SLDG schemes for 1st and 2nd order PDEs

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Joint work with
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ADVERTISMENT : HJ parallel Library

$$\begin{cases} u_t + H(t, x, \nabla u, [D^2 u]) = 0, & x \in \mathbb{R}^{\mathbf{d}} \\ u(0, x) = u_0(x) \end{cases}$$

- C++, parallel (MPI/OpenMP)
- works in any dimension **d** (limited to machine's capacity)

- **Finite Difference solver** (based on ENO): **MPI** / **OpenMP**
- **Semi-Lagrangian schemes** (1 & 2 order HJ PDE) : **OpenMP**
- Explicit schemes, uniform grid, control-oriented equations.
- Development: O. Bokanowski, H. Zidani, A. Desilles (and J. Zhao)

$\Rightarrow$ **www.ensta.fr/$\sim$zidani/BiNoPe-HJ/**

I. Introduction

Could we consider an explicit scheme for

$$u_t - \frac{1}{2} Tr(\sigma\sigma^T D^2 u) + B \cdot \nabla v + rv = 0, \quad x \in \mathbb{R}^d \quad ?$$

PROS

- Only explicit is implementable for high-d !
- Motivated by related HJ equations involving max operators, or obstacle terms, application to optimal control

CONS

- FD, DG methods needs restrictive CFL ($\Delta t \leq C\Delta x^2$, or $\Delta t \leq C\Delta x$, small C) $\Rightarrow$ use SL scheme
- Cannot be high order, and monotone ("Godunov's theorem", Harten, Osterlee and Van Pijl 2012 "negative result")

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II. Schemes

Idea:

- 1d semi-lagrangian for advection
- convex combinations for diffusion
- splitting, splitting, splitting

But :

- In general this strategy may not work practically because the semi-lagrangian scheme may not be precise enough.

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Morton, Priestley, Suli (1988)
Crouzeillles, Mehrenberger, Vecil (2010): SLDG
Qiu and Shu (2011): SLDG + splitting

- Consider the 1d advection equation for $t \in (0, T)$:

$$u_t + b(x) u_x = 0, \quad x \in (0,1) \quad \text{(with periodic b.c.)}$$

- Notice that $\boxed{u(t + \Delta t, x) = u(t, x - b\Delta t)}$ if $b(x) = b = const$
- Introduce DG: mesh intervals I_i partition of $(0,1)$, and

$$V_k := \{v \in L^2(0,1),\ v_{|I_i} \in P_k \text{ for all } i\} \qquad \text{"DG space"}$$

where P_k is the set of polynomials of degree $\leq$ k.

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- **SLDG scheme:** Find $u^{n+1} \in V_k$ such that

$$\boxed{\forall \varphi \in V_k, \quad \int u^{n+1}(x)\varphi(x)dx = \int u^n(x - b\Delta t)\varphi(x)dx}$$

- $u^{n+1} = \Pi(u^n(\cdot - b\Delta t))$ where Π is the L^2 projection on V_k.

Theorem

(i) The scheme is exactly implementable
(ii) High order: $O(\frac{\Delta x^{k+1}}{\Delta t})$
(iii) L^2 *stable*

- $\Rightarrow$ **no CFL !**
- "Immediate" proof
- Implementation : gauss quadrature formula:

$$\int_{-1}^{1} \varphi(x)dx = \sum_{\alpha=0,\dots,k} w_\alpha \varphi(x_\alpha) \quad \text{for any } \varphi \in P_{2k+1}.$$

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where $\frac{d}{dt} y_x(t) = b(y_x(t))$, $y(0) = x$.

