On Whitham's modulated equations for the Euler–Korteweg system

S. Benzoni-Gavage P. Noble L.M. Rodrigues

Institut Camille Jordan
Université Claude Bernard Lyon 1

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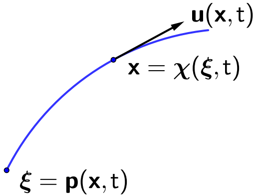
Outline

General model for isothermal capillary fluids

Euler–Lagrange equations for Lagrangian

$$\frac{1}{2}\rho\|\mathbf{u}\|^2 - \mathscr{E}(\rho,\nabla\rho) - \rho\partial_t\varphi - \rho\,\mathbf{u}\cdot\nabla\varphi$$

in coordinates $(\rho,\varphi,\mathbf{p})$



$$\Rightarrow \quad \left\{ \begin{array}{l} \partial_t\rho + \mathrm{div}\,(\rho\mathbf{u}) = 0\,, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(\mathrm{E}_\rho\mathscr{E}) = 0\,, \end{array} \right.$$

ρ = density, $\mathbf{u}$ = velocity, $\mathscr{E}$ = free energy density,
$\mathrm{E}_\rho\mathscr{E}$ = variational derivative of $\mathscr{E}$.

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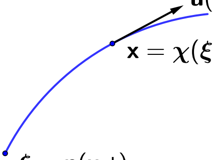

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$\rho =$ density, $\mathbf{u} =$ velocity, $\mathscr{E} =$ free energy density,
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- **Standard compressible fluids:** $\mathscr{E}=\mathscr{E}(\rho)$, $\mathsf{E}_\rho\mathscr{E}=\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}\rho}$.
  - ➡ 'compressible' Euler equations.

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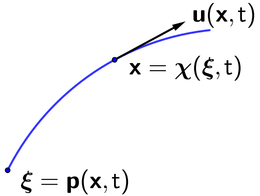

$$\Rightarrow \quad \begin{cases} \partial_t\rho + \mathrm{div}\,(\rho\mathbf{u}) = 0\,, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(\mathrm{E}_\rho\mathscr{E}) = 0\,, \end{cases}$$

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  - ➥ 'compressible' Euler equations.
- **Capillary fluids:** $\mathscr{E} = \mathscr{E}(\rho,\nabla\rho)$, $\mathrm{E}_\rho\mathscr{E} := \dfrac{\partial\mathscr{E}}{\partial\rho} - \sum_{j=1}^{d}\mathrm{D}_{x_j}\left(\dfrac{\partial\mathscr{E}}{\partial\rho_{x_j}}\right)$.

Special cases

Korteweg capillarity theory (after Rayleigh, van der Waals,...), see [Rowlinson & Widom'82]

$$\mathscr{E}(\rho, \nabla\rho) = F(\rho) + \frac{1}{2}\mathscr{K}(\rho)\,\|\nabla\rho\|^2\,.$$

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Quantum fluids (Schrödinger, Madelung, Gross–Pitaevskii)

$$\partial_t\rho + \mathrm{div}\,(\rho\mathbf{u}) = 0\,,\quad \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\left(F'(\rho) - \frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\right) = 0\,.$$

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Vortex-filaments (Levi-Civita & Da Rios, Hasimoto), see [Arnold & Khesin'98]

$$\partial_t\rho + \partial_x(\rho u) = 0\,, \quad \partial_t u + u\partial_x u = \partial_x\left(\frac{\rho}{4} + \frac{\partial_x^2\sqrt{\rho}}{2\sqrt{\rho}}\right)\,.$$

➡ $\rho\,\mathscr{K}(\rho) \equiv \frac{1}{4}\,,\ F(\rho) = -\frac{1}{8}\rho^2\,,\ d = 1.$

Two formulations of 1D model

Eulerian coordinates

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0\,, \\ \partial_t u + u\partial_x u + \partial_x(\mathsf{E}_\rho \mathscr{E}) = 0\,, \end{cases}$$

$\rho =$ density, $u =$ velocity,
$\mathscr{E} = \mathscr{E}(\rho, \rho_x)$ energy density,

