On order conditions of IMEX R-K schemes applied to hyperbolic systems with diffusive relaxation

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Outline

Introduction

Time Discretization, IMEX R-K Schemes

We consider initial value problems for systems of ordinary differential equations (ODEs) of the form

$$y' = f(y) + g(y), \quad y(x_0) = y_0,$$

where $f, g : \mathbb{R}^k \to \mathbb{R}^k$, are sufficiently smooth functions with different stiffness properties. Such system may arise from the spatial discretization of a system of PDEs.

For the numerical integration of such system we use additive Runge-Kutta schemes.

Introduction
What is an IMEX R-K scheme?

The idea is to consider two different s-stage Runge-Kutta scheme,

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{b}^T \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}.$$

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- and use one of them for the function f $(\tilde{A}, \tilde{b})$, and the other one for the function g, (A, b)
- Additive Runge Kutta schemes combining implicit and explicit schemes are known in the literature as IMplicit-EXplicit (IMEX) Runge-Kutta (RK) schemes.

What is an IMEX R-K scheme?
IMEX R-K schemes

An (IMEX) R-K scheme has the form

$$\begin{aligned} Y_i &= y_0 + h\sum_{j=1}^{i-1} \tilde{a}_{ij} f(t_0+\tilde{c}_j h, Y_j) + h\sum_{j=1}^{\nu} a_{ij} g(t_0+c_j h, Y_j), \\ y_1 &= y_0 + h\sum_{i=1}^{\nu} \tilde{b}_i f(t_0+\tilde{c}_i h, Y_i) + h\sum_{i=1}^{\nu} b_i g(t_0+c_i h, Y_i). \end{aligned}$$

$\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = 0$, $j \geq i$ and $A = (a_{ij})$: $\nu \times \nu$ matrices.
Coefficient vectors: $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_\nu)^T$, $\tilde{b}^T = (\tilde{b}_1^T, \dots, \tilde{b}_\nu^T)^T$,
$c = (c_1, \dots, c_\nu)^T$, $b = (b_1, \dots, b_\nu)^T$.

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IMEX R-K schemes

- The IMEX RK scheme $(\tilde{A}, \tilde{b})$ and (A, b) is chosen with the aim of efficiently integrating the previous system with low computational cost. For example, if f represents the nonstiff part of the system and g the stiff part of it, an explicit method can be used for f and an implicit one for g.

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- Sufficient condition to guarantee that f is always evaluated explicitly: the scheme for g is diagonally implicit (*DIRK*).

Classical Order conditions

We report the order conditions for IMEX R-K schemes up to order $p = 3$. We assume that the coefficients c_i, $\tilde{c}_i$, a_{ij}, $\tilde{a}_{ij}$ satisfy conditions

$$\tilde{c}_i = \sum_j \tilde{a}_{i,j}, \quad c_i = \sum_j a_{i,j},$$

then the order conditions are:

First order:

$$\sum_i \tilde{b}_i = 1, \quad \sum_i b_i = 1.$$

Second order:

$$\sum_i \tilde{b}_i \tilde{c}_i = 1/2, \quad \sum_i b_i c_i = 1/2, \quad \sum_i \tilde{b}_i c_i = 1/2, \quad \sum_i b_i \tilde{c}_i = 1/2,$$

Third order:

$$\sum_{ij} \tilde{b}_i \tilde{a}_{ij} \tilde{c}_j = 1/6, \quad \sum_i \tilde{b}_i \tilde{c}_i \tilde{c}_i = 1/3, \quad \sum_{ij} b_i a_{ij} c_j = 1/6, \quad \sum_i b_i c_i c_i = 1/3,$$

Coupling conditions:

$$\sum_{ij} \tilde{b}_i \tilde{a}_{ij} c_j = 1/6, \quad \sum_{ij} \tilde{b}_i a_{ij} \tilde{c}_j = 1/6, \quad \sum_{ij} \tilde{b}_i a_{ij} c_j = 1/6,$$

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$$\sum_i \tilde{b}_i c_i c_i = 1/3, \quad \sum_i \tilde{b}_i \tilde{c}_i c_i = 1/3, \quad \sum_i b_i \tilde{c}_i \tilde{c}_i = 1/3, \quad \sum_i b_i \tilde{c}_i c_i = 1/3.$$

The number of coupling conditions increase dramatically with the order of the schemes.

