An adaptive finite-volume method for a model of two-phase pedestrian flow

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Outline

Systems with elliptic region

We examine initial value problems for first-order systems

$$\begin{aligned} \partial_t\phi_1 + \partial_x f_1(\phi_1,\phi_2) &= 0, \\ \partial_t\phi_2 + \partial_x f_2(\phi_1,\phi_2) &= 0, \end{aligned} \tag{1.1}$$

by applying multi-resolution schemes. We recall that the system (1.1) is called *hyperbolic* at a point (ϕ_1,ϕ_2) if the Jacobian $\mathcal{J}_{\boldsymbol{f}}$ of the flux vector $\boldsymbol{f} = (f_1, f_2)^{\mathrm{T}}$,

$$\mathcal{J}_{\boldsymbol{f}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial \phi_1} & \dfrac{\partial f_1}{\partial \phi_2} \\ \dfrac{\partial f_2}{\partial \phi_1} & \dfrac{\partial f_2}{\partial \phi_2} \end{bmatrix} =: \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

has real eigenvalues, i.e., if the discriminant

$$\Delta(\phi_1,\phi_2) := \big((J_{11} - J_{22})^2 + 4J_{12}J_{21}\big)(\phi_1,\phi_2) \tag{1.2}$$

is positive, and *strictly hyperbolic* if these eigenvalues are moreover distinct. If $\mathcal{J}_{\boldsymbol{f}}(\phi_1,\phi_2)$ has a pair of complex conjugate eigenvalues (i.e., $\Delta(\phi_1,\phi_2) < 0$), then (1.1) is called *elliptic* at that point. The set of all elliptic points is called elliptic region.

Sedimentation of polydisperse suspensions

Vector of unknows: solids concentrations

$$\Phi = (\phi_1, \phi_2, \dots, \phi_N) \tag{1.3}$$

Cumulative solids fraction $\phi := \phi_1 + \cdots + \phi_N$,
Hindered settling factor $V(0) = 1$, $V'(\phi) \leq 0$, $V(1) = 0$, e.g.

$$V(\phi) = \begin{cases} (1-\phi)^{n-2} & \text{if } \Phi \in \mathcal{D}_{\phi_{\max}}, \\ 0 & \text{otherwise}, \end{cases} \qquad n > 2. \tag{1.4}$$

Phase velocity of particle species i

$$v_i(\Phi) = \mu V(\phi) \left[d_i^2 (\varrho_i - \varrho(\Phi)) - \sum_{m=1}^{N} d_m^2 \phi_m (\varrho_m - \varrho(\Phi)) \right], \quad i = 1, \dots, N. \tag{1.5}$$

One-dimensional batch settling of a suspension

$$\begin{aligned} &\partial_t \Phi + \partial_x \boldsymbol{f}(\Phi) = 0, \quad x \in (0, L), \quad t > 0, \\ &\boldsymbol{f}(\Phi) = \bigl(f_1(\Phi), \dots, f_N(\Phi)\bigr)^{\mathrm{T}}, \quad f_i(\Phi) = \phi_i v_i(\Phi), \quad i = 1, \dots, N, \end{aligned} \tag{1.6}$$

Initial and zero-flux boundary conditions

$$\Phi(x, 0) = \Phi_0(x), \quad x \in [0, L], \qquad \boldsymbol{f}|_{x=0} = \boldsymbol{f}|_{x=L} = 0. \tag{1.7}$$

Model of bidisperse suspension

For case $N = 2$, it is convenient to introduce the parameter $\gamma := \bar{\varrho}_2/\bar{\varrho}_1 = (\varrho_2 - \varrho_\mathrm{r})/(\varrho_1 - \varrho_\mathrm{r})$, and we set $\delta := \delta_2 = d_1^2/d_2^2$.

$$f_1(\phi_1, \phi_2) = \phi_1 V(\phi_1 + \phi_2)\Big((1 - \phi_1)(1 - \phi_1 - \gamma\phi_2) - \delta\phi_2\big((1 - \phi_2)\gamma - \phi_1\big)\Big),$$
$$f_2(\phi_1, \phi_2) = \phi_2 V(\phi_1 + \phi_2)\Big(\delta(1 - \phi_2)\big((1 - \phi_2)\gamma - \phi_1\big) - \phi_1(1 - \phi_1 - \gamma\phi_2)\Big).$$

