Low Mach Number Limits to the Navier-Stokes-Smoluchowski System

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Outline

Fluid-Particle System

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0 \tag{1}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \left(p_F(\varrho) + \frac{D}{\zeta} \eta \right)$$

$$= \mu \triangle_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u} - (\eta + \beta \varrho) \nabla_x \Phi \tag{2}$$

$$\partial_t \eta + \mathrm{div}_x (\eta (\mathbf{u} - \zeta \nabla_x \Phi)) - D \triangle_x \eta = 0 \tag{3}$$

where $p_F(\varrho) = a\varrho^\gamma$ for some $a > 0$, $\gamma > 1$, $\beta \neq 0$.
We also impose Newton's law on the fluid

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Boundary and Initial Conditions

- $\mathbf{u} = D\nabla_x \eta \cdot \mathbf{n} + \zeta \eta \nabla_x \Phi \cdot \mathbf{n} = 0$ on $(0, T) \times \partial\Omega$
- $0 \leq \varrho(0, x) = \varrho_0 \in L^\gamma(\Omega)$
- $(\varrho \mathbf{u})(0, x) = \mathbf{m}_0 \in L^{6/5}(\Omega; \mathbb{R}^3)$
- $0 \leq \eta(0, x) = \eta_0 \in L^2(\Omega)$

Smoluchowski Equation and Vlasov-Fokker-Planck Equation

The cloud of particles is described by its distribution function $f_\varepsilon(t, x, \xi)$ on phase space, which is the solution to the dimensionless Vlasov-Fokker-Planck equation

$$\partial_t f_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \left(\xi \cdot \nabla_x f_\varepsilon - \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon \right) = \frac{1}{\varepsilon} \operatorname{div}_\xi \left(\left(\xi - \sqrt{\varepsilon} u_\varepsilon \right) f + \nabla_\xi f_\varepsilon \right).$$

The friction force is assumed to follow Stokes law and thus is proportional to the relative velocity vector, i.e., is proportional to the fluctuations of the microscopic velocity $\xi \in \mathbb{R}^3$ around the fluid velocity field $\mathbf{u}$.

The RHS of the momentum equation in the Navier-Stokes system takes into account the action of the cloud of particles on the fluid through the forcing term

$$F_\varepsilon = \int_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - u_\varepsilon(t,x) \right) f(t,x,\xi) \mathrm{d}\xi.$$

The density of the particles $\eta_\varepsilon(t,x)$ is related to the probability distribution function $f_\varepsilon(t,x,\xi)$ through the relation

$$\eta_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,\xi) \mathrm{d}\xi.$$

Confinement Hypothesis

Let $\Omega \subset \mathbb{R}^3$ be a $C^{2,\nu}$ domain with $\nu > 0$ and $\Phi : \Omega \to \mathbb{R}_0^+$ with $\inf_{x\in\Omega} \Phi(x) = 0$. (Ω, Φ) satisfies the Confinement Hypotheses (HC) iff

- If Ω is bounded, Φ is bounded and Lipschitz continuous on $\overline{\Omega}$.
- If Ω is unbounded, $\Phi \in W^{1,\infty}_{\text{loc}}(\Omega)$, $e^{-\Phi/2} \in L^1(\Omega)$ and

$$|\Delta_x \Phi(x)| \leq c_1 |\nabla_x \Phi(x)| \leq c_2 \Phi(x)$$

for $|x|$ greater than some large R.

Weak Formulation

- $\varrho \geq 0, \mathbf{u}$ is a renormalized solution of the continuity equation, i.e.,

$$\int_0^\infty \int_\Omega B(\varrho)\partial_t \varphi + B(\varrho)\mathbf{u} \cdot \nabla_x \varphi - b(\varrho)\mathrm{div}_x \mathbf{u} \varphi \mathrm{d}x\mathrm{d}t = - \int_\Omega B(\varrho_0)\varphi(0,\cdot)\mathrm{d}x \tag{4}$$

where $b \in L^\infty \cap C[0,\infty), B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z$

- The momentum balance holds in the sense of distributions:

$$\int_0^\infty \int_\Omega \varrho\mathbf{u} \cdot \partial_t \mathbf{v} + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{v} + \left(a\varrho^\gamma + \frac{D}{\zeta}\eta \right) \mathrm{div}_x \mathbf{v} \mathrm{d}x\mathrm{d}t = \int_0^\infty \int_\Omega \mu \nabla_x \mathbf{u} \nabla_x \mathbf{v} \mathrm{d}x\mathrm{d}t - \int_\Omega \mathbf{m}_0 \cdot \mathbf{v}(0,\cdot)\mathrm{d}x + \int_0^\infty \int_\Omega \lambda \mathrm{div}_x \mathbf{u} \mathrm{div}_x \mathbf{v} - (\eta + \beta\varrho)\nabla_x \Phi \cdot \mathbf{v} \mathrm{d}x\mathrm{d}t \tag{5}$$

Weak Formulation (continued)

- $\eta \geq 0$ is a weak solution of the Smoluchowski equation:

$$\int_0^\infty \int_\Omega \eta \partial_t \varphi + \eta \mathbf{u} \cdot \nabla_x \varphi - \zeta \eta \nabla_x \Phi \cdot \nabla_x \varphi - D \nabla_x \eta \cdot \nabla_x \varphi \mathrm{d}x \mathrm{d}t$$
$$= - \int_\Omega \eta_0 \varphi(0, \cdot) \mathrm{d}x \tag{6}$$

