Controllability results for degenerate parabolic operators

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Plan

Degenerate parabolic operator

$\Omega \subset \mathbb{R}^n$ bounded, $A(x) \geqslant 0$ in Ω

$$\begin{cases} u_t - \text{div}[A(x)\nabla u] + b(t,x).\nabla u = f(t,x) & \text{in } \Omega \times (0,T) \\ u(0,x) = u_0(x) & x \in \Omega \\ B.C. & \text{on } (0,T) \times \Gamma \end{cases}$$

Possible degeneracy :

- at the boundary : $A(.)\nabla d_\Gamma(.) = 0$ on $\Gamma_0 \subset \Gamma$,
- in the interior : $A(x)$ is not > 0, $\forall x \in D \subset \Omega$

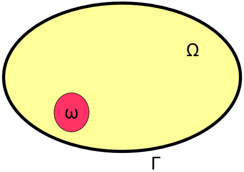
Of particular interest : hypoelliptic operators

Applications : climatology (Budyko-Sellers)
particles systems (Kolmogorov)
But the motivations of this talk are rather theoretical...

The null controllability problem

Let us take a locally distributed control as a source term :

$$\begin{cases} u_t - \text{div}[A(x)\nabla u] + b(t,x).\nabla u = 1_\omega(x) f(t,x) & \text{in } (0,T)\times\Omega \\ u(0,x) = u_0(x) & x \in \omega \\ + \text{B.C.} & \text{on } \partial\Omega \end{cases}$$



Question : Given $u_0 \in L^2(\Omega)$ and $T > 0$ does there exist $f \in L^2((0,T)\times\Omega)$ such that $u(T,.) = 0$?

Classical results about exact controllability

- **Parabolic case :** null controllability of the heat equation

$$u_t - \Delta u = 1_\omega f(t,x), \qquad x \in \Omega$$

holds $\forall \omega \subset \Omega$ and $\forall T > 0$. (infinite propagation speed) [Lebeau-Robbiano, Fursikov-Immanuvilov(1995)]

- **Hyperbolic case :** controllability of the wave eq requires

$$u_{tt} - \Delta u = 1_\omega f(t,x), \qquad x \in \Omega$$

  - **geometric condition** on (Ω, ω) : ω meets every ray of geometric optics
  - $T > T_{min}$= minimal time for an optic ray to meet ω. (finite propagation speed) [Bardos-Lebeau-Rauch(1992)]

Qu : What about degenerate parabolic systems ?

The duality method in an abstract frame (HUM)

control system

$$\begin{cases} \frac{dy}{dt} = Ay + Bu \\ y(0) = y_0 \end{cases}$$

adjoint system

$$\begin{cases} -\frac{d\phi}{dt} = A^*\phi \\ \phi(T) = \phi_T \end{cases}$$

If $y_0 = 0$ then $\langle y(T), \phi_T \rangle = \int_0^T \langle u(t), B^*\phi(t) \rangle dt$

End-point map :

$$\begin{array}{|llll} \mathcal{F}_T : & L^2(0,T) & \rightarrow & H \\ & u & \mapsto & y(T) \end{array} \qquad \begin{array}{|llll} \mathcal{F}_T^* : & H & \rightarrow & L^2(0,T) \\ & \phi_T & \mapsto & B^*\phi(.) \end{array}$$

Approximate control $\Leftrightarrow$ Range($\mathcal{F}_T$) dense $\Leftrightarrow$ $\mathcal{F}_T^*$ injective

Exact control $\Leftrightarrow$ $\mathcal{F}_T$ surjective $\Leftrightarrow$ $\|\mathcal{F}_T^*(\phi_T)\|^2 \geqslant c\|\phi_T\|^2$

Null controllability $\Leftrightarrow$ $\|\mathcal{F}_T^*(\phi_T)\|^2 \geqslant c\|\phi(0)\|^2$

To summarize : approximate controllability

$$\begin{cases} u_t - \text{div}[A(x)\nabla u] = 1_\omega(x) f(t,x) & \text{in } (0,T)\times\Omega \\ u(0,x) = u_0(x) & x \in \omega \\ u(t,.) = 0 & \text{on } \Gamma \end{cases}$$