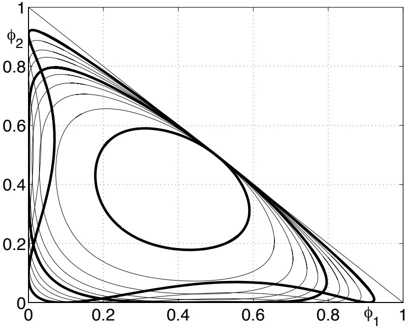
with $\delta \in (0, 1]$ and hindered settling factor $V(\phi) \geq 0$, $V'(\phi) < 0$ on $[0, \phi_{\max})$

Symmetric case $\delta = 1, \gamma = -1$ (B., Bürger, Kozakevicius 2009)

For the symmetric case, where $\delta = 1, \gamma = -1$, there are tangents on the axes $\phi_1 = 0$ and $\phi_2 = 0$ in

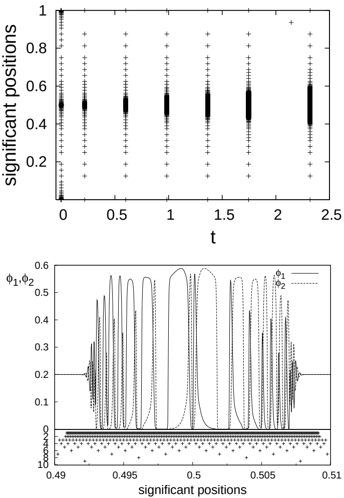
$$\phi_1 = \phi_2 = \frac{1}{2} \pm \frac{\sqrt{n^2 - 8n}}{2n}. \tag{1.8}$$

Depending on n, on the axes we have



$$\begin{cases} n < 8 & \text{no tangent} \\ n = 8 & \text{one tangent} \\ n > 8 & \text{two tangents} \end{cases}$$

Case $n = 8$


significant positions
t
ϕ_2
ϕ_1xϕ_2
initial condition
ϕ_1
ϕ_1,ϕ_2
ϕ_1
ϕ_2
significant positions
ϕ_1,ϕ_2
significant positions

Outline

Two dimensional pedestrian model

• Simulation models for *vehicular traffic* have a more or less one-dimensional character as cars move in lanes on streets allowing cross-directional flow only at distinct crossing points.
• A special property of the Bick-Newell model

$$\begin{aligned} u_t + (u(1-u-\beta v))_x &= 0, \\ v_t + (-v(1-\beta u - v))_x &= 0, \end{aligned} \tag{2.1}$$

is that its phase space contains an elliptic region.
• Pedestrian flow allows a genuine spatial structure: pedestrian movement can be directed principally to any direction and it is strongly influenced by human behavior. Therefore, simulation models for pedestrian traffic are twodimensional, having the form

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x + \boldsymbol{g}(\boldsymbol{u})_y = 0,$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are the fluxes in x and y directions, respectively.

Two-dimensional model

The fluxes $\boldsymbol{f}, \boldsymbol{g}$ in

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x + \boldsymbol{g}(\boldsymbol{u})_y = 0,$$

are composed by a total flux **h** and a direction contribution, which distributes the total local flow of a species as

$$\begin{aligned} f_1(u,v;x,y) &= h_1(u,v)d_1^{\mathrm{x}}(x,y), \quad & g_1(u,v;x,y) &= h_1(u,v)d_1^{\mathrm{y}}(x,y), \\ f_2(u,v;x,y) &= h_2(u,v)d_2^{\mathrm{x}}(x,y), \quad & g_2(u,v;x,y) &= h_2(u,v)d_2^{\mathrm{y}}(x,y). \end{aligned} \tag{2.2}$$