Theorem (B, Simaramata)

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(ii) order: $O(\frac{\Delta x^{k+1}}{\Delta t})$ for $k = 1, 2$ [a]
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[a] (here with $\Delta t \leq \lambda \Delta x$, λ constant)

- same as Liu & Shu 2011 in the linear case.
- $\Rightarrow$ **weak CFL :** $\frac{\Delta x^{k+1}}{\Delta t} \leq const$. (Allows $\Delta t = \lambda \Delta x$ with large λ.)
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ONE PAGE STABILITY PROOF (non constant $b(x)$)

- Scheme definition is : find $u^{n+1} \in V_k$ s.t., $\forall \varphi \in V_k$:

$$\int u^{n+1}(x)\varphi(x)dx \quad \overset{\sim}{=} \quad \int u^n(y_x(-\Delta t))\varphi(x)dx \quad +O(\Delta x^{k+1}) \ ?$$

$$\overset{DEF}{\equiv} \quad \sum_i \sum_{q=0,\dots,p_i} \sum_{\alpha=0}^{k} w^i_{q,\alpha} u^n(y(x^i_{q,\alpha}))\varphi(x^i_{q,\alpha})$$

- $\epsilon_i := \left| \int_{J_i} u^n(y_x(-\Delta t))\varphi(x) - \sum(gauss)\right| \le \|[u^n(y)\varphi]^{(2k+2)}\|_{L^\infty(J_i)}\Delta x^{2k+3}$
- Lemma 1 : $|\frac{d}{dx}y_x(-\Delta t)| \le C$, and $|\frac{d^q}{dx^q}y_x(-\Delta t)| \le C\Delta t$ for $q \ge 2$.
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2) SLDG for diffusion equation with constant $\sigma \in \mathbb{R}$:

$$v_t - \frac{\sigma^2}{2} v_{xx} = 0, \quad x \in \Omega,\ t \in (0,T), \qquad (1)$$

• A first scheme, in semi-discrete form (denoting $h \equiv \Delta t$)

$$u^{n+1}(x) = \frac{1}{2}\Big(u^n(x - \sigma\sqrt{h}) + u^n(x + \sigma\sqrt{h}) \Big) \equiv S_h^0 u^n\,(x). \qquad (2)$$

• **Remark:** this becomes a **tree** method if $\sigma\sqrt{h} = \Delta x$

• **Remark:** Taking $v^n(x) := v(t_n, x)$ where v is solution of (1) the following consistency error estimate holds:

$$\|\frac{v^{n+1} - S_h^0 v^n}{h}\|_{L^2} = O(h\|v^n_{4x}\|_{L^\infty}) = O(h)$$

• **SLDG-RK1 scheme** := weak formulation of (1):
Find u^{n+1} in V_k such that, for all $\varphi \in V_k$:

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First results:

(i) Implementable scheme

(ii) Consistency error : $O(\Delta t) + O(\frac{\Delta x^{k+1}}{\Delta t})$

(iii) L^2 stable

$\Rightarrow$ **We want to improve order in time**

- Let $h := \Delta t$. Using Taylor expansions,

$$
\begin{aligned}
S_h^0 u &= u + h\frac{\sigma^2}{2}u_{xx} + h^2\frac{\sigma^4}{24}u_{4x} + O(h^3), && (3)\\
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- On the other hand, if $v^n = v(t_n, x)$ (where $v_t = \frac{\sigma^2}{2}v_{xx}$):

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v^{n+1} &= v^n + hv_t + \frac{h^2}{2}v_{tt} + O(h^3) && (5)\\
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- $\Rightarrow$ Now we look for coefficients a, b, c such that

$$v^{n+1} = av^n + bS_h^0 v^n + cS_h^0 S_h^0 v^n + O(h^3)$$

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- Therefore a second order scheme is now given by

SLDG-RK2 scheme

$$u^{n+1} = \frac{1}{3}(u^n + S_{\Delta t} u^n + S_{\Delta t} S_{\Delta t} u^n). \qquad (7)$$

- In a similar way, we can identify up to the 3rd order term :

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$$u^{n+1} = \frac{13}{45} u^n + \frac{21}{45} S_{\Delta t} v^n + \frac{9}{45} S_{\Delta t} S_{\Delta t} u^n + \frac{2}{45} S_{\Delta t} S_{\Delta t} S_{\Delta t} u^n.$$

