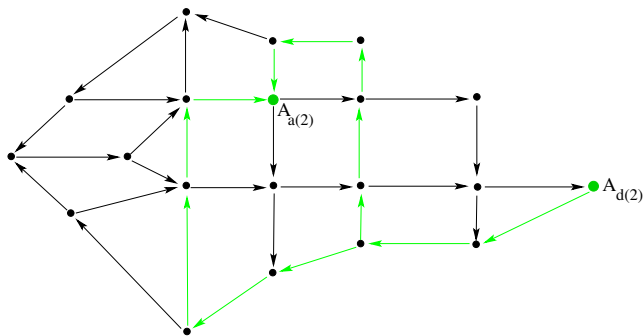
$k$ -drivers depart from  $A_{d(k)}$  and arrive to  $A_{a(k)}$

Departure cost:  $\varphi_k(t)$       arrival cost:  $\psi_k(t)$

**(A2)**  $\varphi' < 0$ ,  $\psi'_k > 0$ ,  $\lim_{|t| \rightarrow \infty} (\varphi(t) + \psi(t)) = \infty$



# Traffic Flow on a Network



drivers can use different paths  $\Gamma_1, \Gamma_2, \dots$  to reach destination

Does there exist a globally optimal solution, and a Nash equilibrium solution for traffic flow on a network ?

# Admissible departure rates

$G_k$  = total number of drivers in the  $k$ -th group,  $k = 1, \dots, n$

$\Gamma_p$  = viable path (concatenation of viable arcs  $\gamma_{ij}$ ),  $p = 1, \dots, N$

$t \mapsto \bar{u}_{k,p}(t)$  = departure rate of  $k$ -drivers traveling along the path  $\Gamma_p$

The set of departure rates  $\{\bar{u}_{k,p}\}$  is **admissible** if

$$\bar{u}_{k,p}(t) \geq 0, \quad \sum_p \int_{-\infty}^{\infty} \bar{u}_{k,p}(t) dt = G_k \quad k = 1, \dots, n$$

Let  $\tau_p(t)$  = arrival time for a driver starting at time  $t$ , traveling along  $\Gamma_p$



# Global optima and Nash equilibria on networks

An admissible family  $\{\bar{u}_{k,p}\}$  of departure rates is **globally optimal** if it minimizes the sum of the total costs of all drivers

$$J(\bar{u}) \doteq \sum_{k,p} \int \left( \varphi_k(t) + \psi_k(\tau_p(t)) \right) \bar{u}_{k,p}(t) dt$$

An admissible family  $\{\bar{u}_{k,p}\}$  of departure rates is a **Nash equilibrium solution** if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.

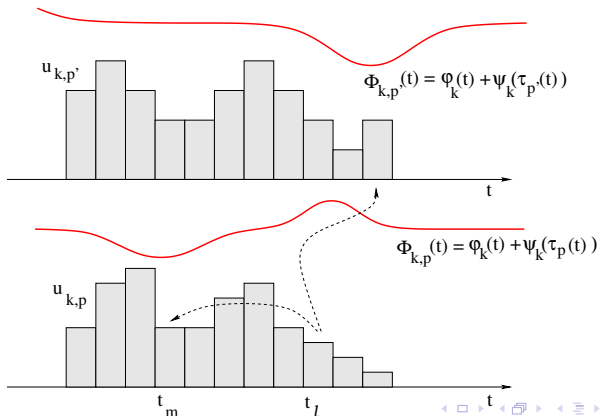
**Theorem** (A.B. - Ke Han, 2012). On a general network of roads, there exists at least one globally optimal solution, and at least one Nash equilibrium solution.

