A positive, entropy preserving, full well-balanced scheme for the shallow-water model

Christophe BERTHON[1] and Christophe CHALONS[2]

1 - LMJL, University of Nantes (France)

2 - LJLL, University Paris 6 (France)

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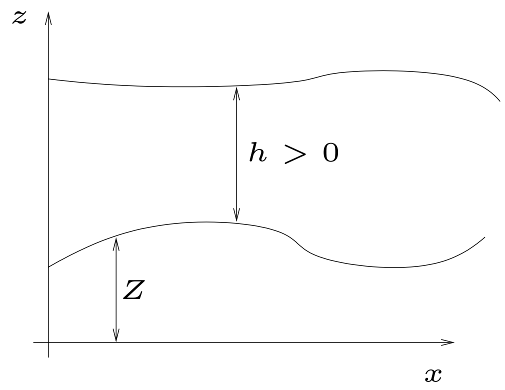
The shallow-water model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + g\dfrac{h^2}{2} \right) = -gh\partial_x Z \end{cases}$$



$$\partial_t w + \partial_x f(w) = S(w)$$

$$\Omega = \{w \in \mathbb{R}^2;\ h > 0\}$$

$Z(x)$ smooth given topography

☐ Main properties

- Steady states

$$\begin{cases} \partial_x hu = 0 \\ \partial_x \left(hu^2 + g\dfrac{h^2}{2} \right) = -gh\partial_x Z \end{cases} \quad \Rightarrow \quad \begin{cases} hu = \text{cste} \\ \dfrac{u^2}{2} + g(h+Z) = \text{cste} \end{cases}$$

- Entropy inequalities

$$\partial_t \left(h\frac{u^2}{2} + g\frac{h^2}{2} \right) + \partial_x \left(h\frac{u^2}{2} + gh^2 \right) u \leq -ghu\partial_x Z$$

Remark: About discontinuous solutions

Steady states $\overset{\text{inconsistent}}{\longleftrightarrow}$ Rankine-Hugoniot relations

$$\begin{cases} h_L u_L = h_R u_R \\ \dfrac{u_L^2}{2} + g h_L = \dfrac{u_R^2}{2} + g h_R \end{cases} \qquad Z_L = Z_R \qquad \begin{cases} h_L u_L = h_R u_R \\ h_L u_L^2 + g\dfrac{h_L^2}{2} = h_R u_R^2 + g\dfrac{h_R^2}{2} \end{cases}$$

Additional condition: Steady states must be smooth

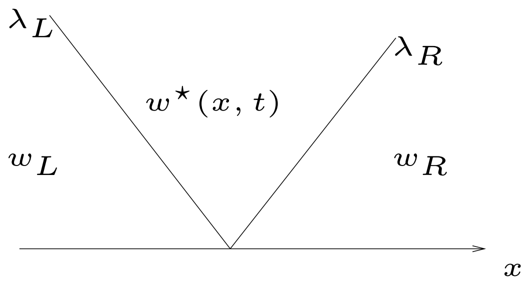
☐ Objectives

Numerical scheme preserving the main properties

Outline

- Godunov type scheme
- Characterization of the approximate Riemann solver
- Robustness and full well-balanced properties
- Discrete entropy inequality

Godunov type scheme



☐ Approximate Riemann solver

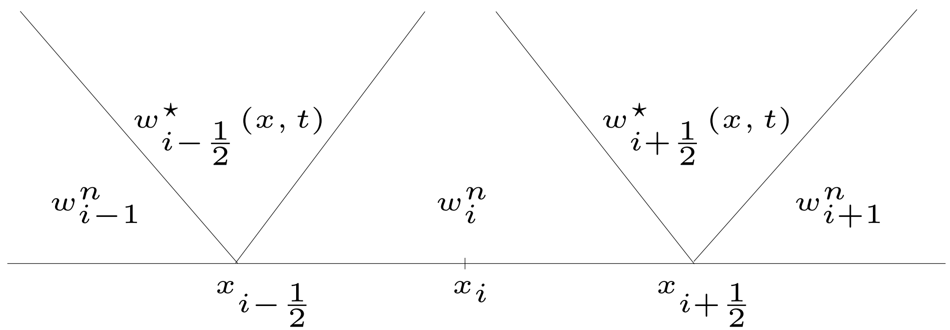
$$\tilde{w}(\frac{x}{t}; w_L, w_R) = \begin{cases} w_L & \text{if } \frac{x}{t} < \lambda_L \\ w^\star(x,t; w_L, w_R) & \text{if } \lambda_L < \frac{x}{t} < \lambda_R \\ w_R & \text{if } \frac{x}{t} > \lambda_R \end{cases}$$