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Mass Lagrangian coordinates

$$\begin{cases} d_t v \;=\; \partial_y u\,, \\ d_t u \;=\; \partial_y(\mathrm{E}_v e)\,, \end{cases}$$

$v =$ specific volume, $u =$ velocity,
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$$\partial_x(\mathrm{E}_\rho \mathscr{E}) \ = \ -\,\partial_y(\mathrm{E}_v e)$$

More special cases

Korteweg / Cahn–Hilliard energy again

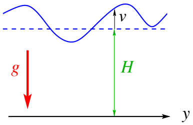
$$\mathscr{E}(\rho,\rho_x) = F(\rho) + \frac{1}{2}\mathscr{K}(\rho)\,\rho_x^2 \quad \Leftrightarrow \quad e(v,v_y) = f(v) + \frac{1}{2}\,\kappa(v)\,v_y^2\,,$$

$$\text{with} \quad F(\rho) = \rho f(v)\,, \quad \kappa(v) := \rho^5\,\mathscr{K}(\rho)\,.$$

Water waves [Boussinesq'72], [Bona & Sachs'88]:

$$\partial_t v = \partial_y u\,,\ \partial_t u - gH\,\partial_y\left(v + \tfrac{3}{2H}\,v^2\right) = \tfrac{1}{3}\,gH^3\,\partial_y^3 v\,,$$

➡ $\kappa = -\frac{1}{3}\,gH^3\,,\ f(v) = \frac{1}{2}\,gH\,v^2\left(1 + \frac{v}{H}\right).$



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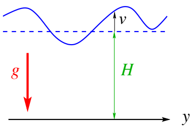
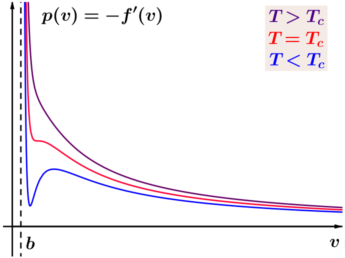


van der Waals fluids



Travelling wave profiles

Eulerian coordinates

$(\rho, u) = (R, U)(x - \sigma t)$ solution of

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) \;=\; 0\,, \\ \partial_t u + u\partial_x u \;+\; \partial_x(\mathrm{E}_\rho \mathscr{E}) \;=\; 0\,, \end{cases}$$

iff

$$\begin{cases} \partial_\xi(R(U - \sigma)) \;=\; 0\,, \\ (U - \sigma)\partial_\xi U \;+\; \partial_\xi(\mathrm{E}_\rho \mathscr{E}) \;=\; 0\,. \end{cases}$$

Lagrangian coordinates

$(v, u) = (V, W)(y + jt)$ solution of

$$\begin{cases} d_t v \;=\; \partial_y u\,, \\ d_t u \;=\; \partial_y(\mathrm{E}_v \mathscr{e})\,, \end{cases}$$

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$$\begin{cases} \partial_\zeta(W \;-\; j\,V) \;=\; 0\,, \\ \partial_\zeta(\mathrm{E}_v \mathscr{e} \;-\; j\,W) \;=\; 0\,. \end{cases}$$

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$$R(\xi)V(Z(\xi)) = 1\,,\; U(\xi) = W(Z(\xi))\,,$$
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Proposition

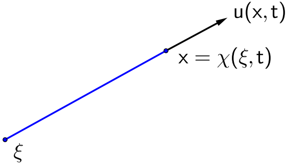
Relationships above yield a one-to-one correspondence between Eulerian travelling waves s.t. $\overline{R(\mathbb{R})}$ is compact in $\mathbb{R}^{+*}$, and Lagrangian travelling waves s.t. $\overline{V(\mathbb{R})}$ is compact in $\mathbb{R}^{+*}$.