Theorem (B., Simarmata, 2012')

Consider SLDG-RKp, $p = 1, 2, 3$:
(i) Consistency order: $O(\Delta t^p) + O(\frac{\Delta x^{k+1}}{\Delta t})$
(ii) L^2 stable **AND** *convex combination of* $\left((S_{\Delta t})^{(q)}\right)_{q=0,\dots,p}$

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Modified SLDG-RK2* scheme for non constant $\sigma(x)$:

- Define

$$S_h^- u(x) := \Pi\left(\frac{1}{2}(u(x+h) - u(x-h))\right): \ L^2\text{-stable}$$

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$$u^{n+1} = \frac{1}{3}(u^n + S_{\Delta t}u^n + S_{\Delta t}S_{\Delta t}u^n)$$
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L^2 stable, second order convergent.

Idea: Stability constant is $(1 + C\Delta t)^N \leq e^{CN\Delta t} \leq e^{CT}$.

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Higher order formulas for non constant $\sigma(x)$:

- Kloeden & Platen 1999 (Springer, Vol 21); Debrabant
- Case $\sigma = const$, second order :

$$u^{n+1}(x) = \frac{1}{6}u^n(x - \sigma\sqrt{3h}) + \frac{2}{3}u^n(x) + \frac{1}{6}u^n(x + \sigma\sqrt{3h}).$$

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$$u^{n+1}(x) = \frac{1}{6}\sum_{\Delta\widehat{W}=\pm\sqrt{3h}} u^n\left(X_x(\Delta\widehat{W})\right) + \frac{2}{3}u^n(x)$$

where

$$\begin{aligned} X_x(\Delta\widehat{W}) \quad := \quad & x + \frac{1}{4}\Big(\sigma(x+\sigma(x)\sqrt{h}) + \sigma(x-\sigma(x)\sqrt{h}) + 2\sigma(x)\Big)\Delta\widehat{W} \\ & + \frac{1}{4}\Big(\sigma(x+\sigma(x)\sqrt{h}) - \sigma(x-\sigma(x)\sqrt{h})\Big)\frac{\Delta\widehat{W}^2 - h}{\sqrt{h}} \end{aligned}$$

3) d-dimensional equations:

For a given $\sigma \in \mathbb{R}^{d \times d}$, let us consider

$$u_t - \frac{1}{2} Tr(\sigma\sigma^T D^2 u) = 0, \quad x \in \Omega, \ t \in (0,T), \tag{9a}$$

$$u(0,x) = u_0(x), \quad x \in \Omega \tag{9b}$$

Rem: (Debrabant and Jakobsen 2012)

$$\sigma = [\Sigma_1, \dots, \Sigma_d], \, \Sigma_k \in \mathbb{R}^d \quad \Rightarrow \quad \sigma\sigma^T = \sum_{k=1}^{d} \Sigma_k \Sigma_k^T$$

Thus (9a) is equivalent to

$$u_t - \frac{1}{2} \sum_{k=1,\dots,d} Tr(\Sigma_k \Sigma_k^T D^2 u) = 0. \tag{10}$$

For the one-directional problem

$$u_t = \frac{1}{2} Tr(\Sigma_k \Sigma_k(x)^T D^2 u) \tag{11}$$

we consider the cheme

$$u^{n+1}(x) = \Pi \frac{1}{2}\Big(u^n(x - \Sigma_k \sqrt{\Delta t}) + u^n(x + \Sigma_k \sqrt{\Delta t}) \Big) =: (S^{\Sigma_k}_{\Delta t} u^n)(x),$$

and for the general problem, we can consider

$$\boxed{u^{n+1} = \frac{1}{d} \sum_{k=1}^{d} S^{\Sigma_k}_{d\Delta t} u^n \quad \equiv: Su^n}$$

OR Trotter's splitting:

$$\boxed{u^{n+1} = S^{\Sigma_d}_{\Delta t} \cdots S^{\Sigma_2}_{\Delta t} S^{\Sigma_1}_{\Delta t} u^n \quad \equiv: Su^n}$$

III. Numerical Examples

$$v_t - \frac{1}{2}\sigma^2 v_{xx} + b v_x = 0, \quad \forall x \in (0,1), \forall t \in (0,T) \tag{12}$$

$$v(0,x) = \cos(2\pi x) + \frac{1}{2}\cos(4\pi x) \tag{13}$$

together with periodic boundary conditions on $(0,1)$, $\sigma = 0.1$, $T = 1$ and $b = 0$ or $b = 0.3$.