The directions can be formally put into a direction matrix

$$D = \begin{pmatrix} d_1^{\mathrm{x}}(x,y) & d_1^{\mathrm{y}}(x,y) \\ d_2^{\mathrm{x}}(x,y) & d_2^{\mathrm{y}}(x,y) \end{pmatrix} = \begin{pmatrix} \boldsymbol{d}_1(\boldsymbol{x}) \\ \hline \boldsymbol{d}_2(\boldsymbol{x}) \end{pmatrix} = \left(\boldsymbol{d}^{\mathrm{x}}(\boldsymbol{x}) \quad \Big| \quad \boldsymbol{d}^{\mathrm{y}}(\boldsymbol{x}) \right),$$

where the subscripts (1 or 2) denote the species and the superscripts (x or y) denote the direction component. E.g., $d_1^y(x,y)$ denotes that fraction of the flux of species 1 that flows in the y direction.

Model of Hughes

The model of Hughes specifies the directions

$$\mathbf{d}_1(\boldsymbol{x}) = \left(d_1^x(x,y) \quad d_1^y(x,y)\right), \qquad \mathbf{d}_2(\boldsymbol{x}) = \left(d_2^x(x,y) \quad d_2^y(x,y)\right),$$

employing the potentials ϕ, ψ associated with phases 1 and 2, respectively, as

$$d_1^x(x,y) = \frac{\phi_x}{\|\nabla\phi\|_2}, \quad d_1^y(x,y) = \frac{\phi_y}{\|\nabla\phi\|_2},$$
$$d_2^x(x,y) = \frac{\psi_x}{\|\nabla\psi\|_2}, \quad d_2^y(x,y) = \frac{\psi_y}{\|\nabla\psi\|_2},$$

where the gradient norms are

$$\|\nabla\phi\|_2 = \sqrt{\phi_x^2 + \phi_y^2}, \qquad \|\nabla\psi\|_2 = \sqrt{\psi_x^2 + \psi_y^2}$$

Outline

Finite volume formulation

Let $\mathcal{T}^0 \subset \cdots \mathcal{T}^\ell \cdots \subset \mathcal{T}^H$ be a family of nested admissible rectangular meshes. Denote the cell averages of u, v on $K^\ell \in \mathcal{T}^\ell$ at time $t = t^n$ by

$$u^n_{K^\ell} := \frac{1}{|K^\ell|} \int_{K^\ell} u(x, t^n)\, dx, \quad v^n_{K^\ell} := \frac{1}{|K^\ell|} \int_{K^\ell} v(x, t^n)\, dx.$$

The resulting finite-volume scheme approximation defined on the resolution level ℓ, assumes values $u^n_{K^\ell}$ and $v^n_{K^\ell}$ for all $K^\ell \in \mathcal{T}^\ell$ at time $t = t^n$ and determines $u^{n+1}_{K^\ell}$ and $v^{n+1}_{K^\ell}$ for all $K^\ell \in \mathcal{T}^\ell$ at time $t = t^{n+1} = t^n + \Delta t$ by the marching formula and using a standard finite-volume approach, the system is discretized as

$$|K^\ell| \frac{\boldsymbol{u}^{n+1}_{K^\ell} - \boldsymbol{u}^n_{K^\ell}}{\Delta t} - \sum_{\sigma \in \mathcal{E}_{\text{int}}(K^\ell)} \frac{|\sigma(K^\ell, L^\ell)|}{d(K^\ell, L^\ell)} \Big(\boldsymbol{F}(\boldsymbol{u}^n_{K^\ell}, \boldsymbol{u}_{L^\ell}; \bar{\boldsymbol{x}}(K^\ell, L^\ell); \boldsymbol{n}(K^\ell, L^\ell)) \cdot \boldsymbol{n}(K^\ell, L^\ell) + \frac{\boldsymbol{b}(\boldsymbol{u}^n_{L^\ell}) + \boldsymbol{b}(\boldsymbol{u}^n_{K^\ell})}{2} (\boldsymbol{u}^n_{L^\ell} - \boldsymbol{u}^n_{K^\ell}) \Big), \tag{3.1}$$

where $\boldsymbol{n}(K^\ell, L^\ell) = \big(n_1(K^\ell, L^\ell), n_2(K^\ell, L^\ell)\big)^{\mathrm{T}}$ is the outer normal vector of cell K^ℓ pointing towards L^ℓ. such that