- **approximate controllability** in time $T > 0$

$$\forall u_0, u_1 \in L^2(\Omega) \quad \forall \epsilon > 0 \quad \exists f \in L^2(Q_T) : \begin{cases} \|u(T,.) - u_1\|_{L^2(\Omega)} < \epsilon \\ \int_{Q_T} |f|^2 \leqslant C_T \int_\Omega |u_0 - u_1|^2 \end{cases}$$

by duality equivalent to

- **unique continuation** from $(0,T)\times\omega$

$$\begin{cases} v_t + \text{div}[A(x)\nabla v] = 0 & \text{in } (0,T)\times\Omega \\ v(t,.) = 0 & \text{on } \Gamma \end{cases}$$

satisfies

$$v \equiv 0 \text{ on } (0,T)\times\omega \quad \Rightarrow \quad v \equiv 0 \text{ in } Q_T$$

To summarize : null controllability

$$\begin{cases} u_t - \mathrm{div}[A(x)\nabla u] = 1_\omega f(t,x) & \text{in } (0,T)\times\Omega \\ u(0,x) = u_0(x) & x\in\omega \\ u(t,.) = 0 & \text{on } \Gamma \end{cases}$$

- **null controllability** in time $T > 0$:

$$\forall u_0 \in L^2(\Omega) \quad \exists f \in L^2(Q_T) : \begin{cases} u(T,.) = 0 \\ \int_{Q_T} |f|^2 \leqslant C_T \int_\Omega |u_0|^2 \end{cases}$$

by duality equivalent to

- **observability inequality** on $(0,T)\times\omega$

$$\begin{cases} v_t + \mathrm{div}[A(x)\nabla v] = 0 & \text{in } (0,T)\times\Omega \\ v(t,.) = 0 & \text{on } \Gamma \end{cases}$$

satisfies

$$\int_\Omega v(0,x)^2 dx \leqslant C_T \int_0^T \int_\omega v(t,x)^2 dxdt$$

This is a 'quantified version' of the unique continuation.

Roadmap to observability for heat equations

- Fattorini-Russel(1971), Russel(1978) : Riesz basis approach
- Lebeau-Robbiano(1995) : Riesz basis + local Carleman estimates
- Fursikov and Imanuvilov(1996) : **global Carleman estimate**

$$\begin{cases} v_t + \mathrm{div}[A(x)\nabla v] = 0 & \text{in } (0,T)\times\Omega \\ v(t,.) = 0 & \text{on } \Gamma \end{cases}$$

$$\int_0^T \int_\Omega \frac{\tau^3}{[t(T-t)]^3} v^2 e^{2\tau\phi(t,x)} dxdt \leqslant C \int_0^T \int_\omega v^2 dxdt$$

where $\tau >> 1$, $\phi(t,x) = \frac{e^{\psi(x)} - e^{2\|\psi\|_\infty}}{t(T-t)}$, $\nabla\psi \neq 0$ in $\Omega - \omega$.

What changes in the degenerate case ?

- observability/null controllability may fail (for violent degeneracy)
- ϕ must be adapted to degeneracy
- Is the Carleman estimate true ? which ϕ ?...

2nd part

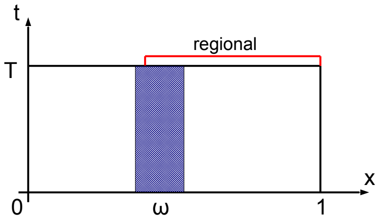
Two examples from the litterature

Ex 1 : Boundary degeneracy, simplest case

$$\begin{cases} u_t + (x^\alpha u_x)_x = 1_\omega(x) f(t,x) & (t,x) \in (0,T) \times (0,1) \\ u(t,1) = 0, \\ u(t,0) = 0 \text{ if } 0 \leqslant \alpha < 1; \quad (x^\alpha u_x)(t,0) = 0 \text{ if } 1 \leqslant \alpha \end{cases}$$

Theorem [Cannarsa-Martinez-Vancostenoble(2008)]

$$\text{Null controllability is} \begin{cases} \text{false} & \alpha \geqslant 2 \quad (\rightarrow \text{regional null controllability}) \\ \text{true} & 0 \leqslant \alpha < 2 \end{cases}$$