- $\mathcal{E}(\varrho, \mathbf{u}, \eta)(t) := \int_\Omega \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{a}{\gamma - 1} \varrho^\gamma + \frac{D}{\zeta} \eta \ln \eta + (\beta \varrho + \eta) \Phi \mathrm{d}x$ satisfies $\mathcal{E}(\varrho, \mathbf{u}, \eta) \leq \mathcal{E}(\varrho_0, \mathbf{u}_0, \eta_0)$ and

$$\int_0^\infty \int_\Omega \mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 + \left| 2 \nabla_x \frac{D}{\sqrt{\zeta}} \sqrt{\eta} + \sqrt{\zeta \eta} \nabla_x \Phi \right|^2 \mathrm{d}x \mathrm{d}t$$
$$\leq \mathcal{E}(\varrho_0, \mathbf{u}_0, \eta_0)$$

Existence of Weak Solutions

- Carrillo, Karper, and Trivisa showed the existence of free energy solutions using an approximation scheme based upon time-discretization, and investigated the asymptotic behavior toward a steady-state solution.
- In a recent paper, we show the existence of *suitable weak solutions* obeying a relative entropy inequality.

Scaling the System

- Begin by defining a reference length, time, density, etc. to the various quantities in the system such that

$$\alpha = \alpha_{\mathsf{ref}} \alpha'.$$

- Substitute into the system and manipulate to obtain a scaled system with various non-dimensional parameters.

We can then make certain parameters small or large (or order 1), and take them to zero or infinity to obtain a system for the limit values $\overline{\varrho}, \overline{\mathbf{u}}, \overline{\eta}$.

Scaled Model

$$\mathsf{Sr}\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{7}$$

$$\mathsf{Sr}\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathsf{Ma}^2}\nabla_x \left(a\varrho^\gamma + \mathsf{Pc}\frac{D}{\zeta}\eta \right)$$

$$= \frac{1}{\mathsf{Re}}(\mu \Delta_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}) - \frac{1}{\mathsf{Fr}^2}(\beta \varrho + \mathsf{Dc}\eta)\nabla_x \Phi \tag{8}$$

$$\mathsf{Sr}\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u}) - \mathsf{Za}\mathrm{div}_x(\zeta \eta \nabla_x \Phi) - \mathsf{Da} D \Delta_x \eta = 0 \tag{9}$$

Scaled Energy Inequality

$$\mathrm{Sr}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \mathrm{Pc}\frac{D\eta}{\zeta}\ln\eta + \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}(\beta\varrho + \mathrm{Dc}\eta)\Phi \mathrm{d}x$$
$$+ \int_\Omega \mathrm{PcDa}D^2\frac{|\nabla_x\eta|^2}{\zeta\eta} + 2\mathrm{Za}D\nabla_x(\eta)\cdot\nabla_x\Phi + \frac{\mathrm{Za}^2}{\mathrm{Da}}\zeta\eta|\nabla_x\Phi|^2 \mathrm{d}x$$
$$+ \int_\Omega \frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\mathrm{d}x \le 0$$

Related Singular Limits Work

Isentropic Navier-Stokes and Navier-Stokes-Fourier systems for compressible fluids for various scalings (for example, Lions and Masmoudi, Desjardins et al., Feireisl et al., . . .).

Low Mach Number Low Stratification Limit

- Ma is taken to be a small parameter $\varepsilon > 0$.
- Za is taken to be equal to Ma.
- Fr is taken to be of order $\sqrt{\varepsilon}$.
- Other parameters are taken to be of order 1.

Low Stratification

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{10}$$

$$\varepsilon^2[\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] + \nabla_x \left(a\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon \right)$$

$$= \varepsilon^2(\mu \Delta_x \mathbf{u}_\varepsilon + \lambda \nabla_x \mathrm{div}_x \mathbf{u}_\varepsilon) - \varepsilon(\beta \varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \tag{11}$$

$$\partial_t \eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon) - \varepsilon \mathrm{div}_x(\zeta \eta_\varepsilon \nabla_x \Phi) - D\Delta_x \eta_\varepsilon = 0 \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma - 1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta}\ln \eta_\varepsilon + \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi \mathrm{d}x$$

$$+ \int_\Omega D^2 \frac{|\nabla_x \eta_\varepsilon|^2}{\zeta \eta_\varepsilon} + 2\varepsilon D \nabla_x \eta_\varepsilon \cdot \nabla_x \Phi + \varepsilon^2 \zeta \eta_\varepsilon |\nabla_x \Phi|^2 \mathrm{d}x$$

$$+ \int_\Omega \varepsilon^2 \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \mathrm{d}x \le 0 \tag{13}$$

Formal Evaluation of the Low Stratification Low Mach Number Limit

- Assume the following expansions:

$$\varrho_\varepsilon = \overline{\varrho} + \sum_{i=1}^{\infty} \varepsilon^i \varrho_\varepsilon^{(i)}$$

$$\eta_\varepsilon = \overline{\eta} + \sum_{i=1}^{\infty} \varepsilon^i \eta_\varepsilon^{(i)}$$

$$\mathbf{u}_\varepsilon = \overline{\mathbf{u}} + \sum_{i=1}^{\infty} \varepsilon^i \mathbf{u}_\varepsilon^{(i)}$$

- By considering the energy inequality, $\nabla_x \overline{\eta} = 0$, so $\overline{\eta} = \frac{1}{|\Omega|} \int_\Omega \eta_0(x) \mathrm{d}x$.
- By equating terms of order 1 in the momentum equation, $\nabla_x \left(a \overline{\varrho}^\gamma + \frac{D}{\zeta} \overline{\eta} \right) = 0$, implying $\overline{\varrho} = \frac{1}{|\Omega|} \int_\Omega \varrho_0(x) \mathrm{d}x$.
- Thus, $\overline{\mathbf{u}}$ satisfies the incompressibility condition $\mathrm{div}_x \overline{\mathbf{u}} = 0$.