Numerical flux

As numerical flux, we choose the *local* Lax-Friedrichs flux, which is defined as

$$\boldsymbol{f}\left(\boldsymbol{u}^n_{K^\ell}, \boldsymbol{u}^n_{L^\ell}; \bar{\boldsymbol{x}}\right) = \frac{\boldsymbol{f}\left(\boldsymbol{u}^n_{K^\ell}; \bar{\boldsymbol{x}}\right) + \boldsymbol{f}\left(\boldsymbol{u}^n_{L^\ell}; \bar{\boldsymbol{x}}\right)}{2} - \frac{a^{\mathrm{x}}(K^\ell, L^\ell)}{2}\left(\boldsymbol{u}^n_{K^\ell} - \boldsymbol{u}^n_{L^\ell}\right),$$
$$\boldsymbol{g}\left(\boldsymbol{u}^n_{K^\ell}, \boldsymbol{u}^n_{L^\ell}; \bar{\boldsymbol{x}}\right) = \frac{\boldsymbol{g}\left(\boldsymbol{u}^n_{K^\ell}; \bar{\boldsymbol{x}}\right) + \boldsymbol{g}\left(\boldsymbol{u}^n_{L^\ell}; \bar{\boldsymbol{x}}\right)}{2} - \frac{a^{\mathrm{y}}(K^\ell, L^\ell)}{2}\left(\boldsymbol{u}^n_{K^\ell} - \boldsymbol{u}^n_{L^\ell}\right),$$

where we used the abbrevation $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(K^\ell, L^\ell)$. The coefficients $a^{\mathrm{x}}, a^{\mathrm{y}}$ are determined as the maximum spectral radius on the cell interface

$$a^{\mathrm{x}}(K^\ell, L^\ell) = \max\left(\varrho(f'(\boldsymbol{u}^n_{K^\ell})), \varrho(f'(\boldsymbol{u}^n_{L^\ell}))\right),$$
$$a^{\mathrm{y}}(K^\ell, L^\ell) = \max\left(\varrho(g'(\boldsymbol{u}^n_{K^\ell})), \varrho(g'(\boldsymbol{u}^n_{L^\ell}))\right),$$

or upper estimates of that radius; here, the flux Jacobians $f'(\boldsymbol{u}_{K^\ell})$, $g'(\boldsymbol{u}_{K^\ell})$ are evaluated for the solution value $\boldsymbol{u}_{K^\ell}$ and $\varrho(f'(\boldsymbol{u}_{K^\ell}))$, $\varrho(g'(\boldsymbol{u}_{K^\ell}))$ are the corresponding spectral radii. The point $\bar{\boldsymbol{x}}(K^\ell, L^\ell)$ is the position of the interface between the cells K^ℓ and L^ℓ.

Multiresolution setting

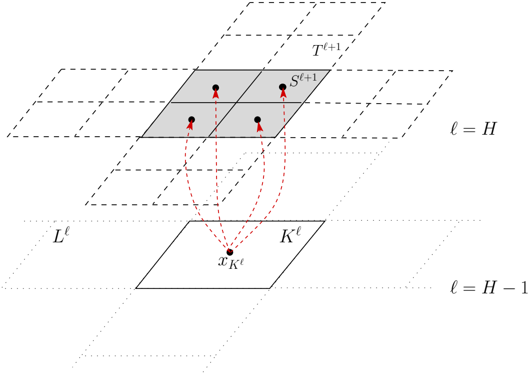


Figure : Sketch of a graded tree structure. Here K^{ℓ} is a parent node on level $\ell = H - 1$, its children nodes (including $S^{\ell+1}$) belong to $\mathcal{L}(\Lambda)$; L^{ℓ} is a virtual node and $T^{\ell+1}$ is a virtual leaf.

