Coupling techniques for nonlinear hyperbolic equations

Benjamin Boutin

IRMAR, Université Rennes 1, France

Joint work with F. Coquel (CNRS & École Polytechnique),
and P. G. LeFloch (CNRS & Université Paris 6)

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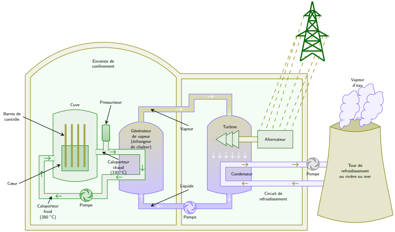
1. **Position of the coupling problem**
   - Practical motivations
   - Mathematical context
   - Some first results and observations
2. **Thin interface regime**
   - Reformulation of the problem
   - Parabolic approximation – existence result
   - Interfacial layer analysis and partial selection criteria
   - Examples
3. **Thick interface regime**
   - A well-posed problem
   - Well-balanced numerical scheme
   - Numerical observations

Outline



NEPTUNE project :

- Two-phase thermal-hydraulic flows
- Direct numerical simulation of multi-scale phenomena

Some collaborators involved in this project (from Paris6 or CEA French Atomic Agency) :
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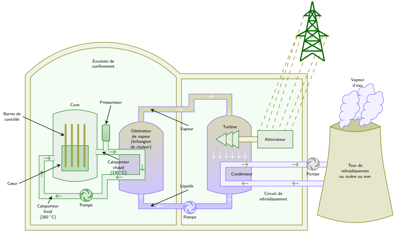

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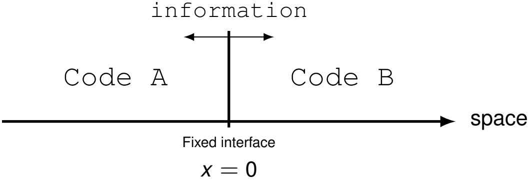
Code coupling



Aims

- numerical solvability
- physically accuracy
- well-posedness ! ?

At the model level, a wide variety of situations :
- different number of unknowns
- same/different primary unknowns
- hierarchy of models (dimensionnal or phenomenological reduction)

In the present study, we restrict our attention to :

The coupling of nonlinear hyperbolic c.l.

$$\begin{cases} \partial_t w + \partial_x f_-(w) = 0, & x < 0,\ t > 0, \\ \partial_t w + \partial_x f_+(w) = 0, & x > 0,\ t > 0, \end{cases}$$

with the same unknown $w(x,t) \in \mathbb{R}^N$, $x \in \mathbb{R}$.

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Possible coupling relation at $x = 0$

- □ Conservation of physical quantities
- □ Continuity of relevant variables
- □ Knowledge about steady states

State coupling

$$w(0^-,t) = w(0^+,t)$$

or more generally $\theta_-(w(0^-,t)) = \theta_+(w(0^+,t))$
where $\theta_\pm$ denote some changes of variable.

[GODLEWSKI & RAVIART], [GODLEWSKI, LE THANH & RAVIART], [AMBROSO et al. 2008]
= a non conservative coupling !

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The situation is essentially different from the **flux coupling** framework where one requires

$$f_-(w(0^-, t)) = f_+(w(0^+, t)), \quad t > 0.$$

i.e. that connects to the *discontinuous flux conservation law* setting.
Some refs : [AUDUSSE, PERTHAME], [SEGUIN, VOVELLE], [TOWERS, KARLSEN, RISEBRO], [PANOV] :
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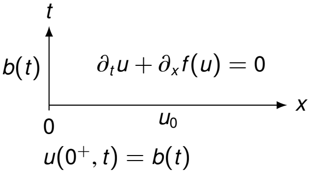
A unified framework is proposed in GALIÉ'S PHD (2009) using a measure-valued source term :

$$\partial_t u + \partial_x f(u,x) = \mathcal{M}(t)\delta_{x=0}.$$

Flux coupling : $\mathcal{M}(t) = 0$.
State coupling : $\mathcal{M}(t) = f_+(w(0^+,t)) - f_-(w(0^-,t))$.

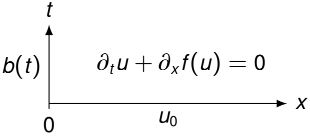
Half-CAUCHY problem [DUBOIS & LEFLOCH, 1988]



Let $\mathcal{W}(x/t, u_\ell, u_r)$ be the self-similar entropy solution for the RIEMANN problem with data (u_ℓ, u_r).

