The Cauchy problem for a stochastic conservation law

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June 28, 2012

Plan of the talk

The Cauchy problem for a stochastic conservation law

A S.P.D.E.

$$du = \operatorname{div} \vec{f}(u)\,dt + h(.,u)\,dW \quad on\ \Omega \times (0,T) \times \mathbb{R}^d,$$

with the initial condition $\quad u_0 \in L^2(\mathbb{R}^d)$,

where

- $\vec{f} : \mathbb{R} \to \mathbb{R}^d$ Lipschitz continuous, $\vec{f}(0) = \vec{0}$,
- $h : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, $h(0) = 0$,
- $W = \{W_t, \mathcal{F}_t, 0 \leq t \leq T\}$ a 1-d adapted continuous Brownian motion on the classical Wiener space $(\Omega, \mathcal{F}, P)$ for the filtration $(\mathcal{F}_t)$.

Remarks

⋆ New term

$\int_0^T h(.,u)dW \rightsquigarrow$ the Itô integral of $h(.,u)$.

⋆ Two types of problem :

Additive noise (a.n.).

$$h(.,u) = h(.)$$

Multiplicative noise (m.n.).

$$h(.,u) = h(u)$$

⋆ Why a stochastic perturbation?

- To **improve** existing model/**compensate** uncertainties...
- To **modeling** phenomena.
- Open problem.

Functional frame

Solution space

$$\mathcal{N}_w^2(0,T,H)$$

$$\llcorner\ e.g.\ L^2(\mathbb{R}^d),\ H^1(\mathbb{R}^d),\cdots$$

$$= \quad L^2((0,T)\times\Omega,\mathcal{P}_T;H)$$

endowed with $dt \otimes dP$
and the predictable σ-field $\mathcal{P}_T$
generated by $]s,t]\times A,\ 0\le s<t\le T,\ A\in\mathcal{F}_s$.

Solution concept

Stochastic entropy solution in $\mathcal{N}_w^2(0,T,L^2(\mathbb{R}^d))$.

Stochastic calculus

Properties of the Itô integral

$$I_t \quad : \quad \mathcal{N}_w^2(0,T;H) \to C([0,T];L^2(\Omega;H))$$
$$\varphi \mapsto \int_0^{\cdot} \varphi(s)\,\mathrm{d}W(s),$$

with φ and ψ predictable

- $$E\left[\int_0^t \varphi(s)\,\mathrm{d}W(s)\right] = 0,$$
- $$E\left[\left\| \int_0^t \varphi(s)\,\mathrm{d}W(s)\right\|^2\right] = E\left[\int_0^t \|\varphi(s)\|^2\,\mathrm{d}s\right]$$ (Itô isometry).

Itô chain rule

$$\begin{aligned}\psi:(t,u) &\in (0,T)\times H \mapsto \tilde{H} \quad (\mathbb{R}) \text{ smooth, then if}\\ \mathbf{du} &= \mathbf{A\,dt + h\ dW},\\ \text{one has} & \\ d\psi(t,u) &= \psi_t(t,u)\,dt + \psi_u(t,u)\overbrace{du}^{Adt+hdW} + \frac{1}{2}\psi_{u,u}(t,u)h^2\,dt\\ &= \left(\psi_t(t,u) + \psi_u(t,u)A + \frac{1}{2}\psi_{u,u}(t,u)h^2\right)dt\\ &+ \psi_u(t,u)h\,dW.\end{aligned}$$

Remark: stochastic energy

"Testing the SPDE with u" is "applying Itô's formula with $\psi(u) = \frac{1}{2}\int_{\mathbb{R}^d} u^2\,dx$".

Former results on "$du = \operatorname{div} \vec{\mathbf{f}}(u)\,dt + h(.,u)\,dW$"

1-D Cauchy problem

1. H. Holden and N. H. Risebro ('91), m.n. with *supph* compact.
2. W. E, K. Khanin, A. Mazel and Ya. Sinai ('00), $f(u) = u^2$ with a.n.
3. J. U. Kim ('03), a.n.