Technics : CVAR/Carleman estimates and Hardy's inequality

Extensions

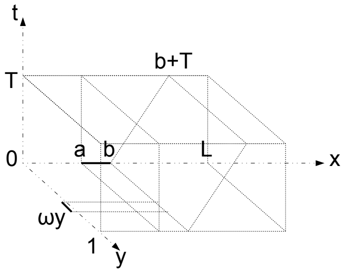
- More general 1D problems
  - divergence form
    - Martinez-Vancostenoble(2006) $u_t - (a(x)u_x)_x = 1_\omega f$
    - Alabau-Cannarsa-Fragnelli(2006)
      $u_t - (a(x)u_x)_x + g(u) = 1_\omega f$
    - Flores-de Teresa(2010) $u_t - (x^\theta u_x)_x + x^\sigma b(x,t)u_x = 1_\omega f$
  - non divergence form Cannarsa-Fragnelli-Rocchetti(2007,2008)
    $u_t - a(x)u_x x - b(x)u_x = 1_\omega f$
  - degenerate/singular problems Vancostenoble-Zuazua(2008, 2009) $u_t - (x^\theta u_x)_x - \frac{\lambda}{x^\sigma} u = 1_\omega f$
  - systems
    - Cannarsa-de Teresa(2009) cascade 2×2
    - Maniar et al.(2001) general 2×2
- Extension in 2D : Cannarsa-Martinez-Vancostenoble(2009)

Ex 2 : Null controllability of Crocco-type equations

$$\begin{cases} u_t + u_x - u_{yy} = 1_\omega(x,y) f(t,x,y) & (t,x,v) \in (0,T) \times (0,L) \times (0,1), \\ \text{Dirichlet in } y, \text{ periodic in } x \end{cases}$$

Decoupling between transport in x and diffusion in y
$\Rightarrow$ Regional null controllability [Martinez-Raymond-Vancostenoble 2003]

$$\omega = (a,b)_x \times \omega_y \qquad \Rightarrow \qquad \Omega_C = (a, b+T)_x \times (0,1)_y$$



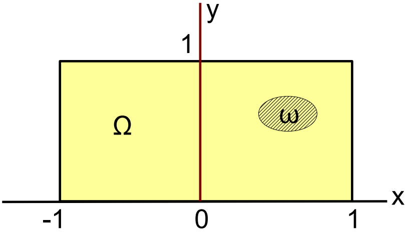
Technics : 1D Carleman estimates along characteristics

3rd part

Controllability of Grushin type operators

Grushin-type operators

$$\gamma > 0 \quad \begin{cases} u_t - u_{xx} - |x|^{2\gamma} u_{yy} = 1_\omega(x,y) f(t,x,y) & \text{in } \Omega \\ u(t,.,.) = 0 & \text{on } \partial\Omega \\ u(0,x,y) = u_0(x,y) \end{cases}$$

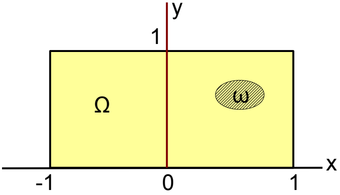


Rk : no dispersion when $\gamma = 1$ for the associated Schrödinger eq on $\mathbb{R}^2$ [Gérard-Gréllier 2010]

Approximate control/unique continuation holds $\forall\gamma$, $\forall\omega$

Elliptic case : [Garofalo 1993]

$$\begin{cases} v_t - v_{xx} - |x|^{2\gamma} v_{yy} = 0 & \text{in } \Omega \\ v(t,.,.) = 0 & \text{on } \partial\Omega \end{cases}$$



$v \equiv 0$ on $(0,T) \times \omega$

$\Rightarrow v \equiv 0$ on $(0,T) \times (0,1) \times (0,1)$ (UC for parabolic operators)

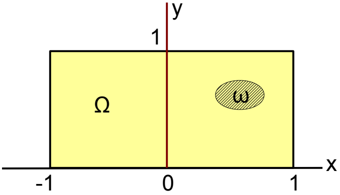
$\Rightarrow v^n \equiv 0$ on $(0,T) \times (0,1)$, $\forall n$

$\Rightarrow v^n \equiv 0$ on $(0,T) \times (-1,1)$, $\forall n$ i.e. $v \equiv 0$ on $(0,T) \times \Omega$

$$v^n(t,x) := \int_0^1 v(t,x,y) \sin(n\pi y) dy \qquad v_t^n - v_{xx}^n + (n\pi)^2 |x|^{2\gamma} v^n = 0$$

Theorem [KB-Cannarsa-Guglielmi, 2012]

$$(G) \quad \begin{cases} u_t - u_{xx} - |x|^{2\gamma} u_{yy} = 1_\omega(x,y) f(t,x,y) & \text{in } \Omega \\ u(t,.,.) = 0 \quad \text{on } \partial\Omega \\ u(0,x,y) = u_0(x,y) \end{cases}$$