Set $\mathcal{O}(b(t)) = \left\{\mathcal{W}(0^+, b(t), \widetilde{u}), \widetilde{u} \in \Omega\right\}$.

Half-CAUCHY problem [DUBOIS & LEFLOCH, 1988]



" $u(0^+, t) = b(t)$ "

$u(0^+, t) \in \mathcal{O}(b(t))$

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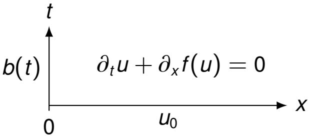

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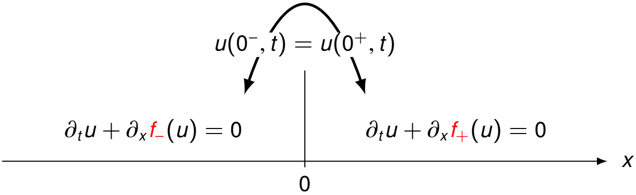
Two half-CAUCHY problems sticked together [GODLEWSKI & RAVIART, 2004]



$$\begin{cases} u(0^-, t) \in \mathcal{O}_-(u(0^+, t)) \\ u(0^+, t) \in \mathcal{O}_+(u(0^-, t)) \end{cases}$$

WEAK FORM FOR THE STATE COUPLING CONDITION (2)

Theorem [B.B., CHALONS, RAVIART, 2010]

In the **scalar case** $N = 1$ with flux functions $f_-, f_+ \in C^1$, There exists a *(non-necessarily unique)* solution for the coupled RIEMANN problem :

$$\begin{cases} \partial_t u + \partial_x f_-(u) = 0, & x < 0,\ t > 0 \\ \partial_t u + \partial_x f_+(u) = 0, & x > 0,\ t > 0 \end{cases} \qquad u(x,0) = \begin{cases} u_\ell, & x < 0 \\ u_r, & x > 0 \end{cases}$$

$$\begin{cases} u(0^-, t) \in \mathcal{O}_-(u(0^+, t)) \\ u(0^+, t) \in \mathcal{O}_+(u(0^-, t)) \end{cases}$$

or more generally

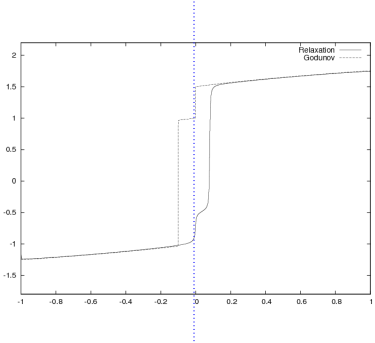
$$\begin{cases} w(0^-, t) \in \mathcal{O}_-(\theta_-^{-1}(\theta_+(w(0^+, t)))) \\ w(0^+, t) \in \mathcal{O}_+(\theta_+^{-1}(\theta_-(w(0^-, t)))) \end{cases}$$

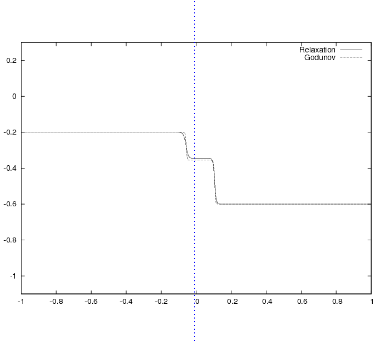
The proof builds on the explicit knowledge of the set of admissible traces, in a constructive way.

Non-uniqueness affects both continuous (at the interface) and discontinuous solutions.

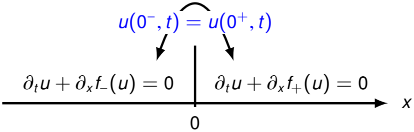
See also [CHALONS, RAVIART, SEGUIN, 2008] :
The interfacial coupling of two EULER systems with different EOS admits a unique solution (for the RIEMANN problem) as soon as $v \ll c_\pm$.
If $v \simeq c_\pm$ then there exist a family of continuous (at the interface) solutions, and at most one discontinuous solution.

Non-uniqueness may occur, the situation is somehow worst : the captured numerical solution may depend on the numerical scheme !





Results obtained with two-fluxes numerical schemes consistent with the state coupling setting [GODLEWSKI & RAVIART, 2004)] (based here on a Godunov flux, and on a relaxation method).