Outline

Numerical examples (B., Ruiz-Baier, Schwandt, Tory (2011))

In the numerical examples, the convection coefficients are normalized to $a_1 = a_2 = 1$. The diffusion matrix is assumed to be constant taking the form

$$\mathbf{b}(\boldsymbol{u}) = \begin{pmatrix} \varepsilon & \delta \\ \delta & \varepsilon \end{pmatrix} \tag{4.1}$$

with self-diffusion rate ε and cross-diffusion rate δ. The cross-diffusion rate δ is assumed to vanish in all examples, except the last one. If not otherwise specified, in the numerical examples we set the velocity function to $V(u, v) = 1 - u - v$, the domain to $\Omega = [-1, 1]^2$, and on the boundary we impose absorbing boundary conditions.

Example 1

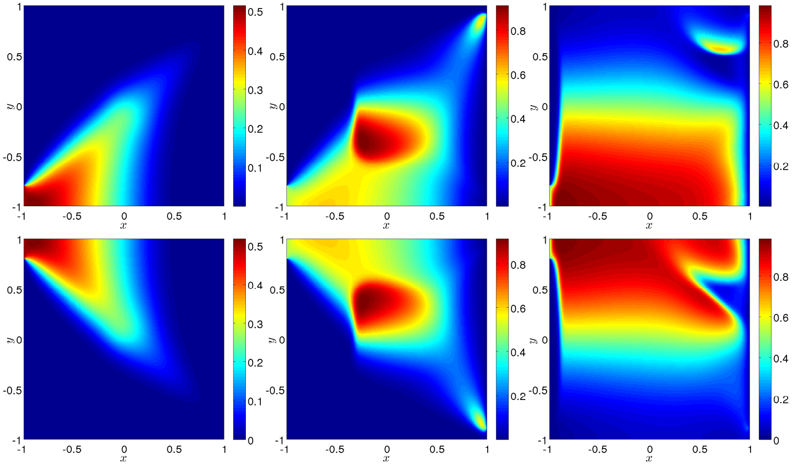


Figure : Example 1. Species' densities u (top) and v (bottom) at times $t = 2.0$, $t = 4.0$, and $t = 18.0$.

The initial data are set to $(0,0)$ all over the domain $\Omega = [-1,1] \times [-1,1]$. The two populations enter the domain through doors of width w, which for the test case is specified to be $w = 0.1$. The two inlets are positioned at $\{-1\} \times [-1, -1+w]$ and $\{-1\} \times [1-w, 1]$. The entrance flow is specified by Neumann boundary conditions as

$$\begin{aligned} f_1(\boldsymbol{u}; \boldsymbol{x}) &= f_{\mathrm{SW}} \quad \text{for} \quad \boldsymbol{x} \in \{-1\} \times [-1, -1+w], \\ f_2(\boldsymbol{u}; \boldsymbol{x}) &= f_{\mathrm{NW}} \quad \text{for} \quad \boldsymbol{x} \in \{-1\} \times [1-w, 1], \end{aligned}$$

with $f_{\mathrm{NW}} = f_{\mathrm{SW}} = 0.5$. The exit fluxes are defined at the outlets $\{1\} \times [1-w, 1]$ and $\{1\} \times [-1, -1+w]$ as

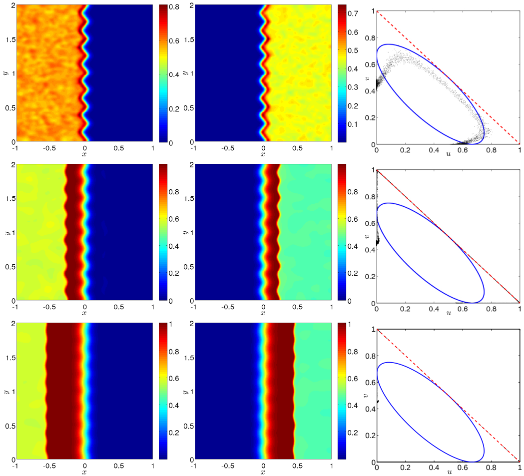
$$\begin{aligned} f_1(\boldsymbol{u}; \boldsymbol{x}) = up,\ f_2(\boldsymbol{u}; \boldsymbol{x}) &= 0 \quad \text{for} \quad \boldsymbol{x} \in \{1\} \times [-1, -1+w], \\ f_2(\boldsymbol{u}; \boldsymbol{x}) = vq,\ f_1(\boldsymbol{u}; \boldsymbol{x}) &= 0, \quad \text{for} \quad \boldsymbol{x} \in \{1\} \times [1-w, 1], \end{aligned}$$