IMEX-RK	**Number of coupling conditions**			
order	General case	$\tilde{b}_i = b_i$	$\tilde{c} = c$	$\tilde{c} = c$ and $\tilde{b}_i = b_i$
1	0	0	0	0
2	2	0	0	0
3	12	3	2	0
4	56	21	12	2
5	252	110	54	15
6	1128	528	218	78

Classification of IMEX R-K schemes

- **Definition 1** (Methods of Type A) The matrix A is invertible.
- **Definition 2** (Methods of Type CK)

$$A = \begin{pmatrix} 0 & 0 \\ a & \hat{A} \end{pmatrix}$$

  The submatrix $\hat{A}$ is invertible.
- **Definition 3** (Methods of Type ARS) Special case of type CK with vector $a = 0$ and submatrix $\hat{A}$ invertible.

Some References:

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- C. A. Kennedy, M. H. Carpenter: *Additive Runge-Kutta schems for convection-diffusion-reaction equations*. Appl. Numer. Math. 44 (2003), no. 1-2, 139–181.
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Diffusive relaxation

A simple prototype of hyperbolic system with relaxation term is given by:

$$\begin{aligned} \partial_\tau u + \partial_\xi V &= 0, \\ \partial_\tau V + \partial_\zeta p(u) &= -\frac{1}{\varepsilon}(V - Q(u)) \end{aligned}$$

where $u = u(x,\tau)$, $V = V(x,\tau) \in \mathbb{R}^M$, $\varepsilon > 0$ is called the relaxation time. Under the rescaling (diffusive scaling)

$$\tau = t/\varepsilon, \quad V = \varepsilon v,$$
$$\xi = x, \quad q(u) = Q(u)/\varepsilon,$$

we obtain a general diffusive relaxation system given by:

$$\begin{aligned} \partial_t u + \partial_x v &= 0, \\ \partial_t v + \frac{1}{\varepsilon^2}\partial_x p(u) &= -\frac{1}{\varepsilon^2}(v - q(u)) \end{aligned}$$

Where $p'(u) > 0$. This system is hyperbolic with two distinct real characteristics speed $\sqrt{p'(u)}/\varepsilon$.

Diffusive relaxation

In the small relaxaction limit, $\varepsilon \to 0$ the system relax towards the system

$$\begin{aligned} \partial_t u + \partial_x q(u) &= \partial_{xx} p(u), \\ v &= q(u) - \partial_x p(u). \end{aligned}$$

Since the equilibrium equation is of parabolic type, the main stability condition for the diffusive relaxation system is

$$|q'(u)|^2 < \frac{p'(u)}{\varepsilon^2}$$

and it is naturally satisfied in the limit $\varepsilon \to 0$.

Motivations

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- To overcome the CFL parabolic restriction obtaining IMEX R-K schemes that have as CFL condition $\Delta t \approx \Delta x$, with corse grid Δt, $\Delta x >> \varepsilon$.

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- To introduce two different approachs.
- To perform the analysis of the behaviour of IMEX RK schemes when $\varepsilon \to 0$, for the two different approachs.
- To have guidelines for the construction of high order schemes.

  To this purpose we reformulate the previous diffusive relaxation system such that it allows us to design a class of IMEX Runge-Kutta (RK) schemes that in the diffusive limit no diffusive restriction is appears on the time step.