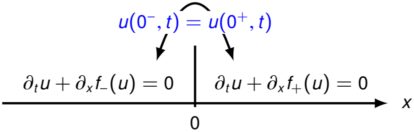
Color function : $v(x,0) = \begin{cases} +1, & x > 0, \\ -1, & x < 0. \end{cases}$

Extended hyperbolic PDE system

$$\begin{cases} \partial_t u + A_1(u,v)\partial_x u = 0, \\ \partial_t v = 0, \end{cases} \qquad x \in \mathbb{R}, t \geq 0.$$

e.g. $A_1(u,v) = \dfrac{1-v}{2}\nabla f_-(u) + \dfrac{1+v}{2}\nabla f_+(u)$,

supposed to be $\mathbb{R}$-diagonalizable with
$\lambda_1(u,v) < \lambda_2(u,v) < \ldots < \lambda_N(u,v)$



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Properties :
The whole system is hyperbolic, in non-conservative form,

- system in u (with size N and given fixed v) is strictly hyperbolic
- system in (u,v) (with size $N+1$) is non-strictly hyperbolic :

  if $A_1(u,v)$ is singular,
  the multiplicity of the eigenvalue $\lambda_0 = 0$ is greater than 2 : $\exists k, u^\star, v^\star : \lambda_k(u^\star, v^\star) = 0$
  $\rightarrow$ resonant coupling at the interface e.g. $\lambda_k^-(u^\star)\lambda_k^+(u^\star) < 0$

  else u consists of RIEMANN invariants for 0-waves : $u(0^-, t) = u(0^+, t)$

EXISTENCE AND CONVERGENCE OF THE SELF-SIMILAR VISCOUS APPROXIMATION

Methodology for the coupling introduced by [B.B., COQUEL & GODLEWSKI (HYP 2006)] :
let analyze the DAFERMOS regularized limit of the extended system

$$\partial_t u^\epsilon + A_1(u^\epsilon, v^\epsilon)\partial_x u^\epsilon = \epsilon \; t \; \partial_x\Big(B_0(u^\epsilon, v^\epsilon)\partial_x u^\epsilon\Big),$$
$$\partial_t v^\epsilon = \epsilon^2 t \, \partial_{xx} v^\epsilon.$$

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Local existence theorem [B.B., COQUEL & LEFLOCH, (2011)]

Under smallness assumptions on the RIEMANN data $|u_r - u_\ell|$ and **structural hypothesis** on the matrix fields A_1, B_0, the coupled RIEMANN problem admits a (self-similar) solution u^ϵ that converges, up to a extraction, to $u \in BV$: a self-similar entropy solution (over each half-space) of

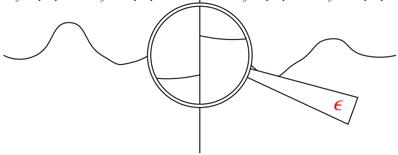
$$\begin{cases} \partial_t u + \partial_x f_-(u) = 0, & x < 0,\ t > 0, \\ \partial_t u + \partial_x f_+(u) = 0, & x > 0,\ t > 0. \end{cases}$$

The result is obtained for the system case $u \in \mathbb{R}^N$ with general coupling functions $\theta_\pm$ and fluxes $f_\pm$ « close enough » , and without smallness assumption on the data in the scalar case ($N = 1$).

The proof follows the works of [TZAVARAS, 1996] and [JOSEPH, LEFLOCH, 2002, 2005] for non-conservative systems (fixed point argument).

$$-\xi d_\xi u + d_\xi f_-(u) = 0 \qquad -\xi d_\xi u + d_\xi f_+(u) = 0$$

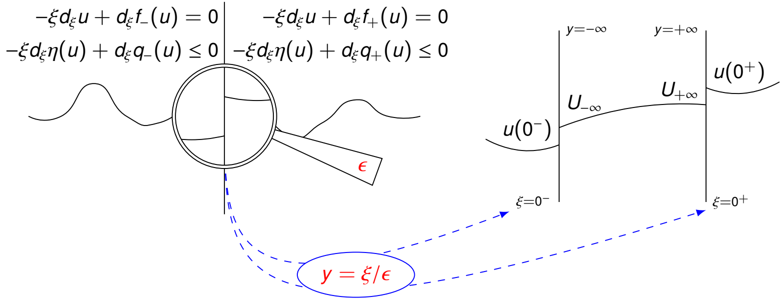
$$-\xi d_\xi \eta(u) + d_\xi q_-(u) \le 0 \qquad -\xi d_\xi \eta(u) + d_\xi q_+(u) \le 0$$



$$y = \xi/\epsilon$$

Blow-up for revealing interfacial layers of solutions :

$$U^\epsilon(y) = u^\epsilon(\epsilon y) \text{ and } V^\epsilon(y) = v^\epsilon(\epsilon y).$$