Consistency condition (Harten-Lax-van Leer) with $w_{\mathcal{R}}$ exact Riemann solution

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}(\frac{x}{t}; w_L, w_R)\, dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w_{\mathcal{R}}(\frac{x}{t}; w_L, w_R)\, dx$$

☐ Approximation scheme: $(w_i^n)_{i\in\mathbb{Z}}$ known at time t^n

CFL restriction $\quad \frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}} |\lambda_{L,R}^{i+\frac{1}{2}}| \leq \frac{1}{2}$



$$w_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_i} \tilde{w}\left(\frac{x - x_{i-\frac{1}{2}}}{\Delta t}; w_{i-1}^n, w_i^n\right) + \frac{1}{\Delta x}\int_{x_i}^{x_{i+\frac{1}{2}}} \tilde{w}\left(\frac{x - x_{i+\frac{1}{2}}}{\Delta t}; w_i^n, w_{i+1}^n\right)$$

Main numerical properties

☐ Robustness

For all $w_{L,R} \in \Omega$, assume $\tilde{h}(\frac{x}{t}; w_L, w_R) > 0$

Then $h_i^{n+1} > 0$ as soon as $h_i^n > 0 \ \forall i \in \mathbb{Z}$

☐ Full well-balanced

For all $w_{L,R} \in \Omega$ such that $$\begin{cases} h_L u_L = h_R u_R \\ \dfrac{u_L^2}{2} + g(h_L + Z_L) = \dfrac{u_R^2}{2} + g(h_R + Z_R) \end{cases}$$

assume $$\tilde{w}(\frac{x}{t}; w_L, w_R) = \begin{cases} w_L & \text{if } \frac{x}{t} < 0 \\ w_R & \text{if } \frac{x}{t} > 0 \end{cases}$$

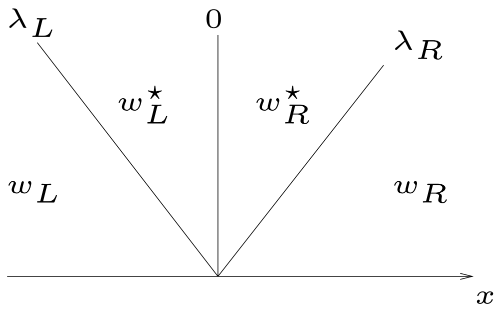
Then the steady states are preserved

$$w_i^{n+1} = w_i^n \qquad \text{as soon as} \quad \begin{cases} h_i^n u_i^n = \text{cste} \\ \dfrac{(u_i^n)^2}{2} + g(h_i^n + Z_i) = \text{cste} \end{cases} \qquad \forall i \in \mathbb{Z}$$

plus smoothness detailed later on

Approximate Riemann solver

☐ Two intermediate constant states



$\lambda_L < 0 < \lambda_R$ HLLC type solver

Source term $\rightarrow$ stationary contact wave

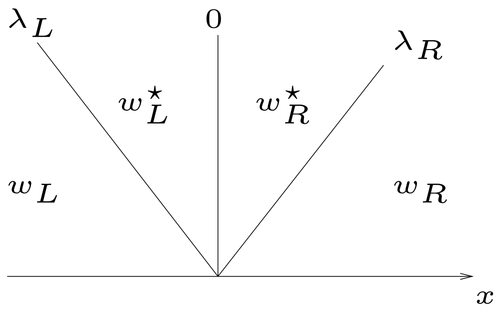
4 unknowns: $h^\star_{L,R}$ and $u^\star_{L,R}$

- Consistency condition

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}(\frac{x}{t}; w_L, w_R)\ dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w_{\mathcal{R}}(\frac{x}{t}; w_L, w_R)\ dx$$

Approximate Riemann solver

□ Two intermediate constant states



$\lambda_L < 0 < \lambda_R$ HLLC type solver

Source term $\longrightarrow$ stationary contact wave

4 unknowns: $h^\star_{L,R}$ and $u^\star_{L,R}$

- Consistency condition

$$\begin{cases} h_L u_L - h_R u_R = \lambda_L (h_L - h_L^\star) + \lambda_R (h_R^\star - h_R) \\ \left(h_L u_L^2 + g\dfrac{h_L^2}{2} \right) - \left(h_R u_R^2 + g\dfrac{h_R^2}{2} \right) - g\Delta x [h\partial_x Z]_L^R = \\ \qquad \lambda_L (h_L u_L - h_L^\star u_L^\star) + \lambda_R (h_R^\star u_R^\star - h_R u_R) \end{cases}$$

where $$[h\partial_x Z]_L^R = \frac{1}{\Delta t \Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \int_0^{\Delta t} h_{\mathcal{R}} \partial_x Z \; dx \, dt$$