Low Stratification Limit

$$\overline{\eta} = \frac{1}{|\Omega|} \int_{\Omega} \eta_0(x) dx \tag{14}$$

$$\overline{\varrho} = \frac{1}{|\Omega|} \int_{\Omega} \varrho_0(x) dx \tag{15}$$

$$\mathrm{div}_x \overline{\mathbf{u}} = 0 \tag{16}$$

$$\overline{\varrho}[\partial_t \overline{\mathbf{u}} + \mathrm{div}_x(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}})] + \nabla_x \Pi = \mu \Delta_x \overline{\mathbf{u}} - (\beta r + \theta) \nabla_x \Phi \tag{17}$$

where r, θ satisfy

$$\nabla_x \left(a r^{\gamma} + \frac{D}{\zeta} \theta \right) = -(\beta \overline{\varrho} + \overline{\eta}) \nabla_x \Phi$$

Low Stratification System Weak Formulation

$\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ form a weak solution to the scaled low stratification equations if:
$\varrho_\varepsilon \geq 0$ and $\mathbf{u}_\varepsilon$ form a renormalized solution of the scaled continuity equation, i.e.,

$$\int_0^T \int_\Omega B(\varrho_\varepsilon)\partial_t \varphi + B(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \varphi - b(\varrho_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon \varphi \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega B(\varrho_0)\varphi(0, \cdot)\mathrm{d}x \tag{18}$$

where $b \in L^\infty \cap C[0, \infty)$, $B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z$.

Low Stratification System Weak Formulation (continued)

The scaled momentum balance holds in the sense of distributions:

$$\int_0^T \int_\Omega \varepsilon^2 \left(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \mathbf{v} + \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{v} \right) + \left(p_F(\varrho_\varepsilon) + \frac{D}{\zeta} \eta_\varepsilon \right) \mathrm{div}_x \mathbf{v} \mathrm{d}x \mathrm{d}t$$
$$= \int_0^T \int_\Omega \varepsilon^2 \left(\mu \nabla_x \mathbf{u}_\varepsilon \nabla_x \mathbf{v} + \lambda \mathrm{div}_x \mathbf{u}_\varepsilon \mathrm{div}_x \mathbf{v} \right) - \varepsilon(\beta \varrho_\varepsilon + \eta_\varepsilon) \nabla_x \Phi \cdot \mathbf{v} \mathrm{d}x \mathrm{d}t$$
$$- \varepsilon^2 \int_\Omega \mathbf{m}_0 \cdot \mathbf{v}(0, \cdot) \mathrm{d}x \tag{19}$$

Low Stratification System Weak Formulation (continued)

- $\eta_\varepsilon \geq 0$ is a weak solution of the scaled Smoluchowski equation:

$$\int_0^T \int_\Omega \eta_\varepsilon \partial_t \varphi + \eta_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi - \zeta \eta_\varepsilon \nabla_x \Phi \cdot \nabla_x \varphi - D \nabla_x \eta_\varepsilon \cdot \nabla_x \varphi \mathrm{d}x \mathrm{d}t$$
$$= - \int_\Omega \eta_0 \varphi(0, \cdot) \mathrm{d}x \tag{20}$$

- The energy inequality is satisfied:

$$\int_\Omega \frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma - 1} \varrho_\varepsilon^\gamma + \frac{D}{\zeta} \eta_\varepsilon \ln \eta_\varepsilon + \varepsilon(\beta \varrho_\varepsilon + \eta_\varepsilon) \Phi \mathrm{d}x(T)$$
$$+ \int_0^T \int_\Omega \varepsilon^2 (\mu |\nabla_x \mathbf{u}_\varepsilon|^2 + \lambda |\mathrm{div}_x \mathbf{u}_\varepsilon|^2) \mathrm{d}x \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \left| 2 \frac{D}{\sqrt{\zeta}} \nabla_x \sqrt{\eta_\varepsilon} + \varepsilon \sqrt{\zeta \eta_\varepsilon} \nabla_x \Phi \right|^2 \mathrm{d}x \mathrm{d}t$$
$$\leq \int_\Omega \frac{\varepsilon^2}{2} \varrho_0 |\mathbf{u}_0|^2 + \frac{a}{\gamma - 1} \varrho_0^\gamma + \frac{D}{\zeta} \eta_0 \ln \eta_0 + \varepsilon(\beta \varrho_0 + \eta_0) \Phi \mathrm{d}x \tag{21}$$

Target System

We say that $\{\overline{\mathbf{u}}, r, s\}$ solve the *Target System* if

$$\mathrm{div}_x \overline{\mathbf{u}} = 0 \text{ weakly on } (0,T) \times \Omega,$$

$$\int_0^T \int_\Omega \overline{\varrho}\overline{\mathbf{u}} \cdot \partial_t \mathbf{v} + \overline{\varrho}\overline{\mathbf{u}} \otimes \overline{\mathbf{u}} : \nabla_x \mathbf{v} \mathrm{d}x \mathrm{d}t$$

$$= \int_0^T \int_\Omega (\mu \nabla_x \overline{\mathbf{u}} - (\beta r + s) \nabla_x \Phi) \cdot \mathbf{v} \mathrm{d}x \mathrm{d}t - \int_\Omega \overline{\varrho}\overline{\mathbf{u}} \cdot \mathbf{v}(0,\cdot) \mathrm{d}x,$$

for any divergence-free text function $\mathbf{v}$ and

$$r = -\frac{1}{a\gamma\overline{\varrho}^{\gamma-1}} \left[(\beta\overline{\varrho} + \overline{\eta})\Phi + \frac{D}{\zeta} s \right]$$

weakly.