Key to one-to-one correspondence

- PDEs solutions, Eulerian *vs* mass Lagrangian coordinates: $x \leftrightarrow \xi \leftrightarrow y$.

$$\rho(\chi(\xi,t),t)\, v(y(\chi(\xi,t),t),t) \,=\, 1\,,\ u(\chi(\xi,t),t) \,=\, w(y(\chi(\xi,t),t),t)\,,$$

$$\begin{cases} \partial_t \chi \,=\, u(\chi,t)\,, \\ \chi(\xi,0) = \xi\,, \end{cases} \qquad \begin{cases} y(\chi(\xi,t),t) = y_0(\xi)\,, \\ {y_0}'(\xi) \,=\, \rho(\xi,0)\,. \end{cases}$$



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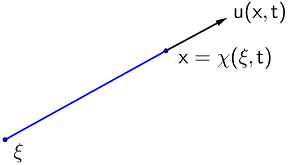

• A travelling wave solution in Eulerian coordinates is such that
$u(\chi,t) = U(\chi - \sigma t) \Rightarrow\; \partial_t(\chi - \sigma t) = U(\chi - \sigma t) \,-\, \sigma \,=\, j/R(\chi - \sigma t)$

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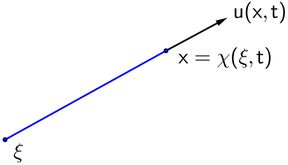

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$u(\chi,t) = U(\chi - \sigma t) \Rightarrow\; \partial_t(\chi - \sigma t) = U(\chi - \sigma t) \;-\; \sigma \;= j/R(\chi - \sigma t)$

$\Rightarrow \partial_t(Z(\chi - \sigma t)) = j$ with $Z' \;=\; R$.

$$\blacktriangleright\; \boxed{Z(\chi(\xi,t) - \sigma t) \;=\; Z(\xi) \,+\, j\, t} \;\blacktriangleleft$$

One-to-one correspondence in practice

Profile equations are Euler–Lagrange equations [Benjamin'72].

$$\begin{cases} R(U-\sigma) \equiv j\,, \\ (U-\sigma)\partial_\xi U + \partial_\xi(\mathsf{E}_\rho \mathscr{E}) = 0\,. \end{cases}$$

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$$\mathscr{L} := \mathscr{E} - \frac{j^2}{2\rho} - \mu\rho\,.$$

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First integrals

$$\mathsf{L}_\rho \mathscr{L} := \rho_x \frac{\partial \mathscr{L}}{\partial \rho_x} - \mathscr{L}\,, \quad \mathsf{L}_v \ell := v_y \frac{\partial \ell}{\partial v_y} - \ell\,.$$

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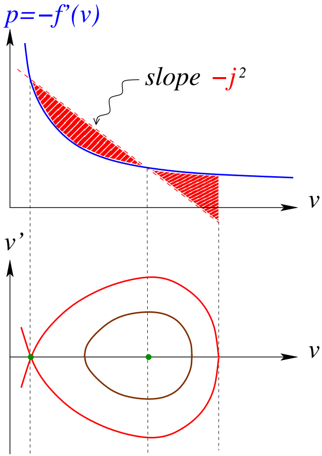
$$\boxed{\mathsf{E}_\rho \mathscr{L} = -v\,\mathsf{E}_v \ell - \mathsf{L}_v \ell - \mu\,, \quad \mathsf{E}_v \ell = -\rho\,\mathsf{E}_\rho \mathscr{L} - \mathsf{L}_\rho \mathscr{L} - \lambda}$$

$$\boxed{\left.\begin{array}{l} \mathsf{E}_\rho \mathscr{L} = 0 \\ \mathsf{L}_\rho \mathscr{L} = -\lambda \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \mathsf{E}_v \ell = 0 \\ \mathsf{L}_v \ell = -\mu \end{array}\right.}$$

Phase portraits for van der Waals pressure law

- **Convex pressure**



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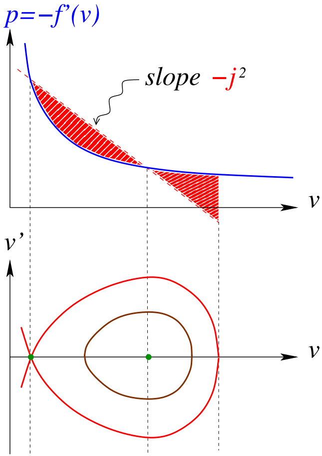