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Viscous profile equation

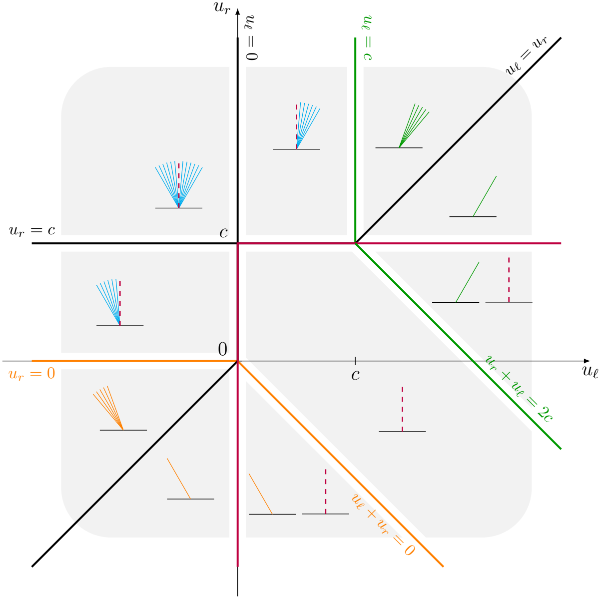
$$A_1(U,V)U_y = (B_0(U,V)U_y)_y$$

Entropy 0-shock waves at the interface

$$f_-(u(0^-)) = f_-(U_{-\infty}), \quad q_-(u(0^-)) \ge q_-(U_{-\infty}),$$
$$f_+(U_{+\infty}) = f_+(u(0^+)), \quad q_+(U_{+\infty}) \ge q_+(u(0^+)).$$

$f_{-}(u) = u^2/2$

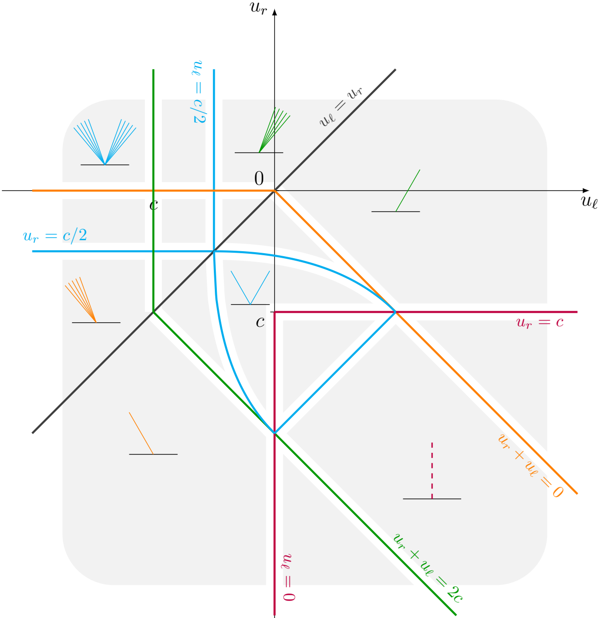
$f_{+}(u) = (u-1)^2/2$



$f_{-}(u) = u^2/2$

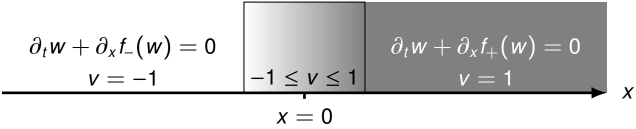
$f_{+}(u) = (u+1)^2/2$

A LAPLACE stability analysis selects the continuous two-waved solutions in a unique way.



Thick interface approach

Aim : Try to understand the sensitiveness of the selected solution with the relation to the structure of the interface. A first step in this direction : fix v to suppress any resonant behavior.



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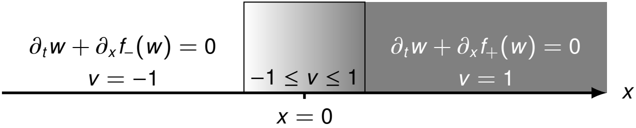

How to understand now the coupling relation ?