We approximate $[h\partial_x Z]_L^R$ $\quad\hookrightarrow\quad$ We get an additional unknown

- Contact preserving

$$\begin{cases} h_L^\star u_L^\star = h_R^\star u_R^\star \\ \dfrac{(u_L^\star)^2}{2} + g(h_L^\star + Z_L) = \dfrac{(u_R^\star)^2}{2} + g(h_R^\star + Z_R) \end{cases}$$

- Steady state preserving: $[h\partial_x Z]_L^R$ is defined by enforcing

$$\begin{cases} h_L u_L = h_R u_R \\ \dfrac{(u_L)^2}{2} + g(h_L + Z_L) = \dfrac{(u_R)^2}{2} + g(h_R + Z_R) \end{cases} \quad \text{and} \quad \begin{cases} h_L^\star = h_L & u_L^\star = u_L \\ h_R^\star = h_R & u_R^\star = u_R \end{cases}$$

to get

$$-g\Delta x[h\partial_x Z]_L^R = \left(h_R u_R^2 + g\frac{h_R^2}{2}\right) - \left(h_L u_L^2 + g\frac{h_L^2}{2}\right)$$

By involving the steady state assumptions

$$\Delta x[h\partial_x Z]_L^R = \frac{h_L h_R}{\bar{h}}(Z_R - Z_L) - \frac{(h_R - h_L)^3}{4\bar{h}} \qquad \text{with} \qquad \bar{h} = \frac{h_L + h_R}{2}$$

$\hookrightarrow$ It is a necessary condition to recover all steady states

- Steady state smoothness condition

$$\Delta x[h\partial_x Z]_L^R = \frac{h_L h_R}{\bar{h}}(Z_R - Z_L) - \frac{(h_R - h_L)^3}{4\bar{h}}$$

To recover steady states but modifying the Rankine-Hugoniot relations

Smoothness correction

$$C_h \quad \text{such that} \quad \sup_{h \text{ smooth}} |\partial_x h| \leq C_h$$

$$\delta h = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C_h \Delta x \\ C_h \Delta x & \text{otherwise} \end{cases}$$

To suggest the approximation

$$\Delta x[h\partial_x Z]_L^R = \frac{h_L h_R}{\bar{h}}(Z_R - Z_L) - \frac{\delta h^3}{4\bar{h}}$$

$\hookrightarrow$ Only smooth steady states can be reached

□ Characterization of the intermediate states

$$\begin{cases} h_L u_L - h_R u_R = \lambda_L (h_L - h_L^\star) + \lambda_R (h_R^\star - h_R) \\ \left(h_L u_L^2 + g\dfrac{h_L^2}{2}\right) - \left(h_R u_R^2 + g\dfrac{h_R^2}{2}\right) - g\Delta x [h\partial_x Z]_L^R = \\ \qquad \lambda_L (h_L u_L - h_L^\star u_L^\star) + \lambda_R (h_R^\star u_R^\star - h_R u_R) \\ h_L^\star u_L^\star = h_R^\star u_R^\star \\ \dfrac{(u_L^\star)^2}{2} + g(h_L^\star + Z_L) = \dfrac{(u_R^\star)^2}{2} + g(h_R^\star + Z_R) \end{cases}$$

To get

$$h_L^\star u_L^\star = h_R^\star u_R^\star = q^\star(w_L, w_R) \qquad \text{(explicit)}$$

and

$$\lambda_L (h_L - h_L^\star) + \lambda_R (h_R^\star - h_R) + q_R - q_L = 0$$

$$\frac{q^\star}{2}\left(\frac{1}{(h_L^\star)^2} - \frac{1}{(h_R^\star)^2}\right) + g(h_L^\star - h_R^\star) + g(Z_L - Z_R) = 0$$

Then $h_L^\star$ solution of a polynomial of degree 5 denoted $p_5(h_L^\star) = 0$