N-D Cauchy problem

1. J. Feng and D. Nualart ('08), m.n., existence in 1-D, uniqueness in $\mathbb{R}^d$.
2. A. Debussche and J. Vovelle ('10), m.n. in a torus with a kinetic formulation.

Entropies

Notations :

- Smooth entropies :

$$\mathcal{E} = \{\eta \in C^{2,1}(\mathbb{R}),\ \eta \geq 0,\ \text{convex},\ \eta(0) = 0,\ \text{supp}\eta'' \text{ compact}\}.$$

- $\forall \eta \in \mathcal{E},\ F^\eta(a,b) = \int_a^b \eta'(\sigma - a)\vec{\mathbf{f}}'(\sigma)\, d\sigma.$
- Entropy flux :

$$F(a,b) = Sgn_0(a-b)[\vec{\mathbf{f}}(a) - \vec{\mathbf{f}}(b)] = \lim_{\delta \to 0^+} F^{\eta_\delta}(a,b).$$

- Uniform approximation of the absolute value function $(\delta \to 0^+)$

$$\eta'_\delta(r) = \begin{cases} 1 & , \quad r \geq \delta \\ \sin\left(\frac{\pi}{2\delta} r\right) & , \quad -\delta < r < \delta \\ -1 & , \quad r \leq -\delta. \end{cases}$$

Our goal: entropy solution

Definition

A function $u \in \mathcal{N}^2_w(0,T,L^2(\mathbb{R}^d))$ is an **entropy solution** with initial condition u_0 if $u \in L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))$ and, dP-a.s.,

$$
\begin{aligned}
0 \quad &\leq \quad \mu^u_{\eta,k}(\varphi) \\
&:= \quad \int_{\mathbb{R}^d} \eta(u_0-k)\varphi(0)\,\mathrm{d}x + \int_Q [\eta(u-k)\varphi_t - F^\eta(u,k)\nabla\varphi]\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_0^T \int_{\mathbb{R}^d} \eta'(u-k)h(u)\varphi\,\mathrm{d}x\,\mathrm{d}W(t) \\
&\quad + \frac{1}{2}\int_Q \eta''(u-k)h^2(u)\varphi\,\mathrm{d}t\,\mathrm{d}x
\end{aligned}
$$

$\forall \eta \in \mathcal{E}, \varphi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d),\ k \in \mathbb{R}$;

$\lim_{t\to 0^+} E\int_K |u(t,x)-u_0|\,\mathrm{d}x = 0$ for any $K \subset \mathbb{R}^d$ compact.

Our strategy

1. Artificial viscosity method
   $\rightsquigarrow u_\epsilon$.
2. Itô formula with u_ϵ
   $\rightsquigarrow$ viscous entropy formulation.
3. **Existence** : $\epsilon \to 0$ + Young measure theory
   $\rightsquigarrow$ measure-valued solution, entropy formulation.
4. **Uniqueness** : adaptation of the Kruzhkov's method (double variables).

Artificial viscosity method

The viscous case : $\epsilon > 0$

$$du_\epsilon = \left[\epsilon \Delta u_\epsilon + \operatorname{div} \vec{\mathbf{f}}(u_\epsilon)\right] dt + h(u_\epsilon)\, dW(t)$$

with $u_\epsilon(0) = u_0^\epsilon \in \mathcal{D}(\mathbb{R}^d)$.