Diffusive relaxation

Overcoming parabolic stiffness

The starting point is to consider the system

$$\begin{aligned} \partial_t u &= -\partial_x v, \\ \varepsilon^2 \partial_t v &= -\partial_x p(u) - (v - q(u)). \end{aligned}$$

We add and subtract the term $\mu(\varepsilon)\partial_{xx}p(u)$ and consider the equivalent system

$$\begin{aligned} \partial_t u &= -\partial_x(v + \mu\partial_x p(u)) + \mu\partial_{xx}p(u), \\ \varepsilon^2 \partial_t v &= -\partial_x p(u) - (v - q(u)). \end{aligned}$$

$\mu(\varepsilon)$ is such that $\mu : \mathbb{R}^+ \to [0,1]$, $\mu(0) = 1$ and $\mu(1) = 0$.

We present two different approach in order to solve numerically the previous system in the diffusive limit, i.e. $\varepsilon \to 0$

Two different approaches
IMEX-I Approach

First approach called IMEX-I approach[‡]

$$\partial_t u = \underbrace{-\partial_x(v + \mu \partial_x p(u))}_{explicit} + \underbrace{\mu \partial_{xx} p(u)}_{implicit},$$
$$\varepsilon^2 \partial_t v = \underbrace{-\partial_x p(u) - (v - q(u))}_{implicit}$$

S. B. , L. Pareschi, G. Russo[‡]: *Implicit-Explicit Runge-Kutta schemes for hyperbolic systems and kinetic equations in the diffusion limit*. submitted to SIAM J. SCI. COMPUT.

IMEX-I Approach

In the IMEX-I approach the diffusive system can be written in the form

- $$\begin{aligned} u' &= f_1(u,v) + f_2(u), \\ \varepsilon^2 v' &= g(u,v). \end{aligned}$$

where $f_1(u,v) = -\partial_x(v + \mu\partial_x p(u))$, $f_2(u) = \mu\partial_{xx}p(u)$ and $g(u,v) = (-\partial_x p(u) - v + q(u))$.

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- When $\varepsilon \to 0$ we get

  $$\begin{aligned} u' &= \hat{f}_1(u) + f_2(u), \\ 0 &= g(u,v). \;\; (\text{MANIFOLD} \;\; \mathcal{M} = \{(u,v) \in \mathbb{R} | g(u,v) = 0\}) \end{aligned}$$

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- i.e. when $\varepsilon \to 0$ the numerical solution is projected onto the manifold $g(u,v) = 0$ ($v = G(u)$, assumed that the Jacobian matrix $g_v(u,v)$ is invertible). with $\hat{f}(u) = f(u, G(u))$. Previous system is called a REDUCED SYSTEM.

IMEX-I Approach

Definition

We say that an IMEX R-K scheme is globally stiffly accurate, **GSA**, if

1. The implicit R-K scheme is *stiffly accurate*, **SA**, if $e_s^T A = b^T$, with $e_s^T = (0, \dots, 0, \underbrace{1}_{sth-comp.})$. This property is important for the *L*-stability of the scheme.
2. The *s*-stage explicit R-K scheme satisfies the condition $e_s^T \tilde{A} = \tilde{b}^T$ (**FSAL**, *First Same As Last*).

- $c_s = \tilde{c}_s = 1$, i.e. the numerical solution is identical to the last internal stage values of the scheme.

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2. The s-stage explicit R-K scheme satisfies the condition $e_s^T \tilde{A} = \tilde{b}^T$ (**FSAL**, *First Same As Last*).

- $c_s = \tilde{c}_s = 1$, i.e. the numerical solution is identical to the last internal stage values of the scheme.
- Initial conditions (u_0, v_0) are *well-prepared* if $g(u_0, v_0) = 0$.

IMEX-I Approach

- Analysis of TYPE A IMEX R-K scheme (for $\varepsilon \to 0$).

