On the long-time behavior of 2D dissipative Euler equations

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The model

We consider the following system 2D in a bounded and smooth domain Ω with boundary Γ

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \chi\, u + \nabla p &= f \\ \nabla \cdot u &= 0 \\ (u \cdot n)_{|\Gamma} &= 0 \end{aligned} \tag{1}$$

with initial condition $u(0, x) = u_0(x)$

This are called dissipative Euler equations. There is a damping term, not a smoothing one.

The model

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2. It is related with Rayleigh friction in the planetary boundary layer (with space-periodic boundary conditions).
3. The constant χ is the Rayleigh friction coefficient (or the Ekman pumping/dissipation constant) or also the sticky viscosity, when the model is used to study motion in presence of rough boundaries.

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Results on the long-time behavior of the damped Navier-Stokes do not directly pass to the limit “viscosity goes to zero,” hence a completely different treatment is required to study the problem without dissipation.

Early studies are in Barcilon Constantin & Titi (SIMA 1988); Hauk (PhD Thesis, Irvine 1997) Gallavotti (Quaderni CNR 1996).

Long time behavior

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The theory is based on the following energy-type estimates.

A priori estimates

Energy estimate: Testing with u itself one obtains

$$\frac{d}{dt}\|u\|^2 + \chi\|u\|^2 \leq \frac{1}{\chi}\|f\|^2$$

hence the estimate

$$\|u(t)\|^2 \leq \|u(t_0)\|^2 e^{-\chi(t-t_0)} + \frac{\|f\|^2}{\chi}\int_{t_0}^{t} e^{-\chi(t-s)}\, ds, \quad \text{a.e. } t \geq t_0 \geq 0,$$

and consequently a UNIFORM bound for the kinetic energy for all positive times.

A priori estimates

Enstrophy estimate: Taking the 2D curl $\xi := \partial_1 u_2 - \partial_2 u_1$
$\phi :=$ curl f we get

$$\partial_t \xi + \chi \xi + (u \cdot \nabla) \xi = \phi \tag{2}$$

one immediately obtain a uniform bound for ξ

$$\|\xi(t)\|^2 \leq \|\xi(t_0)\|^2 e^{-\chi(t-t_0)} + \frac{\|\phi\|^2}{\chi} \int_{t_0}^{t} e^{-\chi(t-s)}\, ds, \quad \text{a. e. } t \geq t_0 \geq 0,$$

Existence of weak solutions

From the above estimates one obtains directly existence of weak solutions, by adapting Yudovich, (USSR Comp. Math. Math. Phys. 1963) and Bardos (JMAA 1972) theorems, based on vanishing viscosity approximation.

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Seemingly this should be enough to construct an attractor in a standard way by taking the ω-limit closure of an absorbing set.

On the global attractor

This approach does not work, since the map

$$u_0 \mapsto u(t)$$

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The mapping

$$u_0 \mapsto u(t)$$

is well defined, but not continuous in this setting, namely in $W^{1,\infty}$!!

On the global attractor

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The well-established techniques for damped hyperbolic equations, as summarized in Temam (Springer 1997) seem to be not applicable

Weak attractor

This explains why in Il'in and Bessaih and Flandoli it is studied a weak attractor, that is the attractor is considered in the path space and the semi-group is made with the time-shifts

$$u(t) \mapsto u(t+h)$$

Links with 2D NSE

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In order to study the pure hyperbolic equations, one probably needs different tools.

Uniqueness?

In order to prove uniqueness, one needs, essentially $\nabla u \in L^{\infty}$ to estimate the nonlinear term.
The link with the vorticity is given by the stream function

$$-\Delta\Psi = \xi \quad \text{in } \Omega \qquad \Psi_{|\Gamma} = 0$$

and

$$\nabla u = \nabla\nabla^{T}\Psi.$$

The fact that one has a precise representation of the velocity in terms of the vorticity is a fundamental tool in incompressible flows and for the 2D Euler is at the basis of the global existence results.

Velocity-Vorticity

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Further developments base on similar ideas can be found also in A.C. Schaeffer (TAMS 1937) and Kato (ARMA 1967)

Bounded gradients

In order to have bounded ∇u one needs to understand essentially the following problem: what are the hypotheses on $\mathcal{F}$ in order that the solution of

$$-\Delta \mathcal{U} = \mathcal{F} \quad \text{in } \Omega \qquad \mathcal{U}_{|\Gamma} = 0$$

are such that

$$D^2\mathcal{U} \quad \text{are bounded??}$$

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Here, clearly $\mathcal{F} = \xi \in L^\infty$ is not enough. Uniqueness in this setting follows in a rather sharp way from Lip-Log estimates and precise behavior of growth of all L^p-norms, proved by Yudovich!!

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Also $\mathcal{F} = \xi \in C^0$ is not enough, while going to Hölder continuous vorticity has again problems in terms of continuity in such spaces.

There is a (known) very narrow class of functions with this property, Dini-continuous

$$\|f\|_{C_D(\overline{\Omega})} := \|f\|_{L^\infty(\Omega)} + \int_0^1 \omega(f,\sigma)\,\frac{d\sigma}{\sigma} < +\infty,$$

where $\omega(f,\sigma)$ is the *modulus of continuity* of f, defined as follows

$$\omega(f,\sigma) = \sup_{0<|x-y|<\sigma, x,y\in\Omega} |f(x) - f(y)|.$$

Introduced by Ulisse Dini for trigonometric series (Italian notes Nistri, 1880), while its application to PDEs appeared first in (Acta Math. 1902), which is taken from a letter of Dini to Mittag-Leffler.