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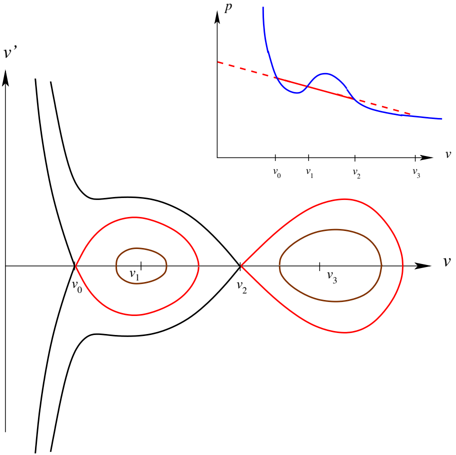

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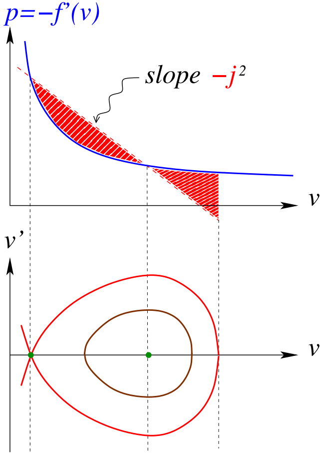

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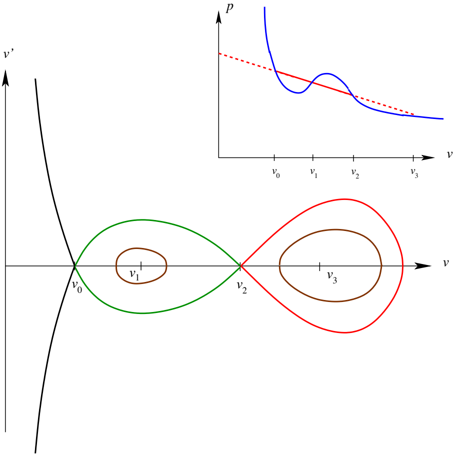

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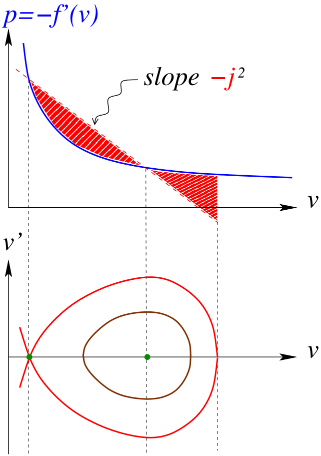

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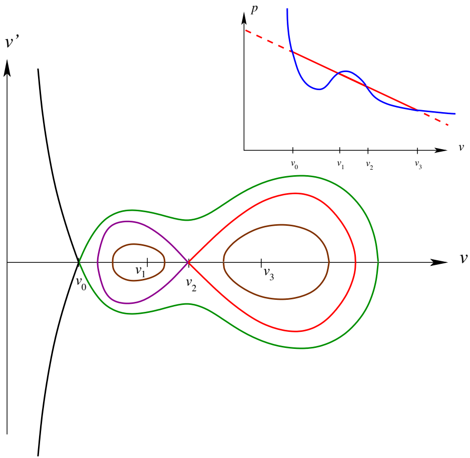

Outline

General Hamiltonian framework

$$(H) \qquad \partial_t \mathbf{U} = \mathscr{J}\left(\mathrm{E}\mathscr{H}[\mathbf{U}]\right), \quad \mathscr{J} = \partial_x \mathbf{J}, \quad \mathscr{H} = \mathscr{H}(\mathbf{U}, \mathbf{U}_x).$$