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- if such scheme is globally stiffly accurate, $g(U_s, V_s) = 0$, i.e. the last stages lie on the manifold then $u_{n+1} = U_s$ and $v_{n+1} = V_s$ and

$$\boxed{g(u_{n+1}, v_{n+1}) = 0} \quad \text{i.e.} \quad \boxed{v_{n+1} = G(u_{n+1})}.$$

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- Analysis of TYPE CK IMEX R-K scheme ($\varepsilon \to 0$).
- If the following condition is satisfied

$$\boxed{\alpha_s = -\hat{e}_{s-1}^T \hat{A}^{-1} a = 0}$$

then $g(U_s, V_s) = 0$, i.e. the last stages lie on the manifold, and if the scheme is globally stiffly accurate we obtain $u_{n+1} = U_s$ and $vn + 1 = V_s$ and

$$\boxed{g(u_{n+1}, v_{n+1}) = 0}, \quad \text{i.e.} \quad \boxed{v_{n+1} = G(u_{n+1})}.$$

IMEX-I Approach

TYPE CK R-K scheme for the IMEX-I Approach.

W-P	$\alpha_s = 0$	**GSA**	$\mathcal{M}$
Yes	Yes	Yes	Yes
Yes	No	Yes	Yes
Yes	Yes/No	No	No
No	Yes	Yes	Yes
No	No	Yes	No
No	Yes/No	No	No

W-P - WELL-PREPARED Initial data

Additional Order Conditions in the diffusive limit (i.e. $\varepsilon \to 0$)

In order to maintain the order accuracy in time of the scheme, by using Taylor's expansion for the exact and numerical solution and by comparing them, we get the following additional order conditions for the reduced system up to third order:

No-SA	**SA**
$b^T A^{-1}\tilde{c} = 1$	$\tilde{c} = 1$
$b^T A^{-1}\tilde{c}^2 = 1$	$\tilde{c} = 1$
$b^T A^{-1}\tilde{A}\tilde{c} = 1/2$	$e_s^T \tilde{A}\tilde{c} = 1/2$

where $\tilde{c} = \tilde{A}e$, with $e = (1, ..., 1)^T$ and $e_s = (0, ..., 0, 1)^T$
If IMEX R-K scheme is **GSA** then the conditions are automatically satisfied, since $e_s^T \tilde{A} = \tilde{b}^T \Rightarrow \tilde{b}^T \tilde{c} = 1/2.$,i.e. the classical second order cond.

IMEX-E Approach

$$\partial_t u = \underbrace{-\partial_x(v + \mu\partial_x p(u))}_{explicit} + \underbrace{\mu\partial_{xx}p(u)}_{implicit},$$

$$\partial_t v = \underbrace{\frac{-\partial_x p(u)}{\varepsilon^2}}_{explicit} - \underbrace{\frac{(v - q(u))}{\varepsilon^2}}_{implicit}$$

We call such approach IMEX-E approach.

- Second equation: $p(u)_x/\varepsilon^2$ is treated explicitly.

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- S. B, G. Russo: *Flux-Explicit IMEX R-K schemes for hyperbolic to parabolic relaxation problems*. submitted to SINUM

IMEX-E Approach

Analysis

We set $p(u) = u$ and $q(u) = 0$. We look for a Fourier solution of the form $u = \hat{u}(t)\exp(i\xi x)$, $v = \hat{v}(t)\exp(i\xi x)$ and inserting the *ansatz* into the system and using the new variable $\hat{w} = -i\hat{v}/\xi$ in place of $\hat{v}$ the evolution equations are

$$\begin{aligned} \hat{u}_t &= \xi^2(\hat{w} + \hat{u}) - \xi^2\hat{u}, \\ \varepsilon^2\hat{w}_t &= -\hat{u} - \hat{w}. \end{aligned}$$

Apply a TYPE A IMEX R-K scheme we obtain

Numerical solution

$$\hat{u}_{n+1} = \hat{u}_n + \theta \sum_{k=1}^{s} \tilde{b}_k (\hat{W}_k + \hat{U}_k) - \theta \sum_{k=1}^{s} b_k \hat{U}_k$$

$$\zeta^2 \hat{w}_{n+1} = \zeta^2 \hat{w}_n - \sum_{k=1}^{s} \tilde{b}_k \hat{U}_k - \sum_{k=1}^{s} b_k \hat{W}_k,$$

where $\theta = \Delta t \xi^2$ and $\zeta = \varepsilon^2 / \Delta t$. Stage values

$$\hat{U}_k = \hat{u}_n + \theta \sum_{j=1}^{k-1} \tilde{a}_{kj} (\hat{W}_j + \hat{U}_j) + \theta \sum_{j=1}^{k} a_{kj} \xi^2 \hat{U}_j$$

$$\zeta^2 \hat{W}_k = \zeta^2 \hat{w}_n - \sum_{j=1}^{k-1} \tilde{a}_{kj} \hat{U}_k - \sum_{j=1}^{k} a_{kj} \hat{W}_j.$$