The expected coupling relation $\theta_-(w(0^-, t)) = \theta_+(w(0^+, t))$ is now ensured by requiring the PDE/scheme to preserve u-constant states, where we set

$$w = C_0(u, v), \quad C_0(u, \pm 1) = \theta_\pm^{-1}(u)$$

Conservative form with smooth source term

$$\partial_t w + \partial_x f(w, v) = \ell(w, v)\partial_x v \quad \text{with } \ell(w, v) = \partial_v f(w, v). \qquad (1)$$

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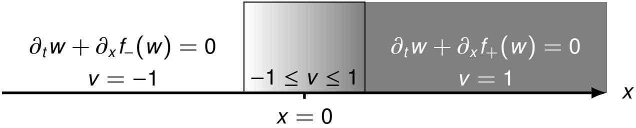

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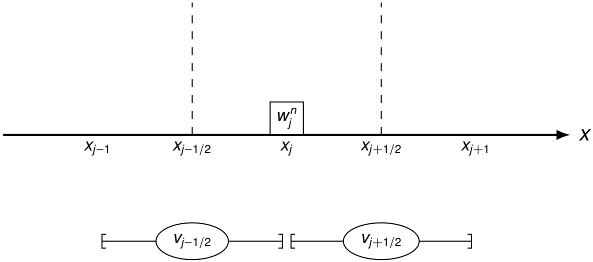
KRUŽKOV theorem (LIPSCHITZ source term)

Let $w_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v \in W^{2,\infty}(\mathbb{R})$ be CAUCHY data, then

$\exists! w \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ entropy solution of (1).

$\partial_t w + \partial_x h(w, x) = S(w, x)$

with $h(w, x) := f(w, v(x))$ and $S(w, x) := \partial_v f(w, v(x))\, \partial_x v(x) = \partial_x h(w, x)$.



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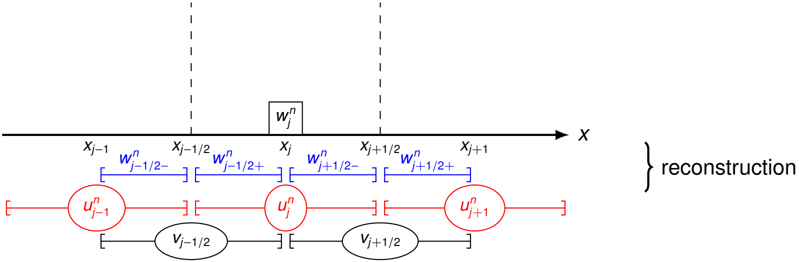


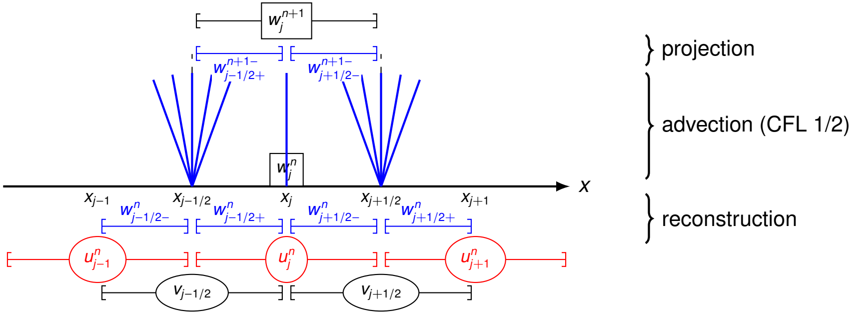
Reconstruction :

$$u_j^n \text{ solution of } w_j^n = \frac{1}{2}\Big(\overbrace{C_0(u_j^n, v_{j-1/2})}^{w_{j-1/2+}^n :=} + \overbrace{C_0(u_j^n, v_{j+1/2})}^{w_{j+1/2-}^n :=}\Big)$$

WELL-BALANCED NUMERICAL SCHEME

$\partial_t w + \partial_x h(w,x) = S(w,x)$

with $h(w,x) := f(w, v(x))$ and $S(w,x) := \partial_v f(w, v(x))\, \partial_x v(x) = \partial_x h(w,x)$.



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Time evolution and projection :

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x}\left(H_{j+1/2}^n - H_{j-1/2}^n\right) - \frac{\Delta t}{\Delta x}\Big(h(w_{j-1/2+}^n, v_{j-1/2}) - h(w_{j+1/2-}^n, v_{j+1/2})\Big)$$

Well-balanced property / Coupling

A numerical data w_j^0 satisfying $u_j^0 := C_0(w_j^0, v_j) = cst$ is a numerical steady state.