- $0 < \gamma < 1$: (G) null controllable $\forall T > 0$, $\forall \omega$
- $\gamma > 1$: (G) not null controllable
- $\gamma = 1$ and $\omega = (a,b) \times (0,1)$: $\exists T^* \geqslant a^2/2$ such that (G) is
  - null controllable $\forall T > T^*$
  - not null controllable $\forall T < T^*$

Null controllability $\Leftrightarrow$ observability inequality

$$(G^*) \quad \begin{cases} v_t - v_{xx} - |x|^{2\gamma} v_{yy} = 0 \\ v(t,.,.) = 0 \quad \text{on } \partial\Omega \\ v(0,x,y) = v_0(x,y) \end{cases}$$

observable in $[0,T] \times \omega$: $\exists C_T > 0$ such that $\forall v_0 \in L^2(\Omega)$

$$\int_\Omega v(T,x,y)^2 dxdy \leqslant C_T \int_0^T \int_\omega v(t,x,y)^2 dxdy$$

Theorem [KB-Cannarsa-Guglielmi, 2012]

- $0 < \gamma < 1$: (G^*) observable $\forall T > 0$, $\forall \omega$
- $\gamma > 1$: (G^*) not observable
- $\gamma = 1$ and $\omega = (a,b) \times (0,1)$: $\exists T^* \geqslant a^2/2$ such that (G^*) is
  - observable $\forall T > T^*$
  - not observable $\forall T < T^*$

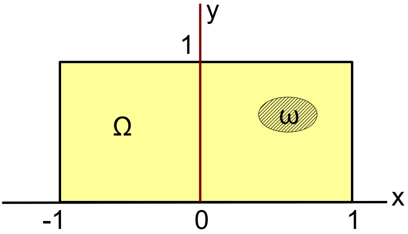
Where the classical approach causes problems

For the heat equation, the Carleman estimate is proved with a weight of the form $\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K}$, where $K > \|\psi\|_\infty$, $\lambda >> 1$ sufficiently large, $\psi > 0$ on Ω and

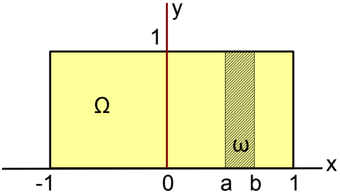
$$\psi_{|\partial\Omega} = 0 \qquad \text{and} \qquad \nabla\psi(x) \neq 0, \forall x \in \overline{\Omega} - \omega.$$

With the same strategy, for Grushin, one would need

$$\begin{pmatrix} \psi_x \\ |x|^\gamma \psi_y \end{pmatrix} \neq 0, \text{ on } \overline{\Omega} - \omega \ \dots$$



Proof when ω is a strip : Fourier decomposition



$$v(t,x,y) = \sum_{n=1}^{\infty} v_n(t,x)\sin(n\pi y)$$

$$\begin{cases} v_t - v_{xx} - |x|^{2\gamma} v_{yy} = 0 \\ v(t,.,.) = 0 \quad \text{on } \partial\Omega \end{cases} \qquad (G_n^*) \begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2 |x|^{2\gamma} v_n = 0 \\ v_n(t,\pm 1) = 0 \end{cases}$$

Goal : $\int_\Omega v(T,x,y)^2 dxdy \leqslant C \int_0^T \int_\omega v(t,x,y)^2 dxdydt$

$\Leftrightarrow \quad \sum_{n=1}^{\infty} \int_{-1}^{1} v_n(T,x)^2 dx \leqslant C \sum_{n=1}^{\infty} \int_0^T \int_a^b v_n(t,x)^2 dxdt$

$\Leftrightarrow \quad$ Uniform observability of (G_n^*) wrt n :

$$\int_{-1}^{1} v_n(T,x)^2 dx \leqslant C \int_0^T \int_a^b v_n(t,x)^2 dxdt$$

Proof of (UO) when $0 < \gamma < 1$ or ($\gamma = 1, T$ large)

$$(G_n^*) \begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2 |x|^{2\gamma} v_n = 0 \\ v_n(t, \pm 1) = 0 \end{cases}$$