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Benjamin's impulse $\mathscr{Q}(\mathbf{U}) := \frac{1}{2}\,\mathbf{U} \cdot \mathbf{J}^{-1}\mathbf{U}$ is such that $\mathscr{J}\,\mathrm{E}\mathscr{Q}[\mathbf{U}] = \partial_x \mathbf{U}\,,$ and solutions of (H) satisfy local conservation law

$$(C) \qquad \partial_t \mathscr{Q}(\mathbf{U}) = \partial_x(\mathscr{S}[\mathbf{U}])\,, \quad \mathscr{S}[\mathbf{U}] := \mathbf{U} \cdot \mathrm{E}\mathscr{H}[\mathbf{U}] + \mathrm{L}\mathscr{H}[\mathbf{U}]\,.$$

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Examples

- (gKdV) $\partial_t v + \partial_x p(v) = -\partial_x^3 v$,

  $$\mathbf{U} = v, \qquad \mathscr{Q} = \tfrac{1}{2}\,v^2, \quad \mathscr{H} = f(v) + \tfrac{1}{2}v_x^2\,, \quad f' = -p\,.$$
- (EK1) $\mathbf{U} = (\rho, u)^T$, $\quad \mathscr{Q} = \rho u\,, \quad \mathscr{H} = \frac{1}{2}\rho u^2 + F(\rho) + \frac{1}{2}\mathscr{K}(\rho)\rho_x^2\,.$
- (EKℓ) $\mathbf{U} = (v, u)^T$, $\quad \mathscr{Q} = vu\,, \quad \mathscr{H} = \frac{1}{2}u^2 + f(v) + \frac{1}{2}\kappa(v)u_x^2\,.$

Wave trains

The profile $\underline{\mathbf{U}}$ of a periodic travelling wave solution to (H) of speed c is a stationary solution in co-moving frame to

$$(H_c) \qquad \partial_t \mathbf{U} = \mathscr{J}\left(\mathrm{E}(\mathscr{H} + c\mathscr{Q})[\mathbf{U}]\right).$$

It is characterized by

$$\mathrm{E}(\mathscr{H} + c\mathscr{Q})[\underline{\mathbf{U}}] = \boldsymbol{\lambda}\,,$$

as well as the integrated equation

$$\mathscr{S}[\underline{\mathbf{U}}] + c\mathscr{Q}(\underline{\mathbf{U}}) = \mu\,.$$

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$\mathrm{E}(\mathscr{H} + c\mathscr{Q})[\underline{\mathbf{U}}] = \boldsymbol{\lambda}\,,$ ☛ N parameters if $\mathbf{U}(t,x) \in \mathbb{R}^N$

as well as the integrated equation

$\mathscr{S}[\underline{\mathbf{U}}] + c\mathscr{Q}(\underline{\mathbf{U}}) = \mu\,.$ ☛ one additional parameter.

Slowly varying wave trains generically evolve in a $(N+2)$-dimensional manifold parametrized by $(c, \boldsymbol{\lambda}, \mu)$.

Derivation of modulated equations

'Two-timing' method. Look for solutions admitting an asymptotic expansion of the form

$$\mathbf{U}(t,x) = \mathbf{U}_0(\underbrace{\varepsilon t}_{T}, \underbrace{\varepsilon x}_{X}, \underbrace{\phi(\varepsilon t, \varepsilon x)/\varepsilon}_{\theta}) + \varepsilon\, \mathbf{U}_1(\varepsilon t, \varepsilon x, \phi(\varepsilon t, \varepsilon x)/\varepsilon, \varepsilon) + o(\varepsilon)\,,$$

with $\mathbf{U}_0$ and $\mathbf{U}_1$ one-periodic in θ.

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Averaging. $\underline{\mathbf{U}}(x) := \mathbf{U}_0(T, X, kx)$ is a $1/k$-periodic travelling wave of speed c profile such that

- $\partial_T \langle \mathbf{U}_0 \rangle \ = \ \mathbf{J}\, \partial_X \langle \mathbf{G}_0 \rangle\,, \quad \mathbf{G}_0 := \mathrm{E}\mathscr{H}_k[\mathbf{U}_0]\,,$
- $\partial_T \langle Q_0 \rangle \ = \ \partial_X \langle S_0 \rangle\,, \quad S_0 := \mathbf{U}_0 \cdot \mathbf{G}_0 \ + \ \mathrm{L}\mathscr{H}_k[\mathbf{U}_0]\,.$

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[Whitham'70]: 'the relation of the stability of the periodic wave with the type of the [modulated equations is] given in the previous papers' ...