Main Result

Theorem: Let (Ω, Φ) satisfy the confinement hypothesis and for each $\varepsilon > 0$, $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ solves (18)-(21). Assume the initial data can be expressed as follows:

$$\varrho_\varepsilon(0,\cdot) = \varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{\varepsilon,0},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{\varepsilon,0},\ \text{and}\ \eta_\varepsilon(0,\cdot) = \eta_{\varepsilon,0} = \overline{\eta} + \varepsilon \eta^{(1)}_{\varepsilon,0}.$$

where $\overline{\varrho}, \overline{\eta}$ are the spatially uniform densities on Ω. Assume also that as $\varepsilon \to 0$,

$$\varrho^{(1)}_{\varepsilon,0} \rightharpoonup \varrho^{(1)}_0, \mathbf{u}_{\varepsilon,0} \rightharpoonup \overline{\mathbf{u}}_0, \eta^{(1)}_{\varepsilon,0} \rightharpoonup \eta^{(1)}_0$$

weakly-$*$ in $L^\infty(\Omega)$ or $L^\infty(\Omega; \mathbb{R}^3)$ as the case may be.

Main Result (continued)

Then up to a subsequence and letting $q := \min\{\gamma, 2\}$,

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^q(\Omega))$$

$$\eta_\varepsilon \to \overline{\eta} \text{ in } L^2(0,T; L^2(\Omega))$$

$$\mathbf{u}_\varepsilon \to \overline{\mathbf{u}} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3))$$

and

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{weakly-} * \text{ in } L^\infty(0,T; L^q(\Omega))$$

$$\eta_\varepsilon^{(1)} = \frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon} \to \eta^{(1)} \text{ weakly in } L^2(0,T; L^2(\Omega))$$

where $\{\overline{\mathbf{u}}, \varrho^{(1)}, \eta^{(1)}\}$ solve the target system mentioned previously.

Free Energy Inequality

Recasting the energy inequality using the free energy

$$
\begin{aligned}
E_F(\varrho) + E_P(\eta) :=& \frac{a}{\gamma - 1}\varrho^\gamma - (\varrho - \overline{\varrho})\frac{a\gamma}{\gamma - 1}\overline{\varrho}^{\gamma-1} - \frac{a}{\gamma - 1}\overline{\varrho}^\gamma \\
&+ \frac{D}{\zeta}\eta \ln \eta - \frac{D}{\zeta}(\eta - \overline{\eta})(\ln \overline{\eta} + 1) - \frac{D}{\zeta}\overline{\eta}\ln \overline{\eta},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(E_F(\varrho_\varepsilon) + E_P(\eta_\varepsilon)) + \frac{1}{\varepsilon}(\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi \mathrm{d}x(T) \\
&+ \int_0^T \int_\Omega \mu|\nabla_x \mathbf{u}_\varepsilon|^2 + \lambda|\mathrm{div}_x \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left|\frac{2D\nabla_x\sqrt{\eta_\varepsilon}}{\sqrt{\zeta}} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right|^2 \mathrm{d}x\mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}(E_F(\varrho_0) + E_P(\eta_0)) + \frac{1}{\varepsilon}(\beta\varrho_0 + \eta_0)\Phi \mathrm{d}x \qquad (22)
\end{aligned}
$$

Momentum Equation

By using the uniform bounds and Sobolev embeddings, $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ converges to a limit $\overline{\varrho \mathbf{u} \otimes \mathbf{u}}$. Thus, the momentum equation converges to becomes

$$\int_0^T \int_\Omega \overline{\varrho}\overline{\mathbf{u}} \cdot \partial_t \mathbf{v} + \overline{\varrho \mathbf{u} \otimes \mathbf{u}} : \nabla_x \mathbf{v} \mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \int_\Omega \mu \nabla_x \overline{\mathbf{u}} : \nabla_x \mathbf{v} - (\beta \varrho^{(1)} + \eta^{(1)}) \nabla_x \Phi \cdot \mathbf{v} \mathrm{d}x\mathrm{d}t - \int_\Omega \overline{\varrho_0}\overline{\mathbf{u}_0} \cdot \mathbf{v} \mathrm{d}x$$

By dividing (19) by ε and taking $\varepsilon \to 0^+$, we have weakly

$$\varrho^{(1)} = -\frac{1}{a\gamma \overline{\varrho}^{\gamma-1}} \left[(\beta \overline{\varrho} + \overline{\eta}) \Phi + \frac{D}{\zeta} \eta^{(1)} \right]$$

Helmholtz Decomposition

Consider a vector $\mathbf{v} \in \mathbb{R}^3$. We can decompose the vector into a gradient part

$$\mathbf{H}^{\perp}[\mathbf{v}] := \nabla_x \Delta_x^{-1} \mathrm{div}_x \mathbf{v}$$

and a divergence-free part

$$\mathbf{H}[\mathbf{v}] := \mathbf{v} - \mathbf{H}^{\perp}[\mathbf{v}]$$

Note that the Helmholtz projections are continuous and linear.