with $p = q = 1$. Moreover $a_1 = a_2 = 1$, $\varepsilon = 0.01$, $\delta = 0$.
The crowd dynamics are oriented towards exit targets which are located in the centers of the respective exit doors and are located at $(x_1, y_1) = (1, 1 - w/2)$, $(x_2, y_2) = (1, -1 + w/2)$. The directions towards the targets (exit points) (x_1, y_1), (x_2, y_2) for species 1 and 2, respectively, are given by

$$\boldsymbol{d}_i(\boldsymbol{x}) = \frac{\tilde{\boldsymbol{d}}_i(\boldsymbol{x})}{\|\tilde{\boldsymbol{d}}_i(\boldsymbol{x})\|_2}, \quad \tilde{\boldsymbol{d}}_i(\boldsymbol{x}) = \begin{pmatrix} x - x_i & y - y_i \end{pmatrix}, \qquad i = 1, 2.$$

Example 2

Species' densities u (left), v (right), and corresponding phase diagram where the elliptic region is depicted, for times $t = 0.1$ (top), $t = 0.5$ (center) and $t = 1.5$ (bottom).



Example 2

Example 2 gives an account of the Battle of Agincourt, 1415, a relatively well documented medieval war. The flow is assumed to be in opposite directions

$$\boldsymbol{d}_1(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \boldsymbol{d}_2(\boldsymbol{x}) = \begin{pmatrix} -1 & 0 \end{pmatrix}.$$

For this countercurrent flow, the governing equations are specified as

$$u_t + \big(u(1-u-v)\big)_x = \varepsilon \Delta u, \qquad v_t - \big(v(1-u-v)\big)_x = \varepsilon \Delta v.$$

The boundary conditions are absorbing. Introducing the zig-zag curve

$$z(y) = 4A\left(\left|Fy - \lfloor Fy \rfloor - \frac{1}{2}\right| - \frac{1}{4}\right),$$

where $\lfloor \cdot \rfloor$ gives the next lower integer, the domain $\Omega = \Omega_u \cup \Omega_v$ is splitted as

$$\Omega_u = \{-1 \le x \le z(y),\ 0 \le y \le 2\}, \quad \Omega_v = \{z(y) < x \le 1,\ 0 \le y \le 2\},$$

The initial conditions are constant in each subdomain, with small perturbations.

Example 3

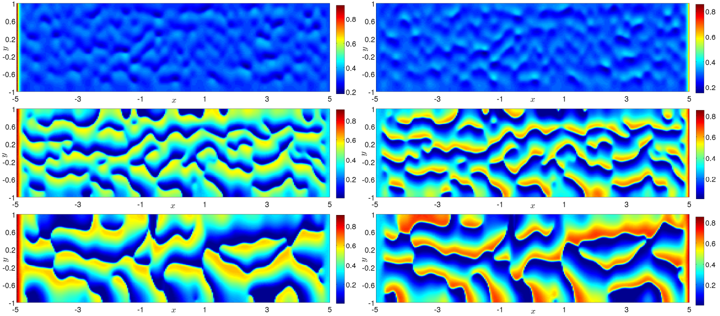


Figure : Example 3. Species' densities u, v at times $t = 2$, $t = 4$, $t = 8$.