Convergence of the 1D scheme (BOUTIN, COQUEL, LEFLOCH)

The numerical solution converges to the unique KRUŽKOV solution of

$$\partial_t w + \partial_x f(w, v) = \ell(w, v)\partial_x v.$$

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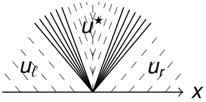
Sketch of the proof :

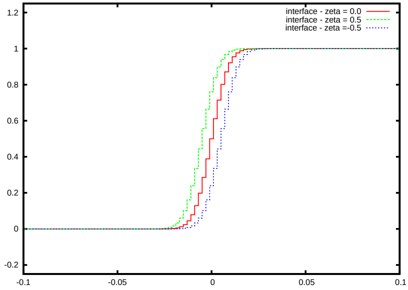
- L^∞ stability (under CFL 1/2) : $\min_k u_k^0 \leq u_j^n \leq \max_k u_k^0, \quad \forall n \in \mathbb{N}, j \in \mathbb{Z}$.
- Consistency with some entropy-like inequalities for : $\partial_t w + \partial_x h(w, x) = S(w, x)$
- But no uniform *TV* bound (due to the reconstruction step) except for the case $C_0(u, v) = u$
- Weak estimate on the discrete derivatives (*BV* weak estimate)
- Use of entropy measure valued solutions of DIPERNA, and limit in the sense of YOUNG measures to the KRUŽKOV solution, due to the uniqueness of entropy measure valued solution.

Remark :
This scheme works also for the multiD case, under suitable assumptions on the meshes.

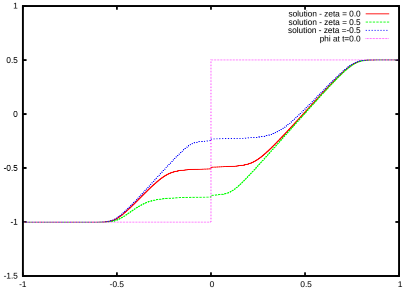


1-parameter set of solutions
$u^\star \in [u_\ell, u_r]$

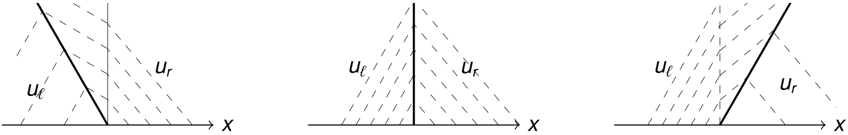
$$v(x) = \frac{\text{erf}(x/\eta + \zeta) + 1}{2}.$$



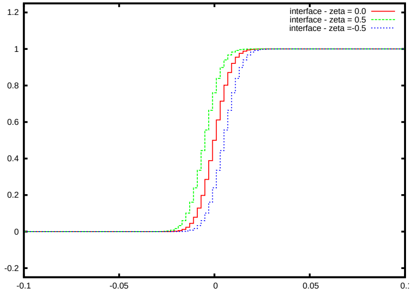
Interfaces v

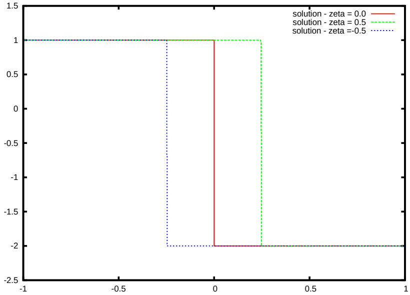


Corresponding solutions ($N = 1000$ points)



$$v(x) = \frac{\text{erf}(x/\eta + \zeta) + 1}{2}.$$





Interfaces v

Corresponding solutions ($N = 1000$ points)

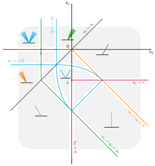
Conclusions and perspectives

Mathematical difficulties :

- (localized) non-conservative term
- characteristic boundary at the interface : *"resonant coupling"*

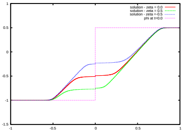
Extended PDE formalism and thin interface regime :

- Use of the self-similar viscous approximation of Dafermos as a selection criterion for the self-similar solutions to the coupled Riemann problem.
- Reduced non-uniqueness, Laplace stability analysis.



Thick interface regime :

- Well-posed Cauchy problem (smoothed non-conservative term).
- Convergent well-balanced scheme, preserving the steady states of the coupling problem.
- Numerical sensitiveness of the solution according to the interface profile.



Perspectives :

- study of the sensitiveness of solution according to the interface profile
- link with the formalism of Dal Maso, LeFloch, Murat for non-conservative products
- the self-similar Dafermos approximation as a numerical prospective tool (solving then the interfacial boundary layers)
- stability of these interfacial profiles