1. **Explicit decay rate** for the Fourier components

$$\int_{-1}^{1} v_n(T,x)^2 dx \leqslant e^{-cn^{\frac{2}{1+\gamma}}(T-t)} \int_{-1}^{1} v_n(t,x)^2 dx, \quad \forall t \in [0,T]$$

2. **Carleman estimate** with weight $e^{\frac{nT^2\psi(x)}{t(T-t)}}$ (adapted $0 < \gamma < 1/2$)

$$\int_{T/3}^{2T/3} \int_{-1}^{1} v_n(t,x)^2 dx dt \leqslant e^{Cn} \int_0^T \int_a^b v_n(t,x)^2 dx dt \quad \forall n \geqslant n_T$$

$$\frac{T}{3} \int_{-1}^{1} v_n(T,x)^2 dx \leqslant e^{Cn - cn^{\frac{2}{1+\gamma}} \frac{T}{3}} \int_0^T \int_a^b v_n(t,x)^2 dx dt \quad \forall n \geqslant n_T$$

$Cn - cn^{\frac{2}{1+\gamma}} \frac{T}{3} \to -\infty$ when $(0 < \gamma < 1, \forall T > 0)$ or $(\gamma = 1, T > 3/c)$

When ω is not a strip : construction of the control

Lebeau-Robbiano's strategy : $0 = T_0 < T_1 < ... < T_j \to T$,
$T_{j+1} = T_j + 2a_j$

- on $[T_j, T_j + a_j]$, one applies a control that steers the 2^j-firt components to zero : cost $\leqslant e^{C2^j}$
- on $[T_j + a_j, T_{j+1}]$, no control $\to$ dissipation $e^{-c(2^j)^{\frac{2}{1+\gamma}} T_j}$

Key point : $\exists C > 0$, st $\forall N$, $\sum_{k=1}^N |b_k|^2 \leqslant e^{CN} \int_c^d \left| \sum_{k=1}^N b_k \sin(ky) \right|^2 dy$

$$\begin{aligned}
\int_\Omega v(T,x,y)^2 dxdy &= \sum_{n=1}^{2^j} \int_{-1}^1 v_n(T,x)^2 dx \\
&\leqslant C \sum_{n=1}^{2^j} \int_0^T \int_a^b v_n(t,x)^2 dxdt \\
&\leqslant e^{C2^j} \int_0^T \int_a^b \int_c^d \left| \sum_{n=1}^{2^j} v_n(t,x) \sin(k\pi y) \right|^2 dydxdt \\
&= e^{C2^j} \int_0^T \int_\omega v(t,x,y)^2 dxdydt
\end{aligned}$$

Failure of UO when $\gamma > 1$ or ($\gamma = 1$, T small) : strategy

- Take the first eigenfunctions w_n with λ_n

$$\left\{ \begin{array}{l} -w_n''(x) + [(n\pi)^2|x|^{2\gamma} - \lambda_n] w_n(x) = 0 \,, \quad x \in (-1,1) \,,\; n \in \mathbb{N}^* \,, \\ w_n(\pm 1) = 0 \,, \quad w_n \geq 0 \,, \quad \|w_n\|_{L^2(-1,1)} = 1 \end{array} \right.$$

- $v_n(t,x) := w_n(x) e^{-\lambda_n t}$ is a particular solution of

$$\left\{ \begin{array}{ll} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2 |x|^{2\gamma} v_n = 0 & (t,x) \in (0,T) \times (-1,1) \,, \\ v_n(t, \pm 1) = 0 & t \in (0,T) \,, \end{array} \right.$$

- (UO) fails if we can provide upper bound such that

$$\frac{\int_0^T \int_a^b v_n(t,x)^2 dx dt}{\int_{-1}^1 v_n(T,x)^2 dx} = \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b w_n(x)^2 dx \underset{n \to +\infty}{\longrightarrow} 0$$

This is technical because $[(n\pi)^2|x|^{2\gamma} - \lambda_n]$ changes sign in $(-1,1)$.

Failure of UO when $\gamma = 1$ and $T \leqslant a^2/2$

We can perform explicit computations because

$$\begin{cases} -w_n'' + (n\pi)^2 |x|^2 w_n = \lambda_n w_n, \quad x \in (-1,1) \\ w_n(\pm 1) = 0 \end{cases}$$

is 'close' to the harmonic oscillator :

$$w_n(x) \sim \sqrt[4]{n} e^{-\frac{n\pi x^2}{2}}, \qquad \lambda_n \sim n\pi.$$

Thus

$$\frac{\int_0^T \int_a^b v_n(t,x)^2 dx dt}{\int_{-1}^1 v_n(T,x)^2 dx} \sim \frac{e^{n\pi(2T-a^2)}}{2a\pi^2 n^{3/2}}$$

tends to zero when $T \leqslant a^2/2$.