Example 3

In Example 3, the domain is specified to be a long channel having the domain $\Omega = [-5, 5] \times [-1, 1]$. The parameters are set to $a_1 = a_2 = 1$, $\varepsilon = 2.5 \times 10^{-3}$, $\delta = 0$. Initially, the domain is assumed to be empty with $u(\boldsymbol{x}, t = 0) = v(\boldsymbol{x}, t = 0) = 0$ for all $\boldsymbol{x} \in \Omega$. Both populations move in opposite directions

$$\boldsymbol{d}_1(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \boldsymbol{d}_2(\boldsymbol{x}) = \begin{pmatrix} -1 & 0 \end{pmatrix},$$

such that they are expected to meet somewhere in the middle. The two populations access the domain at two opposite edges; at the left and right edges of the domain, an inflow is imposed:

$$\boldsymbol{f}(\boldsymbol{u}, \boldsymbol{x}) - \varepsilon \partial_x \boldsymbol{u} = \begin{pmatrix} f_{\mathrm{W}}(t) & 0 \end{pmatrix}^{\mathrm{T}} \quad \text{at} \quad \boldsymbol{x} \in \{-5\} \times [-1, 1],$$

$$\boldsymbol{f}(\boldsymbol{u}, \boldsymbol{x}) - \varepsilon \partial_x \boldsymbol{u} = \begin{pmatrix} 0 & f_{\mathrm{E}}(t) \end{pmatrix}^{\mathrm{T}} \quad \text{at} \quad \boldsymbol{x} \in \{5\} \times [-1, 1].$$

The boundaries of the longer edges are assigned with zero flux, i.e.,

$$\boldsymbol{g}(\boldsymbol{u}; \boldsymbol{x}) - \varepsilon \partial_y \boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{x} \in [-5, 5] \times \{-1, 1\}.$$

The populations finally leave the domain with a constant rate at the respective target side

$$\boldsymbol{f}(\boldsymbol{u}, \boldsymbol{x}) - \varepsilon \partial_x \boldsymbol{u} = -\boldsymbol{u} \quad \text{at} \quad \boldsymbol{x} \in \{-5, 5\} \times [-1, 1].$$

Example 4

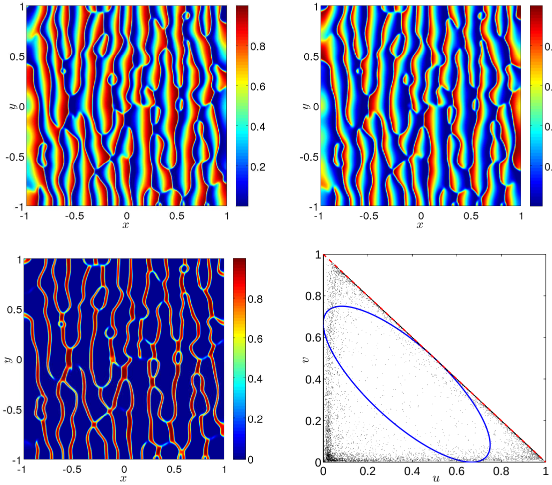


Figure : Example 4. Species' densities u, v (top), void fraction $1 - u - v$ (bottom-left), and corresponding phase diagram where the elliptic region is depicted (bottom-right). The simulated time is $t = 2$.