L^2 error	SLDG-RK1		SLDG-RK2		SLDG-RK3	
N	error	order	error	order	error	order
10	3.52E-03	-	3.27E-05	-	1.37E-07	-
20	1.73E-03	1.02	8.05E-06	2.02	1.69E-08	3.01
40	8.61E-04	1.01	1.99E-06	2.01	2.10E-09	3.00
80	4.29E-04	1.01	4.96E-07	2.00	2.62E-10	3.00
160	2.14E-04	1.00	1.23E-07	2.00	3.27E-11	3.00

Table: [with neglictable spatial error : $k = 4$ and $M = 1000$.]

L^2 error		SLDG-RK1		SLDG-RK2		SLDG-RK3	
M	N	error	order	error	order	error	order
20	10	4.10E-03	-	4.88E-05	-	8.27E-07	-
40	20	1.75E-03	1.23	2.36E-05	1.05	5.78E-07	0.52
80	40	8.68E-04	1.01	2.23E-06	3.40	3.82E-09	7.24
160	80	4.30E-04	1.01	5.58E-07	2.00	5.10E-10	2.91
320	160	2.14E-04	1.00	1.38E-07	2.01	4.01E-11	3.67
640	320	1.07E-04	1.00	3.44E-08	2.01	4.74E-12	3.08

Table: with $\Delta t \equiv \Delta x$.

		$k=3, RK3$, with $\Delta t \gg \Delta x$		
M	N	L^1-Error	L^2-Error	L^∞-Error
10	10	5.129E-05	1.443E-05	2.166E-05
20	20	2.465E-06	9.191E-07	1.088E-06
40	30	9.307E-08	5.002E-08	5.588E-08
80	40	6.591E-09	3.360E-09	3.823E-09
160	50	1.923E-09	1.079E-09	1.209E-09

Table: "SLDG-RK3" + P3 and large time steps $\Delta t \gg \Delta x$.

Example 2 (1D diffusion with nonconstant $\sigma(x)$).

$$v_t - \frac{1}{2}\sigma^2(x) v_{xx} = f(t,x); \quad \sigma(x) = \sin(2\pi x) \tag{14}$$

$$v(0,x) = 0 \qquad x \in (0,1), \tag{15}$$

with periodic boundary conditions

L^2 error		SLDG-RK1		SLDG-RK2*	
M	N	error	order	error	order
10	10	8.60E-02	-	4.13E-02	-
20	20	3.52E-02	1.29	7.30E-03	2.50
40	40	1.59E-02	1.15	1.39E-03	2.39
80	80	7.54E-03	1.08	3.03E-04	2.20
160	160	3.67E-03	1.04	7.17E-05	2.08
320	320	1.81E-03	1.02	1.80E-05	2.00

Table: $T = 0.2$, $\Delta t \equiv \Delta x$.

We consider the following rotation example:

$$u_t + 2\pi(-x_2, x_1)\cdot\nabla u = 0, \qquad x = (x_1, x_2) \in \Omega,\ t \in (0, T),$$
$$u(0, x) = 1 - e^{-20((x_1-1)^2 + x_2^2 - r_0^2)},$$

with $T = 0.9$ and $r_0 = 0.25$.