$\longrightarrow$

G. Da Prato - J. Zabczyk ('92) W. Grecksch - C. Tudor ('95) C. Prévôt - M. Röckner ('07) ../..

$\exists ! u_\epsilon \in \mathcal{N}_w^2(0,T;H^1(\mathbb{R}^d)) \cap L^\infty(0,T;L^p(\Omega \times \mathbb{R}^d))$ *with* $\forall p \geq 2$

$$\left[u_\epsilon - \int_0^\cdot h(u_\epsilon)\right]_t, \Delta u_\epsilon \in L^2(\Omega \times Q)$$

and

A priori estimates

$$\|u_\epsilon\|^2_{L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))} + \epsilon \|u_\epsilon\|^2_{L^2((0,T)\times\Omega;H^1(\mathbb{R}^d))} \leq Cte.$$

Viscous entropy formulation

The **Itô chain rule** with $\psi(t, u_\epsilon) = \int_{\mathbb{R}^d} \eta(u_\epsilon - k)\varphi \, dx$, $\eta \in \mathcal{E}$, $k \in \mathbb{R}$ and $\varphi \in \mathcal{D}^+([0, T] \times \mathbb{R}^d)$, yields

$$
\begin{aligned}
0 \leq \; & \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\,dx + \int_Q \eta(u_\epsilon - k)\varphi_t \,dt\,dx \\
& \underbrace{-\epsilon \int_Q |\nabla u_\epsilon|^2 \eta''(u_\epsilon - k)\varphi \,dt\,dx}_{\leq 0} \underbrace{- \epsilon \int_Q \nabla u_\epsilon \nabla \varphi \eta'(u_\epsilon - k)\,dt\,dx}_{\to 0^+} \\
& - \int_Q \underbrace{\eta'(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon)\nabla\varphi + \varphi\eta''(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon)\nabla u_\epsilon}_{F^\eta(u_\epsilon, k)\nabla\varphi} \,dt\,dx \\
& + \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi \,dx\,dW(t) + \frac{1}{2}\int_Q \eta''(u_\epsilon - k)h^2(u_\epsilon)\varphi \,dt\,dx.
\end{aligned}
$$

Viscous entropy formulation

Passage to the limit $\epsilon \to 0$**:** $\exists \mathbf{u} \in L^2\big((0,T)\times\Omega\times\mathbb{R}^d\times(0,1)\big)$ s.t. dP-a.s.,

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^d} \eta(u_0 - k)\varphi(0)\,\mathrm{d}x + \int_0^1 \int_Q \eta(\mathbf{u}(.,\alpha) - k)\varphi_t\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\alpha \\ & - \int_0^1 \int_Q F^\eta(\mathbf{u}(.,\alpha), k)\nabla\varphi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\alpha \\ & + \int_0^1 \int_0^T \int_{\mathbb{R}^d} \eta'(\mathbf{u}(.,\alpha) - k)h(\mathbf{u}(.,\alpha))\varphi\,\mathrm{d}x\,\mathrm{d}W(t)\,\mathrm{d}\alpha \\ & + \frac{1}{2}\int_0^1 \int_Q \eta''(\mathbf{u}(.,\alpha) - k)h^2(\mathbf{u}(.,\alpha))\varphi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\alpha. \end{aligned}$$

Existence of a measure-valued solution

Definition

A function $\mathbf{u} \in \mathcal{N}_w^2(0,T,L^2(\mathbb{R}^d \times (0,1)))$ is a **measure-valued entropy solution** with initial condition $u_0 \in L^2(\mathbb{R}^d)$ if $\mathbf{u} \in L^\infty\big(0,T,L^2(\Omega \times \mathbb{R}^d \times (0,1))\big)$ and, dP-a.s.,

$$0 \leq \int_0^1 \mu_{\eta,k}^{\mathbf{u}(.,\alpha)}(\varphi)\,\mathrm{d}\alpha \qquad \forall \eta \in \mathcal{E},\ \varphi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d),\ k \in \mathbb{R};$$

$$\lim_{t\to 0^+} E\int_{K\times(0,1)} |\mathbf{u}(t,x,\alpha) - u_0|\,\mathrm{d}(\alpha,x) = 0 \text{ for any } K \subset \mathbb{R}^d \text{ compact.}$$

Theorem

There exists a measure-valued entropy solution.