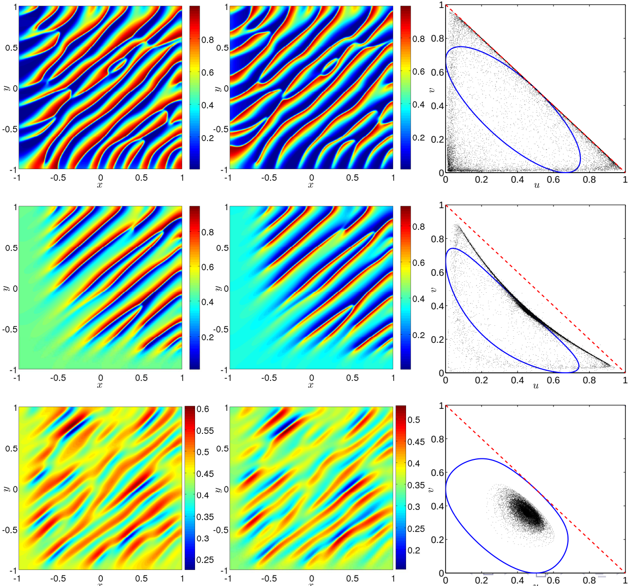
Example 4

In Example 4, countercurrent flow is modelled, where two groups of pedestrians move in the opposite directions $\boldsymbol{d}_1(\boldsymbol{x}) = (0, 1)$ and $\boldsymbol{d}_2(\boldsymbol{x}) = (0, -1)$. Inside the domain $\Omega = [-1, 1]^2$, the initial data are set to be randomly perturbed around a state $\boldsymbol{u}_0 = (0.4, 0.35)^{\mathrm{T}}$ which is located inside the elliptic region. More specifically, initial conditions are set to

$$u(\boldsymbol{x}, t = 0) = u_0 + \eta_u(\boldsymbol{x}), \quad v(\boldsymbol{x}, t = 0) = v_0 + \eta_v(\boldsymbol{x}), \text{ for } \quad \boldsymbol{x} \in \Omega = [-1, 1]^2, \tag{4.2}$$

where η_u, η_v are uniformly distributed random noise with variations of 10% and 1.5% for u and v, respectively. The boundary conditions are set to be absorbing. Here the diffusion matrix has the values $\varepsilon = 1.5 \times 10^{-3}$, whereas $\delta = 0$. For the multiresolution setting, $L = 10$ resolution levels are used with a reference tolerance of $\varepsilon_{\mathrm{ref}} = 1.25 \times 10^{-2}$.

Example 5

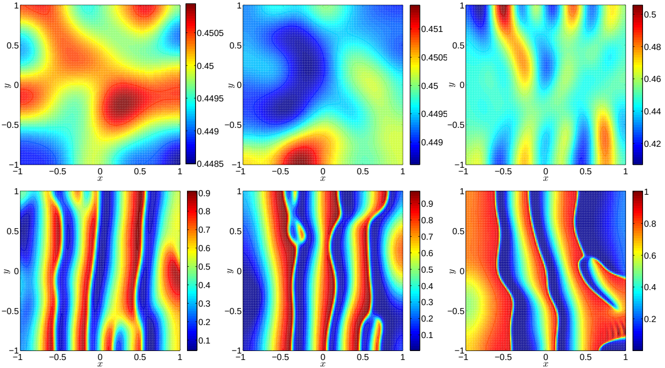


Example 5

In Example 5, a crossing with perpendicular flow directions $\boldsymbol{d}_1(\boldsymbol{x}) = (1, 0)$, $\boldsymbol{d}_2(\boldsymbol{x}) = (0, 1)$ is considered in the domain $\Omega = [-1, 1]^2$. Self-diffusion and cross-diffusion are set to $\varepsilon = 1 \times 10^{-3}$, $\delta = 0$, respectively. As in Example 4, there are absorbing boundary conditions and homogeneous initial data (4.2) with $\boldsymbol{u}_0 = (0.4, 0.35)^{\mathrm{T}}$ are taken inside the elliptic region. Moreover, we consider different velocity functions which are intended to describe real and hypothesized forces during interactions between pedestrians. The velocity function $V(u, v) = 1 - u - v$ assumes a slow-down that is proportional to $u + v$. This choice falls in the more general class of velocity functions that have the desirable property that $V(u, v)$ is convex and satisfies

$$V(0,0) = 1, \quad V(1,0) = 0, \quad V(0,1) = 0. \tag{4.3}$$

Example 6



Densities for species u at time $t = 2.0$. Here, $\varepsilon = 0.01$, and, from top-left, $\delta \in \{0, 0.01, 1, 2.5, 5, 10\}$

Example 7

Behavior of the numerical solution depending on the diffusion and cross-diffusion parameters ε and δ.

ε	δ	Unstable	Stable patterns	Steep patterns
0	0, 0.01, 1, 2.5, 5, 10	•		
0.001	0, 1		•	
0.001	1.5, 2.5			•
0.001	5, 10	•		
0.01	0, 0.01, 0.1		•	
0.01	1, 2.5, 5, 10			•
0.1	0, 0.01, 0.1		•	
0.1	1, 2.5, 5, 10	•		