Convective Term

We decompose the tensor $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ using the Helmholtz projections into

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon + \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] + \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon]$$

Using the properities of the Helmholtz projections and the convergence results earlier,

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}[\overline{\varrho}\overline{\mathbf{u}}] = \overline{\varrho}\overline{\mathbf{u}}$$

in $C_{\text{weak}}([0,T]; L^{2q/q+1}(\Omega;\mathbb{R}^3))$,
and $\mathbf{H}[\mathbf{u}_\varepsilon] \to \overline{\mathbf{u}}$ in $L^2(0,T; L^2(\Omega;\mathbb{R}^3))$, so

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon \to \overline{\varrho}\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}$$

$$\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] \to 0$$

weakly in $L^2(0,T; L^{6q/4q+3}(\Omega;\mathbb{R}^{3\times 3}))$.

After some manipulations, the scaled NSS system becomes

$$\int_0^T \int_\Omega \varepsilon r_\varepsilon \partial_t \phi + \mathbf{V}_\varepsilon \cdot \nabla_x \phi \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \mathbf{h}^2_\varepsilon \cdot \nabla_x \phi \mathrm{d}x\mathrm{d}t \tag{23}$$

$$\int_0^T \int_\Omega \varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \mathbf{v} + \omega r_\varepsilon \mathrm{div}_x \mathbf{v} \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega [\beta(\overline{\varrho} - \varrho_\varepsilon) + (\overline{\eta} - \eta_\varepsilon)] \nabla_x \Phi \cdot \mathbf{v} + h^1_\varepsilon : \nabla_x \mathbf{v} - h^3_\varepsilon \mathrm{div}_x \mathbf{v} \mathrm{d}x\mathrm{d}t \tag{24}$$

where

$$\mathbf{V}_\varepsilon := \varrho_\varepsilon \mathbf{u}_\varepsilon$$
$$r_\varepsilon := \varrho^{(1)}_\varepsilon + \frac{D}{a\gamma\overline{\varrho}^{\gamma-1}\zeta}\eta^{(1)}_\varepsilon + \frac{(\beta\overline{\varrho} + \overline{\eta})\Phi}{a\gamma\overline{\varrho}^{\gamma-1}}$$
$$\omega := a\gamma\overline{\varrho}^{\gamma-1}$$

equal to Ma and h^1_ε, $\mathbf{h}^2_\varepsilon$, and h^3_ε are quantities converging to zero. Note also that $[\beta(\overline{\varrho} - \varrho_\varepsilon) + (\overline{\eta} - \eta_\varepsilon)]\nabla_x \Phi$ converges weakly to zero.

In light of (23)-(24), we consider the eigenvalue problem

$$-\Delta_x q = \Lambda q$$
$$\nabla_x q \cdot \mathbf{n}|_{\partial\Omega} = 0$$
$$-\Lambda = \frac{\lambda^2}{\omega}$$

with a countable system of eigenvalues
$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \ldots$ and eigenvectors $\{q_n\}_{n=0}^{\infty}$.

Decomposition of $L^2(\Omega;\mathbb{R}^3)$

Defining

$$\mathbf{w}_{\pm n} := \pm i\sqrt{\frac{\omega}{\Lambda_n}}\nabla_x q_n$$

where q_n, Λ_n are defined from the previous eigenvalue problem. Thus, we decompose the space

$$L^2(\Omega;\mathbb{R}^3) = L^2_\sigma(\Omega;\mathbb{R}^3) \oplus L^2_g(\Omega;\mathbb{R}^3)$$

where

$$L^2_g(\Omega;\mathbb{R}^3) := \text{closure}_{L^2}\left\{\text{span}\left\{\frac{-i}{\omega}\mathbf{w}_n\right\}_{n=1}^{\infty}\right\}$$
$$L^2_\sigma(\Omega;\mathbb{R}^3) := \text{closure}_{L^2}\{\mathbf{v}\in C_c^\infty(\Omega;\mathbb{R}^3)|\mathrm{div}_x\mathbf{v}=0\}$$

and define the projection

$$\mathbf{P}_M : L^2(\Omega;\mathbb{R}^3) \to \mathrm{span}\left\{\frac{-i}{\sqrt{\omega}}\mathbf{w}_n\right\}_{n\le M}$$

Note that we define $\mathbf{H}^\perp_M := \mathbf{P}_M\mathbf{H}^\perp = \mathbf{H}^\perp\mathbf{P}_M$ since the operators $\mathbf{H}^\perp$ and $\mathbf{P}_M$ commute.

Return to the Singular Term

Rewriting the singular term and noting convergences, the problem of showing the singular term converges weakly to a gradient reduces to showing

$$\int_0^T \int_\Omega \mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] : \nabla_x \mathbf{v} \mathrm{d}x \mathrm{d}t \to 0$$

for each fixed $M \in \mathbb{N}$ as $\varepsilon \to 0$.