# Finite dimensional approximations

Fix a time step  $\Delta t > 0$

Consider piecewise constant departure rates  $u = (u_{k,p})$ , with bounded support

Solving a variational inequality on a compact finite dimensional set  $\mathcal{K}$ , we obtain a Galerkin approximation to a Nash equilibrium



# Existence of a Nash equilibrium on a network

Letting the discretization step  $\Delta t \rightarrow 0$ , taking subsequences:

departure rates:  $\bar{u}_{k,p}^\nu(\cdot) \rightharpoonup \bar{u}_{k,p}(\cdot)$  weakly

arrival times:  $\tau_p^\nu(\cdot) \rightarrow \tau_p(\cdot)$  uniformly

The departure rates  $\bar{u}_{k,p}(\cdot)$  provide a Nash equilibrium

# Stability of Nash equilibrium ?

To justify the practical relevance of a Nash equilibrium, we need to

- analyze a suitable dynamic model
- check whether the rate of departures asymptotically converges to the Nash equilibrium

Assume: drivers can change their departure time on a day-to-day basis, in order to decrease their own cost (one group of drivers, one single road)

Introduce an additional variable  $\theta$  counting the number of days on the calendar.

$\bar{u}(t, \theta) \doteq$  rate of departures at time  $t$ , on day  $\theta$ ]

$\Phi(t, \theta) \doteq$  [cost to a driver starting at time  $t$ , on day  $\theta$ ]

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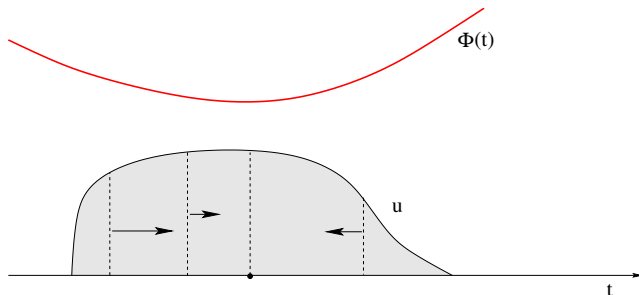
$\Phi(t, \theta) \doteq$  [cost to a driver starting at time  $t$ , on day  $\theta$ ]

# A conservation law with non-local flux

**Model 1:** drivers gradually change their departure time, drifting toward times where the cost is smaller.

If the rate of change is proportional to the gradient of the cost, this leads to the conservation law

$$\bar{u}_\theta + [\Phi_t \bar{u}]_t = 0$$

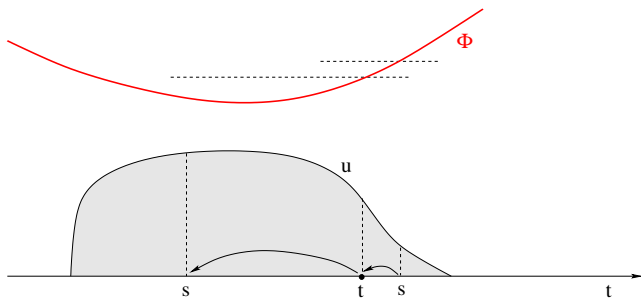


# An integral evolution equation

**Model 2:** drivers jump to different departure times having a lower cost.

If the rate of change is proportional to the difference between the costs, this yields

$$\frac{d}{d\theta} \bar{u}(t) = \int \bar{u}(s) [\Phi(s) - \Phi(t)]_+ ds - \int \bar{u}(t) [\Phi(t) - \Phi(s)]_+ ds$$

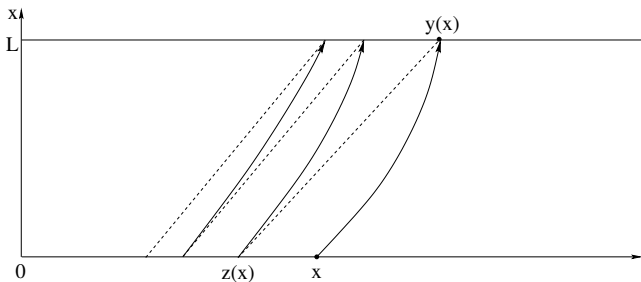


**Question:** as  $\theta \rightarrow \infty$ , does the departure rate  $\bar{u}(t, \theta)$  approach the unique Nash equilibrium?

Flux function:  $f(\rho) = \rho(2 - \rho)$

Departure and arrival costs:  $\varphi(t) = -t$ ,  $\psi(t) = e^t$





main difficulty: non-local dependence

linearized equation: 
$$\frac{d}{d\theta} Y(x) = \left[ \alpha(x) \left( \beta(x) Y(x) - Y(z(x)) \right) \right]_x$$