Theorem

Assume w_L and w_R in Ω

There exists $\lambda_L < 0$ and $\lambda_R > 0$ (large enough) such that p_5 admits 5 roots exactly. The third root, denotes $h_L^\star$, satisfies

$$0 < h_L^\star < \frac{\lambda_R - \lambda_L}{-\lambda_L}\left(\frac{\lambda_R h_R - \lambda_L h_L}{\lambda_R - \lambda_L} - \frac{q_R - q_L}{\lambda_R - \lambda_L}\right)$$

$$h_L^\star = h_L \qquad \text{if } w_L \text{ and } w_R \text{ define a steady state}$$

In addition $h_R^\star$ satisfies

$$0 < h_R^\star < \frac{\lambda_R - \lambda_L}{\lambda_R}\left(\frac{\lambda_R h_R - \lambda_L h_L}{\lambda_R - \lambda_L} - \frac{q_R - q_L}{\lambda_R - \lambda_L}\right)$$

$$h_R^\star = h_R \qquad \text{if } w_L \text{ and } w_R \text{ define a steady state}$$

Remark

The steady state property is not satisfied by the other roots

Corollary

The resulting Godunov type scheme is positive and full well-balanced

Discrete entropy inequality

□ General principle (Gallice 03, Chalons et al 10)

Conservation laws with source term $\partial_t w + \partial_x f(w) = S(w)$

Entropy inequalities $\partial_t \eta(w) + \partial_x G(w) \leq \sigma(w)$

Approximate Riemann solver (with constant intermediate states w_ℓ)

consistent with the entropy inqualities if

$$\sum_{k=1}^{\ell} \lambda_\ell(\eta(w_{\ell+1}) - \eta(w_\ell)) \geq G(w_R) - G(w_L) - \Delta x \tilde{\sigma}(\Delta x, \Delta t, w_L, w_R)$$

• Objective: Establish

$$\Delta\eta := \lambda_L\left(\eta(w_L^\star) - \eta(w_L)\right) + \lambda_R\left(\eta(w_R) - \eta(w_R^\star)\right) \leq G(w_R) - G(w_L) - \Delta x \tilde{\sigma}(\Delta x, \Delta t, w_L, w_R)$$

where

$$\eta(w) = h\frac{u^2}{2} + g\frac{h^2}{2} \qquad G(w) = \left(h\frac{u^2}{2} + gh^2\right)u \qquad \lim_{\substack{\Delta x,\, \Delta t \,\to\, 0 \\ w_L,\, w_R \,\to\, w}} \tilde{\sigma} = -ghu\partial_x Z$$

□ Behaviors of the intermediate states

Smoothness of Z imposes

$$h_L^\star = h^{HLL} - \alpha(Z_R - Z_L)\frac{(h^{HLL})^3}{\tilde{q}^2/2 - (h^{HLL})^3} + (Z_R - Z_L)\varepsilon(Z_R - Z_L)$$

$$h_R^\star = h^{HLL} + (1-\alpha)(Z_R - Z_L)\frac{(h^{HLL})^3}{\tilde{q}^2/2 - (h^{HLL})^3} + (Z_R - Z_L)\varepsilon(Z_R - Z_L)$$

$$q^\star = \tilde{q} - \frac{g}{\lambda_R - \lambda_L}\frac{h_L h_R}{\bar{h}}(Z_R - Z_L) \qquad \tilde{q} = q^{HLL} + \frac{g}{\lambda_R - \lambda_L}\frac{\delta h^3}{4\bar{h}}$$

$w^{HLL} = (h^{HLL}, q^{HLL})$ constant intermediate state coming from HLL scheme

To obtain

$$\Delta\eta = (\lambda_R \eta(w_R) - \lambda_L \eta(w_L)) - (\lambda_R - \lambda_L)\eta(w^{HLL}) + g\frac{h_L h_R}{\bar{h} h^{HLL}} q^{HLL}(Z_R - Z_L)$$
$$+ (Z_R - Z_L)\varepsilon(Z_R - Z_L) + \mathcal{O}(\Delta x^3)$$

But we have $\quad \eta(w^{HLL}) \leq \dfrac{\lambda_R \eta(w_R) - \lambda_L \eta(w_L)}{\lambda_R - \lambda_L} - \dfrac{1}{\lambda_R - \lambda_L}(G(w_R) - G(w_L))$

To obtain the required entropy inequality up to

$$(Z_R - Z_L)\varepsilon(Z_R - Z_L) = \Delta x \varepsilon(\Delta x)$$

Thanks for your attention