Weak Convergence to a Gradient

Rewriting (23) and (24) in terms of the Helmholtz projectors and some manipulations yield

$$\varepsilon\partial_t[r_\varepsilon]_M + \mathrm{div}_x(\mathbf{H}_M^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]) = \chi^3_{\varepsilon,M} \tag{25}$$

$$\varepsilon\partial_t\mathbf{H}_M^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon] + \omega\nabla_x[r_\varepsilon]_M = \chi^4_{\varepsilon,M} \tag{26}$$

where

$$[r_\varepsilon]_M := \sum_{n=1}^{M} b_n[r_\varepsilon]q_n$$

Analysis in the spirit of Feireisl and Novotný, . . . of the above shows the singular term converges to a gradient

Scaled Model

$$\mathsf{Sr}\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{27}$$

$$\mathsf{Sr}\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathsf{Ma}^2}\nabla_x \left(a\varrho^\gamma + \mathsf{Pc}\frac{D}{\zeta}\eta \right)$$

$$= \frac{1}{\mathsf{Re}}(\mu \Delta_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}) - \frac{1}{\mathsf{Fr}^2}(\beta\varrho + \mathsf{Dc}\eta)\nabla_x \Phi \tag{28}$$

$$\mathsf{Sr}\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u}) - \mathsf{Za}\mathrm{div}_x(\zeta\eta\nabla_x\Phi) - \mathsf{Da} D \Delta_x \eta = 0 \tag{29}$$

Scaled Energy Inequality

$$\mathrm{Sr}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \mathrm{Pc}\frac{D\eta}{\zeta}\ln\eta + \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}(\beta\varrho + \mathrm{Dc}\eta)\Phi \mathrm{d}x$$

$$+ \int_\Omega \mathrm{PcDa}\,D^2\frac{|\nabla_x\eta|^2}{\zeta\eta} + 2\mathrm{Za}\,D\nabla_x(\eta)\cdot\nabla_x\Phi + \frac{\mathrm{Za}^2}{\mathrm{Da}}\zeta\eta|\nabla_x\Phi|^2\mathrm{d}x$$

$$+ \int_\Omega \frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\mathrm{d}x \leq 0$$

Strong Stratification

- Ma is taken to be a small parameter $\varepsilon > 0$.
- Za, Da are taken to be ε^{-1}.
- Fr is taken to be of order ε.
- Other parameters are taken to be of order 1.
- We also assume that $\Phi = gx_3$ where g is a constant (gravity/buoyancy).

Strong Stratification

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{30}$$

$$\varepsilon^2[\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] + \nabla_x \left(a\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon \right)$$

$$= \varepsilon^2(\mu \Delta_x \mathbf{u}_\varepsilon + \lambda \nabla_x \mathrm{div}_x \mathbf{u}_\varepsilon) - (\beta \varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \tag{31}$$

$$\varepsilon\left[\partial_t \eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon)\right] - \mathrm{div}_x(\zeta \eta_\varepsilon \nabla_x \Phi) - D\Delta_x \eta_\varepsilon = 0 \tag{32}$$

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma - 1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta} \ln \eta_\varepsilon + (\beta \varrho_\varepsilon + \eta_\varepsilon)\Phi \mathrm{d}x$$

$$+ \varepsilon \int_\Omega \varepsilon^2 \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \mathrm{d}x + \int_\Omega \left| D\frac{\nabla_x \eta_\varepsilon}{\sqrt{\zeta \eta_\varepsilon}} + \sqrt{\zeta \eta_\varepsilon}\nabla_x \Phi \right|^2 \mathrm{d}x \leq 0 \tag{33}$$

Formal Evaluation

Assuming that

$$\varrho_\varepsilon = \tilde{\varrho} + \sum_{i=1}^{\infty} \varepsilon^i \varrho_\varepsilon^{(i)}$$

$$\eta_\varepsilon = \tilde{\eta} + \sum_{i=1}^{\infty} \varepsilon^i \eta_\varepsilon^{(i)}$$

$$\mathbf{u}_\varepsilon = \tilde{\mathbf{u}} + \sum_{i=1}^{\infty} \varepsilon^i \mathbf{u}_\varepsilon^{(i)}$$

and using (30)-(33), we formally obtain

$$g\tilde{\eta} = -\frac{D}{\zeta}\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}x_3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[a\tilde{\varrho}^\gamma\right] = -\beta g \tilde{\varrho}$$

$$\mathrm{div}_x(\tilde{\varrho}\tilde{\mathbf{u}}) = 0$$

$$\tilde{\varrho}\partial_t\tilde{\mathbf{u}} + \mathrm{div}_x(\tilde{\varrho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}) + \nabla_x \Pi = \mu\Delta_x\tilde{\mathbf{u}} + \lambda\nabla_x\mathrm{div}_x\tilde{\mathbf{u}} - \left(\beta\varrho^{(2)} + \eta^{(2)}\right)\nabla_x\Phi$$

Target System: Weak Formulation

$$\int_0^T \int_\Omega \tilde{\varrho}\tilde{\mathbf{u}} \cdot \nabla_x \phi \mathrm{d}x\mathrm{d}t = 0 \tag{34}$$

for all $\phi \in C_C^\infty((0,T) \times \Omega)$

$$g\tilde{\eta} = -\frac{D}{\zeta}\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}x_3} \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[a\tilde{\varrho}^\gamma\right] = -\beta g\tilde{\varrho} \tag{36}$$

$$\begin{aligned}\int_0^T \int_\Omega \tilde{\varrho}\tilde{\mathbf{u}} \cdot \mathbf{w} + \tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \nabla_x \mathbf{w} \mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_\Omega \mu \nabla_x \tilde{\mathbf{u}} \nabla_x \mathbf{w} - \left(\beta\varrho^{(2)} + \eta^{(2)}\right) \nabla_x \Phi \cdot \mathbf{w} \mathrm{d}x\mathrm{d}t\end{aligned} \tag{37}$$

for all $\mathbf{w} \in C_C^\infty((0,T) \times \Omega; \mathbb{R}^3)$ such that $\mathrm{div}_x \mathbf{w} = 0$