Uniqueness of the measure-valued solution

Theorem (Local Kato inequality)

Let $\mathbf{u_i}$ $i = 1, 2$ two measure-valued entropy solutions for two initial conditions $u_{0,i}$. Then, for any $\varphi \in \mathcal{D}^+([0,T] \times \mathbb{R}^d)$,

$$\begin{aligned} 0 \quad \leq \quad & E \int_{Q\times(0,1)^2} |\mathbf{u_1}(.,\alpha) - \mathbf{u_2}(.,\beta)| \varphi_t \, d(t,x,\alpha,\beta) \\ & - E \int_{Q\times(0,1)^2} F(\mathbf{u_1}(.,\alpha), \mathbf{u_2}(.,\beta)) \nabla \varphi \, d(t,x,\alpha,\beta) \\ & + \int_{\mathbb{R}^d} |u_{0,1} - u_{0,2}| \varphi(0,.) \, dx \end{aligned}$$

Consequence:

$u_{0,1} = u_{0,2}$ and $\varphi(t,x) = \psi(|x| + \omega t) \frac{(T-t)^+}{T}$ yields

$$E \int_{Q\times(0,1)^2} |\mathbf{u_1}(.,\alpha) - \mathbf{u_2}(.,\beta)| \psi(|x| + \omega t) \, \mathrm{d}(t,x,\alpha,\beta) = 0$$

$$\rightsquigarrow \mathbf{u}_1(t,x,\alpha) = \mathbf{u}_2(t,x,\beta) = u(t,x).$$

Proof

We would like to use Kruzhkov's method (double variables):

$\mathbf{u_1}(t,x,\alpha), k := \mathbf{u_2}(s,y,\beta)$, test with $\varphi\rho_n(t-s)\rho_m(x-y) := \Psi_n^m(s,t,x,y)$,

$$0 \leq \int_0^1 \mu_{\eta_\delta,\mathbf{u_2}(s,y,\beta)}^{\mathbf{u_1}(\cdot,\alpha)}[\varphi\rho_n(.-s)\rho_m(.-y)]\,\mathrm{d}\alpha$$

then,

$$\begin{aligned} \eta_\delta &\to |.|, & & \int_{Q\times(0,1)}, \\ n &\to +\infty & i.e. \quad & t = s \\ \text{and } m &\to +\infty & i.e. \quad & x = y. \end{aligned}$$

Proof

We would like to use Kruzhkov's method (double variables):

$\mathbf{u_1}(t,x,\alpha)$, $k := \mathbf{u_2}(s,y,\beta)$, test with $\varphi\rho_n(t-s)\rho_m(x-y) := \Psi_n^m(s,t,x,y)$,

$$0 \leq \int_0^1 \mu_{\eta_\delta,\mathbf{u_2}(s,y,\beta)}^{\mathbf{u_1}(\cdot,\alpha)}[\varphi\rho_n(.-s)\rho_m(.-y)]\,d\alpha$$

$$\begin{aligned}0 \leq\ & \cdots\text{classical terms}\cdots \\ & + \int_0^1\int_0^T\int_{\mathbb{R}^d} \eta_\delta'(\mathbf{u_1}(t,x,\alpha) - \mathbf{u_2}(s,y,\beta))h(\mathbf{u}_1(t,x,\alpha))\Psi_n^m\,dx\,dW(t)\,d\alpha \\ & + \frac{1}{2}\int_0^1\int_Q \eta_\delta''(\mathbf{u}_1(t,x,\alpha) - \mathbf{u}_2(s,y,\beta))h^2(\mathbf{u}_1(t,x,\alpha))\Psi_n^m\,dt\,dx\,d\alpha.\end{aligned}$$

Pb_1 : ↖ not $\mathcal{F}_t$ measurable when $s > t$!