Solve for the stage values, insert in the numerical solution for the variable $\hat{w}$, and obtain

$$\zeta \hat{w}_{n+1} = \zeta (1 - b^T A^{-1} \mathbf{1}) \hat{w}_n + (b^T A^{-1} \tilde{A} - \tilde{b}^T) \hat{U} - \zeta b^T A^{-1} A^{-1} \tilde{A} \hat{U} + \mathcal{O}(\zeta^2),$$

IMEX-E Approach
Additional Conditions

- Consistency as $\zeta \to 0$ implies

$$\boxed{b^T A^{-1}\tilde{A} - \tilde{b}^T = 0}. \quad (1)$$

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- Furthermore, if

$$\boxed{1 - b^T A^{-1} e = 0} \quad (2), \quad \Longrightarrow \boxed{\hat{w}_{n+1} = -b^T A^{-2}\tilde{A}\hat{U}},$$

IMEX-E Approach

Additional Conditions

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- Note that if the implicit scheme is **SA**, i.e. $e_s^T = b^T A^{-1}$, the condition (2) is satisfied, then condition (1) is equivalent to $e_s^T \tilde{A} = \tilde{b}^T$, which means that the scheme is **GSA**.

IMEX-E Approach
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Remark

Then a sufficient condition to guarantee that both (1) and (2) are satisfied is that the IMEX R-K of type A is **GSA**.

IMEX-E Approach

Higher Order IMEX R-K schemes

- The construction of higher order schemes that capture the correct behavior of the solution in the limit $\varepsilon \to 0$ is more complicated. The condition we found in fact only prevents divergence of the numerical solution $\hat{w}_{n+1}$.

IMEX-E Approach
Higher Order IMEX R-K schemes

- The construction of higher order schemes that capture the correct behavior of the solution in the limit $\varepsilon \to 0$ is more complicated. The condition we found in fact only prevents divergence of the numerical solution $\hat{w}_{n+1}$.
- From this we derive the new additional order conditions on the IMEX R-K schemes of type A.

IMEX-E Approach

Additional Conditions up to second order

By using Taylors expansion for the exact and numerical solution and by comparing them, we get additional order conditions for the TYPE A

Additional consistency conditions: w-component

$$b^T A^{-2}\tilde{A}\mathbf{1} = 1,$$

first order conditions: for u and w component

$$(b^T - \tilde{b}^T\mathcal{C})\mathbf{1} = 1, \qquad b^T A^{-2}\tilde{A}\mathbb{A}\mathbf{1} = -1,$$

where $\mathbb{A} = (\tilde{A}\mathcal{C} - A)$, and $\mathcal{C} = I - A^{-1}\tilde{A}$

Additional second order conditions: for u and w component

$$(\tilde{b}^T\mathcal{C} - b^T)\mathbb{A}\mathbf{1} = \frac{1}{2}, \quad b^T A^{-2}\tilde{A}\mathbb{A}^2\mathbf{1} = \frac{1}{2},$$

IMEX-E Approach
Second order IMEX R-K schemes

Negative results:

Theorem

There are no second order GSA IMEX R-K schemes of type A with three stages.

Theorem

There are no second order GSA IMEX R-K schemes of type A with four stages where the implicit part is singly diagonally implicit (SDIRK).

IMEX-E Approach

Generalization.

$q(u) \neq 0$ and $p(u)$ non-linear. In this case the system relaxes to a convection-diffusion equation.

As $\varepsilon \to 0$, the scheme becomes an IMEX R-K scheme for the limit convection-diffusion equation, where the convection term is treated explicitly and the diffusion term is treated implicitly.