Strong Stratification Free Energy

Using (33) and defining

$$E_F(\varrho,\tilde\varrho) := \frac{a}{\gamma-1}\varrho^\gamma - (\varrho-\tilde\varrho)\frac{a\gamma}{\gamma-1}\tilde\varrho^{\gamma-1} - \frac{a}{\gamma-1}\tilde\varrho^\gamma$$

$$E_P(\eta,\tilde\eta) := \frac{D}{\zeta}\eta\ln\eta - \frac{D}{\zeta}(\eta-\tilde\eta)(\ln\tilde\eta+1) - \frac{D}{\zeta}\tilde\eta\ln\tilde\eta$$

we can determine that

$$\int_\Omega \frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left[E_F(\varrho_\varepsilon,\tilde\varrho) + E_P(\eta_\varepsilon,\tilde\eta)\right]\mathrm{d}x(T)$$
$$\int_0^T\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) : \nabla_x\mathbf{u}_\varepsilon \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon^3}\int_0^T\int_\Omega\left|\frac{D\nabla_x\eta_\varepsilon}{\sqrt{\zeta\eta_\varepsilon}} + \sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right|^2\mathrm{d}x\mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}[E_F(\varrho_0,\tilde\varrho) + E_P(\eta_0,\tilde\eta)]\mathrm{d}x$$

Strong Stratification Main Result

Theorem: Let (Ω, Φ) satisfy the confinement hypothesis and for each $\varepsilon > 0$, $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ solves (30)-(33) in the renormalized weak sense. Assume the initial data can be expressed as follows:

$$\varrho_\varepsilon(0,\cdot) = \varrho_{\varepsilon,0} = \tilde{\varrho} + \varepsilon \varrho^{(1)}_{\varepsilon,0},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{\varepsilon,0}, \text{ and } \eta_\varepsilon(0,\cdot) = \eta_{\varepsilon,0} = \tilde{\eta} + \varepsilon \eta^{(1)}_{\varepsilon,0}.$$

where $\tilde{\varrho}, \tilde{\eta}$ are the densities defined by (34)-(36). Assume also that as $\varepsilon \to 0$,

$$\varrho^{(1)}_{\varepsilon,0} \rightharpoonup \varrho^{(1)}_0, \mathbf{u}_{\varepsilon,0} \rightharpoonup \overline{\mathbf{u}}_0, \eta^{(1)}_{\varepsilon,0} \rightharpoonup \eta^{(1)}_0$$

weakly-$*$ in $L^\infty(\Omega)$ or $L^\infty(\Omega; \mathbb{R}^3)$ as the case may be.

Strong Stratification Main Result (continued)

Then up to a subsequence and letting $q := \min\{\gamma, 2\}$,

$$\varrho_\varepsilon \to \tilde{\varrho} \text{ in } C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^q(\Omega))$$

$$\eta_\varepsilon \to \tilde{\eta} \text{ in } L^2(0,T; L^2(\Omega))$$

$$\mathbf{u}_\varepsilon \to \tilde{\mathbf{u}} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3))$$

where $\{\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta}\}$ solve the target system (34)-(37).

References

1 J. Ballew and K. Trivisa, Suitable weak solutions and low stratification singular limit for a fluid particle interaction model. *Quart. Appl. Math.*, posted on May 10, 2012, PII S 0033-569X(2012)01310-2 (to appear in print).

2 J. Ballew and K. Trivisa. Weak-strong uniquenss for the Navier-Stokes-Smoluchowski system. Preprint, 2012.

3 J.A. Carrillo, T. Karper, and K. Trivisa. On the dynamics of a fluid-particle interaction model: The bubbling regime. *Nonlinear Analysis*, **74**:2778-2801, 2011

4 B. Desjardins, E. Grenier, P.-L. Lions, and N. Masmoudi. Incompressible limit for solutions of the isentropic Navier-Stokes equations with Dirichlet boundary conditions. *J. Math. Pures Appl.*, **78**:461-471, 1999.

5 E. Feireisl. *Dynamics of viscous compressible fluids.* Oxford University Press, Oxford, 2004.

References

6 E. Feireisl and A. Novotný. On the low Mach number limit for the full Navier-Stokes-Fourier system. *Arch. Rational Mech. Anal.*, **186**:77-107, 2007.

7 E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids.* Birkhauser, Basel, 2009.

8 E. Feireisl, A. Novotný, and Y. Sun. *Suitable weak solutions to the Navier-Stokes equations of compressible viscous fluids.* Nečas Center for Mathematical Modeling, Preprint 2010-019, 2010.

9 Y.-S. Kwon and K. Trivisa. On the incompressible limit problems for multicomponent reactive flows. *Quart. Appl. Math.*, (to appear) 2013.