Stability of periodic waves *vs* type of modulated equations

[Serre'05] [Bronski–Johnson–Zumbrun'09-11] [Pogan–Scheel–Zumbrun'12]

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Theorem (SBG–Noble–Rodrigues)

- *If the set of periodic travelling wave profiles is a* $(N+2)$*-dimensional manifold parametrized by* $(k, \mathbf{M} := \langle \mathbf{U} \rangle, P := \langle Q \rangle)$ *near a reference profile* $\underline{\mathbf{U}}$ *(of period* $1/\underline{k}$ *and speed* $\underline{c}$*),*
- *and the linearized operator* $\mathscr{A}$ *about* $\underline{\mathbf{U}}$ *has an* $(N+2)$*-dimensional generalized kernel in space of* $1/\underline{k}$*-periodic functions,*

then a necessary condition for $\underline{\mathbf{U}}$ *to be stable is that the modulated system be 'weakly hyperbolic' at* $(\underline{k}, \underline{\mathbf{M}}, \underline{P})$*.*

Sketch of proof

- By Floquet–Bloch theory, the spectrum of $\mathscr{A} := \mathscr{J}\,\mathrm{Hess}(\mathscr{H} + c\mathscr{Q})[\underline{\mathbf{U}}]$ in L^p is made of the eigenvalues of $\mathscr{A}^\nu := \mathscr{A}(\underline{k}(\partial_\theta + i\nu))$ in L^∞_{per}, $\nu \in \mathbb{R}/2\pi\mathbb{Z}$.

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- Expanding $\mathscr{A}^\nu$ in powers of ν, we see the leading operator $\mathscr{A}^0$ has a generalized kernel spanned by $\partial_\theta \mathbf{U}_0$, $\partial_{M_\alpha} \mathbf{U}_0$, $\partial_P \mathbf{U}_0$.

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- Spectral projector Π^0 onto $\mathrm{span}(\partial_\theta \mathbf{U}_0, \partial_{M_\alpha} \mathbf{U}_0, \partial_P \mathbf{U}_0)$ extends to spectral projector Π^ν for $\mathscr{A}^\nu$. Kato's perturbation argument then yields dual bases $(\mathbf{\Phi}^\nu_\beta)$ and $(\mathbf{\Psi}^\nu_\beta)$ of g-ker $\mathscr{A}^\nu$ and g-ker$(\mathscr{A}^\nu)^*$ depending analytically on ν.

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- Matrix $D^\nu := i\underline{k}\nu\,\left(\mathbf{\Psi}^\nu_\alpha \cdot \mathscr{A}^\nu \mathbf{\Phi}^\nu_\beta\right)$ is similar to a matrix whose leading order part turns out to coincide with $A + c\mathbf{I}_{N+2}$, where A is the matrix of linearized modulated equations about $(\underline{k}, \underline{\mathbf{M}}, \underline{P})$.

Modulated equations for Euler–Korteweg systems

Theorem (SBG–Noble–Rodrigues)

The following diagram is commutative

$$
\begin{array}{cccc}
 & & \textit{mass Lagrangian} & \\
 & & \textit{change of coordinates} & \\
 & (EK1) & \longrightarrow & (EK\ell) \\
\textit{Whitham's averaging} & \downarrow & & \downarrow \\
 & \langle EK1 \rangle & \longrightarrow & \langle EK\ell \rangle
\end{array}
$$

Proof

Recall that (R, U) is a $1/K$-periodic solution of (EK1) if and only if (V, W) is a $1/k$-periodic solution of (EKℓ), with

$$R(\xi)V(Z(\xi)) = 1,\ U(\xi) = W(Z(\xi)), \frac{\mathrm{d}Z}{\mathrm{d}\xi} = R = \frac{1}{V(Z)},\ \frac{1}{k} = Z\Big(\frac{1}{K}\Big).$$