Failure of UO when $\gamma > 1$: comparison argument

$$\begin{cases} -w_n''(x) + [(n\pi)^2|x|^{2\gamma} - \lambda_n]w_n(x) = 0, x \in (-1,1) \\ w_n(\pm 1) = 0 \end{cases}$$

- restrict to $[x_n, 1]$ where $x_n = [\lambda_n/(n\pi)^2]^{1/2\gamma} \to 0$
- equation yields upper bound $|w_n'(x_n)| \leqslant \sqrt{x_n}\lambda_n$
- by comparison argument

$$\begin{cases} -W_n''(x) + [(n\pi)^2|x|^{2\gamma} - \lambda_n]W_n(x) \geqslant 0 \\ W_n(1) \geqslant 0 \\ W_n'(x_n) < -\sqrt{x_n}\lambda_n \end{cases} \Rightarrow w_n \leqslant W_n \text{ on } [x_n, 1]$$

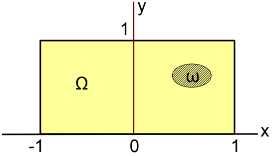
- construct $C_n > 0$ such that $W_n(x) := C_n e^{-C_\gamma n x^{\gamma+1}}$ satisfies

$$\frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b w_n(x)^2 dx \leqslant \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b W_n(x)^2 dx \leqslant e^{2n\left(\frac{\lambda_n}{n} T - C_\gamma\right)} R(n)$$

- conclude with dissipation speed $\lambda_n \leqslant c n^{\frac{2}{1+\gamma}}$

Concluding remarks and open problems about Grushin

$$u_t - u_{xx} - |x|^{2\gamma} u_{yy} = 1_\omega(x,y) f(t,x,y)$$



We have proved that null controllability

- holds $\forall T > 0$, $\forall \omega$ when $\gamma \in (0,1)$
- holds only for $T >> 1$ when $\gamma = 1$ and $\omega = (a,b) \times (0,1)$
- does not hold when degeneracy is too strong, i.e. $\gamma > 1$.

Open problems :

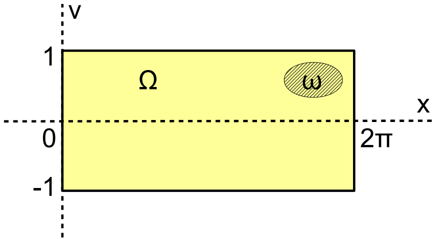
- when $\gamma = 1$: Is $T_{min} = a^2/2$? When $T > a^2/2$, $\forall \omega$?
- more general and/or multidimensional Ω ? boundary control ?

4th part

Controllability of Kolmogorov-type equations

Kolmogorov-type equations

$\gamma > 0$ $\quad \begin{cases} u_t + v^\gamma u_x - u_{vv} = 1_\omega(x,v) f(t,x,v) & x \in \mathbb{T} \quad v \in (-1,1) \\ \text{B.C. at } v = \pm 1 \end{cases}$



Rk : If $\gamma = 1$, $u(t,x,v) = h(t, x - vt, v)$ where

$$\partial_t h - (\partial_v - t\partial_x)^2 h = 0$$

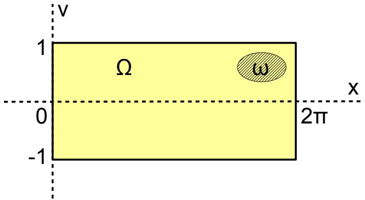
Unique continuation for Kolmogorov-type equations

Alinhac-Zuily(1980) :

- **In the elliptic case** $(\partial_v^2 + v\partial_x)$, unique continuation holds.
- **In the parabolic case**, there exists a zero-order C^∞-perturbation without unique continuation : $\forall \gamma \in \mathbb{N}^*$, $\exists a(t,x,v)$, $u(t,x,v)$
  - C^∞-functions on a neighborhood V of 0 in $\mathbb{R}^3$
  - that vanish for $v < 0$
  - such that $u_t - v^\gamma u_x - u_{vv} - au = 0$ in V
  - $0 \in \text{Supp}(u)$.