(Dini)-Continuous vorticity

These space are of interest for the following reason. If $\xi \in L^\infty$, then streamlines are well-defined

$$\frac{d}{ds} U(s,t,x) = u(s, U(s,t,x)) \qquad U(t,t,x) = x, \tag{3}$$

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$$\xi(t,x) = \xi_0(U(0,t,x)) + \int_0^t \phi(U(s,t,x))\, ds.$$

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In the case of dissipative Euler one has to make a change of variables in time, but representation formulas are essentially the same.

(Dini)-Continuous vorticity

The main point is that

1. if $\xi_0, \phi \in C(\overline{\Omega})$ then $\xi \in C([0, T]; C(\overline{\Omega}))$;
2. if $\xi_0, \phi \in C_D(\overline{\Omega})$ then $\xi \in C([0, T]; C_D(\overline{\Omega}))$.

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Again this is not new, since it has been first observed by Beirão da Veiga (JDE 1984) and then it has been independently used by H. Koch (Math. Ann. 2002).

(Dini)-Continuous vorticity

In the first paper it is proved that one can construct unique strong solutions, with continuous data dependence in the space

$$E(\Omega) = \left\{ u : \ \Omega \to \mathbf{R}^2 : \ \nabla \cdot u = 0, \ (u \cdot n)_{|\Gamma} = 0, \ \operatorname{curl} u \in C(\overline{\Omega}) \right\}$$

and also construct unique classical solutions in the space

$$F(\Omega) = \left\{ u : \ \Omega \to \mathbf{R}^2 : \ \nabla \cdot u = 0, \ (u \cdot n)_{|\Gamma} = 0, \ \operatorname{curl} u \in C_D(\overline{\Omega}) \right\}$$

The proof is based on representation formulas with suitable applications of Ascoli-Arzelà compactness theorem.

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This can be then balanced by a damping term which implies exponential decay.

This seems among the sharpest norms having this property. Maybe this can happen also for some Besov spaces as in Vishik (ARMA 1998), but for the moment I am not considering this.

Attractor in the phase space

The first results is the following:

Theorem: (B. 2012) Let be given f such that $\phi = \operatorname{curl} f \in C_D(\overline{\Omega})$. Then, there exists $\chi_0 = \chi_0(\|f\|_{C_D}) > 0$ such that for all $\chi > \chi_0$ there exists a global attractor $\mathcal{A} \subset E(\Omega)$ for the dissipative 2D Euler equations (1).

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The existence is based on the usual representation

$$\mathcal{A} := \bigcup_{t\geq 0} \overline{\bigcap_{s\geq t} S(t)\,\mathcal{B}}^X$$

we need an Y-bounded absorbing set $\mathcal{B}$, a couple of Banach spaces such that Y is compactly embedded in X and that the semigroup is continuous for all $t \geq 0$ as a mapping from Y into itself.

Attractor in the phase space

Two main points are the following lemma
Lemma: Let be given $f \in C_D(\overline{\Omega})$, then f is uniformly continuous and

$$|f(x) - f(y)| \leq \frac{C_D}{\log|x-y|}, \quad \forall\, x, y \in \overline{\Omega}, \text{ s. t. } |x-y| \leq e.$$

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The fact that $\mathcal{A}$ is attracting need some work based on the continuous data dependence

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The bound for the Hausdorff dimension of the attractor are probably not optimal.

Almost periodic solutions

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In a work still in progress we are considering a slightly different approach hence to find almost periodic solutions.

Almost periodic solutions

One main difficulty is then showing a sort of contraction principle making possible to use well-established techniques as those in Foias, (Rend. Sem. Mat. Padova 1962), Foias & Prodi (Rend. Sem. Mat. Padova 1967) Amerio & Prouse, (Van Nostrand 1971).

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Again the approach with $\operatorname{curl} f \in S^2(\mathbf{R}; H^1(\Omega))$ or $\operatorname{curl} f \in S^2(\mathbf{R}; W^{1,\infty}(\Omega))$ seems not working since we do not have enough (uniform) control on ∇u, hence control on the convective term.

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We recall that $f : \mathbf{R} \to X$ is Stepanov p-almost periodic (denoted by $f \in S^p(\mathbf{R}; X)$) if $f \in L^p_{\text{loc}}(\mathbf{R}; X)$ and if the set of its translates is relatively compact in the $L^p_{\text{uloc}}(\mathbf{R}; X)$ topology defined by the norm

$$\|f\|_{L^p_{\text{uloc}}(\mathbf{R},X)} := \sup_{t\in\mathbf{R}} \left[\int_t^{t+1} \|f(s)\|_X^p \, ds\right]^{1/p}.$$

Almost periodic solutions

Theorem: (Joint work with L. Bisconti, in preparation) Let be given f such that curl f $\in C(\mathbf{R}; C_D(\overline{\Omega}))$ and curl $f \in S^2(\mathbf{R}; C(\overline{\Omega}))$. Then there exists $\chi_0(f)$ such that for all $\chi > \chi_0$ solutions of (1) are almost periodic such that $u \in S^2(\mathbf{R}; L^2(\Omega))$.

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Preliminary result probably improvable; the same problem maybe can be handled in the smaller space of Bohr *UAP* classical almost periodic functions.

Thank you for your attention!