L^2 error	Trotter		Strang		3rd or. split.		4th or. split.	
$M = N$	error	order	error	order	error	order	error	order
10	6.89E-01	-	2.91E-01	-	1.94E+00	-	2.26E-02	-
20	3.90E-01	0.82	6.62E-02	2.13	1.81E-01	3.42	8.10E-04	4.80
40	1.92E-01	1.02	1.60E-02	2.05	1.99E-02	3.18	3.46E-05	4.55
80	9.49E-02	1.02	3.99E-03	2.01	2.45E-03	3.02	1.80E-06	4.27
160	4.71E-02	1.01	9.96E-04	2.00	3.06E-04	3.00	1.07E-07	4.07

Table: Rotation example, $T = 0.9$, various splittings

We consider the following rotation example:

$$u_t + 2\pi(-x_2, x_1) \cdot \nabla u = 0, \qquad x = (x_1, x_2) \in \Omega,\ t \in (0, T),$$
$$u(0, x) = 1 - e^{-20((x_1-1)^2 + x_2^2 - r_0^2)},$$

with $T = 0.9$ and $r_0 = 0.25$.

L^2 error	Trotter		Strang		3rd or. split.		4th or. split.	
$M = N$	error	order	error	order	error	order	error	order
10	6.89E-01	-	2.91E-01	-	1.94E+00	-	2.26E-02	-
20	3.90E-01	0.82	6.62E-02	2.13	1.81E-01	3.42	8.10E-04	4.80
40	1.92E-01	1.02	1.60E-02	2.05	1.99E-02	3.18	3.46E-05	4.55
80	9.49E-02	1.02	3.99E-03	2.01	2.45E-03	3.02	1.80E-06	4.27
160	4.71E-02	1.01	9.96E-04	2.00	3.06E-04	3.00	1.07E-07	4.07

Table: Rotation example, $T = 0.9$, various splittings

- Set $\Omega := (0,1)^2$ with periodic boundary conditions, and:

$$u_t - \frac{1}{2}(5u_{xx} - 4u_{xy} + u_{yy}) = 0, \quad x \in \Omega,\ t \in (0,1),$$
$$u(0,x) = u_0(x), \quad x \in \Omega$$

- Non obvious initial data[1]
- In order to define the numerical scheme, we use the fact that

$$A := \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \sum_{k=1,2} \Sigma_k \Sigma_k^T, \text{ with } \Sigma_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \Sigma_2 := \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

[1] $u_0(x) = u_{01}(x+2y) + u_{02}(-y)$, with $u_{0i}(x) := \sum_{q=1,2} c_q^i \cos(2\pi q x) \quad (c_q^i = \frac{1}{i+q})$

L^2 error		SLDG-RK1		SLDG-RK2		SLDG-RK3	
M	N	error	order	error	order	error	order
10	10	6.66E-03	-	1.86E-04	-	2.20E-06	-
20	20	3.26E-03	1.02	4.52E-05	2.04	3.10E-07	2.83
40	40	1.61E-03	1.01	1.08E-05	2.06	3.20E-08	3.27
80	80	8.04E-04	1.00	2.69E-06	2.01	4.34E-09	2.88
160	160	4.01E-04	1.00	6.66E-07	2.01	4.90E-10	3.14

Table: Example 2 ($2D$ diffusion equation), error table with $\Delta t \sim \Delta x$.

CONCLUSION: (1)

• New stability proof for SLDG with non constant $b(x)$

• New SLDG schemes for diffusion equations, with some nice "monotony properties".

CONCLUSION: (2)

• **Generalizations:** monotony / order OK up to $p = 5$! (Bokanowski/Bonnans, in progress)

• **Applications:** american option, HJB for stochastic optimal control, nonlinear PDE ....: for completely monotone scheme, we can replace Π by P_1 interpolation : gives a building block

$$u^{n+1} = \tilde{S}(u^n)$$

that is monotone and $O(\Delta t^p) + O(\frac{\Delta x^2}{\Delta t})$ consistent.

• **Potential adaptivity** (polynomial degree) / **parallelization**

• **Challenging problem:** develop efficient PDE solvers for HJ for **stochastic control problems in high dimension**;