Theorem [KB, 2012]

$$(K): \qquad u_t + v^\gamma u_x - u_{vv} = 1_\omega(x,v) f(t,x,v) \quad x \in \mathbb{T} \quad v \in (-1,1)$$



- **Periodic-type BC** and $\gamma = 1$: (K) is null controllable $\forall T > 0$, $\forall \omega$
- **Dirichlet BC** and $\omega = \mathbb{T} \times (a, b)$
  - $\gamma = 1$: (K) is null controllable $\forall T > 0$
  - $\gamma = 2$: there exists $T^* \geqslant a^2/2$ s.t.
    - (K) is null controllable $\forall T > T^*$
    - (K) is not null controllable $\forall T < T^*$

Proof

$$u_t + v^\gamma u_x - u_{vv} = 0 \qquad x \in \mathbb{T} \quad v \in (-1,1) \quad + BC$$

Fourier decomposition : $u(t,x,v) = \sum u_n(t,v)e^{inx}$ where

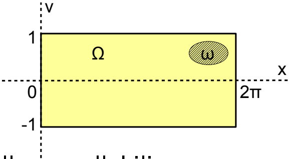
$$\partial_t u_n + inv^\gamma u_n - \partial_v^2 u_n = 0 \qquad v \in (-1,1) + BC$$

- **Carleman estimate** with weight $e^{\frac{\sqrt{n}T^2\psi(v)}{t(T-t)}}$: $\to$ cost $e^{C\sqrt{n}}$
- **Dissipation estimate** :
  - Periodic, $\gamma = 1$: $e^{-n^2t^3/12}$ (explicit solution)
  - Dirichlet, $\gamma = 1$: $e^{-n^{2/3}t}$ (Lyapunov fn)
  - Dirichlet, $\gamma = 2$: $e^{-\delta\sqrt{n}t}$ (Lyapunov fn)
- **Failure of UO ($\gamma = 2$, $T \leqslant a^2/2$) :** $G_n(t,v) := e^{-\sqrt{in}(t+v^2/2)}$

$$\frac{\int_0^T \int_a^b |G_n(t,v)|^2 dvdt}{\int_{-1}^1 |G_n(T,v)|^2 dv} \underset{n\to+\infty}{\sim} \frac{e^{\sqrt{2n}(T-a^2/2)}}{n^{3/4}}$$

Concluding remarks and open problems about Kolmogorov

$$u_t - v^\gamma u_x - u_{vv} = 1_\omega(x,y) f(t,x,y)$$



We have proved that null controllability

- holds $\forall T > 0$, $\forall \omega$ with $\gamma = 1$ and periodic BC
- holds $\forall T > 0$ with $\omega = \mathbb{T} \times (a, b)$, $\gamma = 1$ and Dirichlet BC
- holds for $T > T^* > 0$ with $\omega = \mathbb{T} \times (a, b)$, $\gamma = 2$ and Dirichlet BC

Open problems :

- with Dirichlet BC : $\forall \omega$?
- when $\gamma = 2$, $T^* = a^2/2$?
- when $\gamma \geqslant 3$? more general and/or multidimensional Ω ?

Concluding rk : Hörmander's condition for hypoellipticity

Theorem [Hörmander, 1967] Let $P := \sum_{j=1}^r X_j^2 + X_0 + c$, where $X_0, ..., X_r$ are 1st order homogeneous differential operators with C^∞ coefficients in an open set $\Omega \subset \mathbb{R}^n$ and $c \in C^\infty(\Omega)$. If there exists n operators among X_{j_1}, $[X_{j_1}, X_{j_2}]$,...,$[X_{j_1}, [X_{j_2}, [X_{j_3}, [..., X_{j_k}]...]]]$,... where $j_i \in \{0, 1, ..., r\}$, which are linearly independent at any point in Ω, then, P is **hypoelliptic** : any distribution u in Ω, is a C^∞ function where so is Pu.

Grushin and Kolmogorov are prototypes of hypoelliptic operators :

$$X_1(x,y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x,y) := \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}.$$

$$[X_1, X_2](x,y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, [X_1,[X_1,X_2]](x,y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix}, \dots$$

- Grushin : NC holds only when the first bracket is sufficient
- Kolmogorov : NC holds when the 2 first brackets are sufficient
- Link NC/nb of iterated Lie brackets in Hörmander's cdt ?