Essential and Residual Spaces

We divide $\Omega \times (0, T)$ into an essential set, where the densities are close to the uniform densities, and a residual set:

$$\mathcal{O}_{\text{ess}} := \{(\varrho, \eta) \in \mathbb{R}^2 | \overline{\varrho}/2 \leq \varrho, \eta \leq 2\overline{\eta}\}$$
$$\mathcal{M}^{\varepsilon}_{\text{ess}} := \{(x, t) \in (0, T) \times \Omega | (\varrho_\varepsilon(t, x), \eta_\varepsilon(t, x)) \in \mathcal{O}_{\text{ess}}\}$$
$$\mathcal{M}^{\varepsilon}_{\text{res}} := ((0, T) \times \Omega) - \mathcal{M}^{\varepsilon}_{\text{ess}}$$

By using (22) and the coercivity of $\mathcal{H}$, it can be shown that

$$\operatorname*{ess\,sup}_{t \in (0,T)} |\mathcal{M}^{\varepsilon}_{\text{res}}[t]| \leq \varepsilon^2 c.$$

By the strong convexity of ϱ^γ and $\eta \ln \eta$ on $\mathcal{M}^{\varepsilon}_{\text{ess}}$, we have

$$\mathcal{H}(\varrho_\varepsilon, \eta_\varepsilon) := E_F(\varrho_\varepsilon) + E_P(\eta_\varepsilon) \geq C(|\varrho_\varepsilon - \overline{\varrho}|^2 + |\eta_\varepsilon - \overline{\eta}|^2)$$

on $\mathcal{M}^{\varepsilon}_{\text{ess}}$.

Uniform Bounds

From (22) and the bounds on the essential and residual sets, we have

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$$

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$$

$$\left\{\frac{1}{\varepsilon}\left(\frac{2D\nabla_x\sqrt{\eta_\varepsilon}}{\sqrt{\zeta}} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right)\right\}_{\varepsilon>0} \in_b L^2(0,T;L^2(\Omega;\mathbb{R}^3))$$

$$\left\{\left[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\right]_{\mathrm{ess}}\right\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega))$$

$$\left\{\left[\frac{\eta_\varepsilon-\overline{\eta}}{\varepsilon}\right]_{\mathrm{ess}}\right\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega))$$

$$\{[\varrho_\varepsilon]_{\mathrm{res}}\}_{\varepsilon>0} \in_b L^\infty(0,T;L^\gamma(\Omega))$$

Convergence

Based on the bounds on the last slide there exist $\varrho^{(1)}, \eta^{(1)} \in L^2(0,T;L^2(\Omega))$ and $\overline{\mathbf{u}} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$ such that,

$$\left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\text{ess}} \to \varrho^{(1)} \text{ weakly in } L^2(0,T;L^2(\Omega))$$

$$\left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\text{res}} \to 0 \text{ weakly-}* \text{ in } L^\infty(0,T;L^\gamma(\Omega))$$

$$\left[\frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon}\right]_{\text{ess}} \to \eta^{(1)} \text{ weakly in } L^2(0,T;L^2(\Omega))$$

$$\left[\frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon}\right]_{\text{res}} \to 0 \text{ weakly in } L^2(0,T;L^2(\Omega))$$

$$\mathbf{u}_\varepsilon \to \overline{\mathbf{u}} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$$

Convergence (continued)

As a result of the convergence results on the last slide,

$$\varrho_\varepsilon \to \overline{\varrho} \text{ weakly in } L^2(0,T;L^q(\Omega)) \text{ for } q := \min\{2,\gamma\}$$
$$\eta_\varepsilon \to \overline{\eta} \text{ weakly in } L^2(0,T;L^2(\Omega))$$
$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{ weakly in } L^2(0,T;L^q(\Omega))$$
$$\frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon} \to \eta^{(1)} \text{ weakly in } L^2(0,T;L^2(\Omega))$$

In addition, using (18),

$$\int_0^T \int_\Omega \overline{\mathbf{u}} \cdot \nabla_x \varphi \mathrm{d}x = 0$$

for any test function φ.

Setting the test functions

$$\phi(t,x) = \psi(t) q_n(x)$$

and

$$\mathbf{v}(t,x) = \psi(t) \frac{1}{\sqrt{\Lambda_n}} \nabla_x q_n(x)$$

in (23) and (24) yields

$$\varepsilon \partial_t b_n[r_\varepsilon] - \sqrt{\Lambda_n} a_n[\mathbf{V}_\varepsilon] = \chi^1_{\varepsilon,n} \tag{38}$$

$$\varepsilon \partial_t a_n[\mathbf{V}_\varepsilon] + \omega \sqrt{\Lambda_n} b_n[r_\varepsilon] = \chi^2_{\varepsilon,n} \tag{39}$$

where

$$a_n[\mathbf{V}_\varepsilon] := \frac{1}{\Lambda_n} \int_\Omega \mathbf{V}_\varepsilon \cdot \nabla_x q_n \mathrm{d}x$$

$$b_n[r_\varepsilon] := \int_\Omega r_\varepsilon q_n \mathrm{d}x$$

Since this system is defined, we define $\Psi_{\varepsilon,M} := \mathbf{H}_M^{\perp}[\varrho_\varepsilon \mathbf{u}_\varepsilon]$ and can reformulate the problem to showing

$$\begin{aligned}
&\int_0^T \int_\Omega \Delta_x \Psi_{\varepsilon,M} \nabla_x \Psi_{\varepsilon,M} \cdot \mathbf{v} \mathrm{d}x \mathrm{d}t \\
&= \varepsilon \int_0^T \int_\Omega [r_\varepsilon]_M \nabla_x \Psi_{\varepsilon,M} \cdot \partial_t \mathbf{v} \mathrm{d}x \mathrm{d}t \\
&+ \int_0^T \int_\Omega \chi^3_{\varepsilon,M} \mathbf{H}_M^{\perp}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{v} + [r_\varepsilon]_M \chi^4_{\varepsilon,M} \cdot \mathbf{v} \mathrm{d}x \mathrm{d}t
\end{aligned}$$

goes to zero as $\varepsilon \to 0$ for each divergence-free and zero-normal-trace test function $\mathbf{v}$. From the convergence of the χ terms, this follows.