Additional Order Conditions. We give explicitly first and second order conditions

$$\begin{array}{llll} order & 1 & w - component: & b^T A^{-2}\tilde{A}\tilde{c} = 1 \\ order & 2 & u - component: & \tilde{b}^T \tilde{A} A^{-1}\tilde{c} = 1/2, \quad (b^T - \tilde{b}^T \mathcal{C})\tilde{c} = 1/2 \end{array}$$

Of course classical order conditions, as well as new algebraic order conditions as introduced before have to be satisfied as well.

Numerical Results

Convergence Results

We consider the prototype problem

$$\begin{aligned} \partial_t u + \partial_x v &= \partial_{xx} u - \partial_{xx} u, \\ \partial_t v + \frac{1}{\varepsilon^2}\partial_x u &= -\frac{1}{\varepsilon^2} v, \end{aligned}$$

in the limit case this lies to the linear problem

$$u_t(x,t) = u_{xx}(x,t)$$

- periodic boundary condition;
- initial data: $u(x,0) = \cos(x)$ and $v(x,0) = \sin(x)$;
- Set $\varepsilon^2 = 10^{-6}$ with $\mu(\varepsilon) = 1$, final time $T = 1$ and $\Delta t \approx \Delta x$. The system is integrated for $x \in [-\pi, \pi]$.

- We consider for the space discretization central difference schemes,

S. B, G. Russo: *Flux-Explicit IMEX R-K schemes for hyperbolic to parabolic relaxation problems*[‡]. submitted to SINUM

- We consider for the space discretization central difference schemes,
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- We consider for the space discretization central difference schemes,
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- The term $\mu(\varepsilon)p(u)_{xx}$ is discretized by central differencing.

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Numerical Results

N	Error	Order
40	1.2014e-02	–
80	3.4484e-03	1.8007
160	9.3359e-04	1.8851
320	2.418e-04	1.9490
640	6.1579e-05	1.9733
1280	1.5557e-05	1.9849

L^∞-norm of the error and convergence rates of u in with $\varepsilon^2 = 10^{-6}$, CFL $= 0.5$ and $\Delta t = CFL\Delta x$

Numerical Results
First Numerical Test

Now we test the second order IMEX scheme to this system by solving a Riemann problem

$$\begin{aligned} \rho_t + j_x &= \mu\rho_{xx} - \mu\rho_{xx} \\ \varepsilon^2 j_t + \rho_x &= -j \end{aligned}$$

- Initial data

$$\begin{aligned} \rho_L &= 2.0 \quad j_L = 0, \quad -1 < x < 0, \\ \rho_R &= 1.0 \quad j_R = 0, \quad 0 < x < 1. \end{aligned}$$

$\mu(\varepsilon) = \exp(-\varepsilon^2/\Delta x)$,

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- we compute the scheme in the rarefied regime ($\varepsilon = 0.7, \;\mu = 0$) and in the diffusive regime (or stiff regime) for $\varepsilon^2 = 10^{-6}, \;\mu = 1$.

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- The boundary condition are of reflecting type. Final time $t = 0.25$ in the rarefied regime and $t = 0.04$ in the diffusive regime.

- We consider for the space discretization Finite difference schemes such us WENO schemes, e.g. third-second order WENO(3,2) reconstruction.

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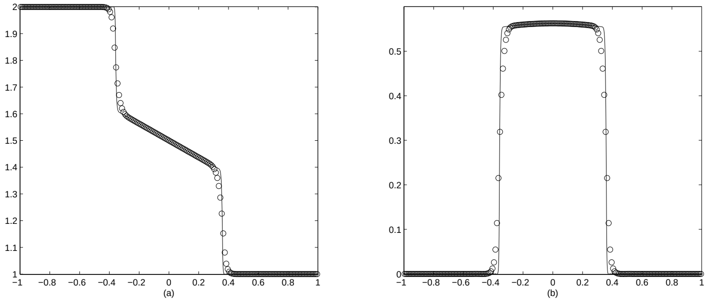
Numerical Results