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Hence $\langle\rho_0\rangle = \dfrac{K}{k}\,,\ \langle v_0\rangle = \dfrac{k}{K}\,,\ \langle w_0\rangle = \dfrac{\langle\rho_0 u_0\rangle}{\langle\rho_0\rangle}\ .$

$$\boxed{\mathrm{d}Y \ = \ \langle\rho_0\rangle\mathrm{d}X \ - \ \langle\rho_0 u_0\rangle\mathrm{d}T \overset{S=T}{\Longleftrightarrow} \mathrm{d}X \ = \ \langle v_0\rangle\mathrm{d}Y \ + \ \langle w_0\rangle\mathrm{d}S}$$

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- $K\,\mathrm{d}X-\sigma K\,\mathrm{d}T=k\mathrm{d}Y+jk\mathrm{d}S$
- $\langle u_0\rangle\mathrm{d}X+(\frac{1}{2}\sigma^2-\mu-\sigma\langle u_0\rangle)\mathrm{d}T=(\frac{1}{2}\sigma^2-\mu+j\sigma\langle v_0\rangle+j^2\langle v_0^2\rangle)\mathrm{d}S$ $+\langle v_0w_0\rangle\,\mathrm{d}Y$

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- $K\,\mathrm{d}X - \sigma K\,\mathrm{d}T = k\mathrm{d}Y + jk\mathrm{d}S$
- $\langle u_0\rangle\mathrm{d}X + (\frac{1}{2}\sigma^2 - \mu - \sigma\langle u_0\rangle)\mathrm{d}T = (\frac{1}{2}\sigma^2 - \mu + j\sigma\langle v_0\rangle + j^2\langle v_0^2\rangle)\mathrm{d}S + \langle v_0 w_0\rangle\,\mathrm{d}Y$
- $\langle\rho_0 u_0\rangle\mathrm{d}X + \left(-\langle\rho_0 u_0^2\rangle + \lambda + j^2\langle 1/\rho_0\rangle\right)\mathrm{d}T = (\lambda + j^2\langle v_0\rangle^2)\mathrm{d}S + \langle w_0\rangle\,\mathrm{d}Y$

Sufficient condition for hyperbolicity

Theorem ([Gavrilyuk–Serre'95])

The strict convexity of the averaged energy

$$\mathrm{e} := \langle \mathscr{e}_0 \rangle + \frac{1}{2} \langle w_0^2 \rangle - \frac{1}{2} \langle w_0 \rangle^2$$

as a function of $(\langle v_0 \rangle, k, (\langle v_0 w_0 \rangle - \langle v_0 \rangle \langle w_0 \rangle)/k)$ *implies the hyperbolicity of modulated equations for* (EKℓ).

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Remark

The convexity of e is equivalent to the convexity of

$$\langle \rho_0 \rangle \, \mathrm{e} = \langle \mathscr{E}_0 \rangle + \tfrac{1}{2} \langle \rho_0 u_0^2 \rangle - \tfrac{1}{2} \frac{\langle \rho_0 u_0 \rangle^2}{\langle \rho_0 \rangle}$$

as a function of $(\langle \rho_0 \rangle, K, (\langle \rho_0 \rangle \langle u_0 \rangle - \langle \rho_0 u_0 \rangle)/K)$.

Concluding remarks

- Periodic waves with $j = 0$ are unstable [Serre'94], but there are also stable ones, *e.g.* small-amplitude periodic waves for defocussing cubic (NLS) ($\rho\mathscr{K}(\rho) \equiv 4$, $F'(\rho) = \rho$) [Gallay–Hărăguș'07].

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- For (EK) with van der Waals pressure law, numerical computation of eigenvalues display nontrivial regions of hyperbolicity for the modulated equations (away from unstable constant states)...

Further reading

S. Benzoni-Gavage. Planar travelling waves in capillary fluids.
To appear in special volume of *Diff. Int. Eq.*

S. Benzoni-Gavage, P. Noble, and L.M. Rodrigues. Slow modulations of periodic waves in Hamiltonian PDEs, with application to capillary fluids.
In preparation.