Proof

We would like to use Kruzhkov's method (double variables):

$\mathbf{u_1}(t,x,\alpha), k := \mathbf{u_2}(s,y,\beta)$, test with $\varphi\rho_n(t-s)\rho_m(x-y) := \Psi_n^m(s,t,x,y)$,

$$0 \leq \int_0^1 \mu_{\eta_\delta,\mathbf{u_2}(s,y,\beta)}^{\mathbf{u_1}(\cdot,\alpha)}[\varphi\rho_n(.-s)\rho_m(.-y)]\,\mathrm{d}\alpha$$

$$\begin{aligned} 0 \leq\ & \cdots\text{classical terms}\cdots \\ &+ \int_0^1\int_0^T\int_{\mathbb{R}^d}\eta'_\delta(\mathbf{u}_1(\cdot,\alpha)-\mathbf{u}_2(s,y,\beta))h(\mathbf{u}_1(\cdot,\alpha))\Psi_n^m\,\mathrm{d}x\,\mathrm{d}W(t)\,\mathrm{d}\alpha \\ &+ \frac{1}{2}\int_0^1\int_Q\eta''_\delta(\mathbf{u}_1(\cdot,\alpha)-\mathbf{u}_2(s,y,\beta))h^2(\mathbf{u}_1(\cdot,\alpha))\Psi_n^m\,\mathrm{d}(t,x)\,\mathrm{d}\alpha. \end{aligned}$$

Pb_2 : $\nwarrow \eta_\delta \to |\cdot|$! Tanaka formula - local time?

Proof

We would like to use Kruzhkov's method (double variables):

$\mathbf{u_1}(t,x,\alpha)$, $\mathbf{u_2}(s,y,\beta)$, test with $\varphi\rho_n(t-s)\rho_m(x-y) := \Psi_{n,m}(s,t,x,y)$,

$$0 \leq E\int_{Q\times(0,1)}\int_{\mathbb{R}}\int_0^1 \mu_{\eta_\delta,k}^{\mathbf{u_1}(\cdot,\alpha)}\Psi_{n,m}(s,.,.,y)\,\mathrm{d}\alpha\,\rho_l(\mathbf{u_2}(s,y,\beta)-k)\,\mathrm{d}k\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}\beta.$$

Then,

- $n \to +\infty$ (time),
- $l \to +\infty$ $(k = \mathbf{u_2}(t,y,\beta))$,
- $\eta_\delta \to |.|$,
- $m \to +\infty$.

Proof

Still difficulties to treat the stochastic integral:

$\mathbf{u_1}(t,x,\alpha)$, ~~$\mathbf{u_2}$~~ $\rightarrow u_\epsilon$ (viscous regular solution),

$$0 \leq E\int_Q \int_{\mathbb{R}} \int_0^1 \mu_{\eta_\delta,k}^{\mathbf{u_1}(\cdot,\alpha)}[\varphi\rho_n(.-s)\rho_m(.-y)]\,\mathrm{d}\alpha\,\rho_l(u_\epsilon(s,y)-k)\,\mathrm{d}k\,dsdyd\beta$$
$$+O(\sqrt{\epsilon})$$

Then,

- Stochastic calculus with u_ϵ,
- $n \rightarrow +\infty$ (time),
- $l \rightarrow +\infty$ $(k = u_\epsilon(t,y))$,
- $\eta_\delta \rightarrow |.|$,
- $\epsilon \rightarrow 0$,
- $m \rightarrow +\infty$.

Extension and open problem

- **Dirichlet problem**

$$du = \operatorname{div} \vec{f}(u)\, dt + h(u)\, dW \qquad \text{in } \Omega \times (0, T) \times D$$

with $u = 0$ on ∂D, $D \subset \mathbb{R}^d$ a bounded domain.

Problem:

- use of semi-Kruzhkov entropies $(\cdot - k)^+$ and $(k - \cdot)^-$ (non-smooth, not pair).

- **Degenerate stochastic hyperbolic-parabolic equations**

$$du = [\Delta\phi(u) + \operatorname{div} \vec{f}(u)]\, dt + h(u)\, dW \qquad \text{in } \Omega \times (0, T) \times D$$

with ϕ nondecreasing.

Problem:

- missing regularity of u_ϵ of viscous problem,
- infinite propagation speed in $\mathbb{R}^d$.