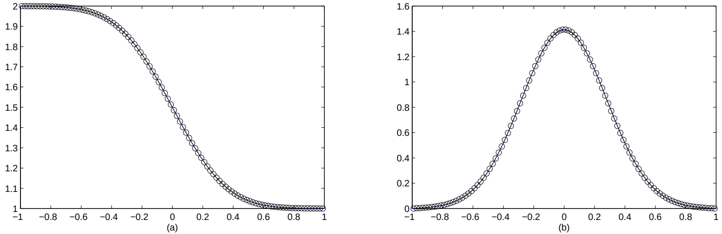
Rarefied Regime, ($\varepsilon = 0.7$)



Numerical solution at time $t = 0.25$, in the rarefied regime $(\varepsilon = 0.7)$ with $\Delta t = 0.0025$, $CFL = 0.25$ and $\Delta x = 0.01$ and $N = 200$. From the left to the right the mass density ρ (a) and the flow j, (b). Solid line reference solution with $N = 2000$ cells.

IMEX-E Approach

Parabolic Regime, ($\varepsilon = 10^{-6}$)



Numerical solution at time $t = 0.04$ in the parabolic regime ($\varepsilon^2 = 10^{-6}$) with $\Delta t = 0.001$, $CFL = 0.05$ and $\Delta x = 0.02$ and $N = 100$. From the left to the right the mass density (a) ρ and the flow j (b). Solid line reference solution with $N = 2000$ cells.

Numerical Results

Second Numerical Test

Finally we take the generalized Carlemann model with $m = -1$,

$$\begin{aligned} \rho_t &= -j_x + \mu(\varepsilon)\frac{\partial_{xx}\rho}{2\rho^m} - \mu(\varepsilon)\frac{\partial_{xx}\rho}{2\rho^m} \\ \varepsilon^2 j_t &= -\rho_x - 2\rho^m j, \end{aligned}$$

when $\varepsilon \to 0$, the local equilibrium is given by $j = -\frac{\partial_x \rho}{2\rho^m}$ and $\mu(\varepsilon) \to 1$, thus the system relaxes to the porous media equation.

The exact Barenblatt solution for the porous media equation,

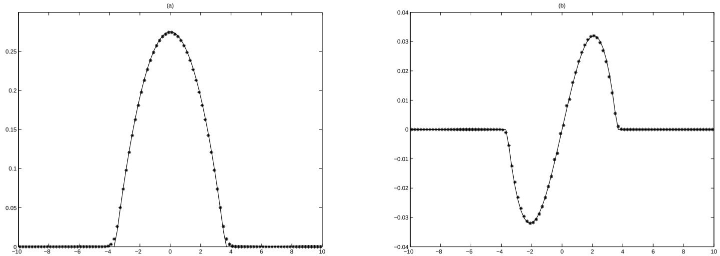
$$\rho(x,t) = \frac{1}{R(t)}\left[1 - \left(\frac{x}{R(t)}\right)^2\right], \quad j(x,t) = \rho\frac{2x}{R(t)^3}, \quad |x| < R(t),$$

$$\rho(x,t) = 0, \quad j(x,t) = 0, \quad |x| > R(t),$$

where $R(t) = [12(t+1)]^{1/3}$, $t \geq 0$. We take $\Delta x = 0.2$ and $x \in]-10, 10[$.

Numerical Results

Second Numerical Test



Numerical solution at time $t = 3.0$ for the Barenblatt problem in the parabolic regime $\varepsilon^2 = 10^{-6}$ with $\Delta x = 0.2$, $CFL = 0.5$ and $\Delta t = 0.1$.
(a) The mass density ρ and (b) the flow j.

Work in Progress

- Study the uniform accuracy

Work in Progress

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- Develop really high order schemes

Work in Progress

- Study the uniform accuracy
- Develop really high order schemes
- Capture the Navier-Stokes limit of hyperbolic systems with relaxation.

Thank you for your attention!