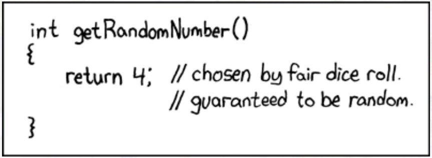
Penalty methods for edge plasma transport in a tokamak

Thomas Auphan, LATP, June 28, 2012

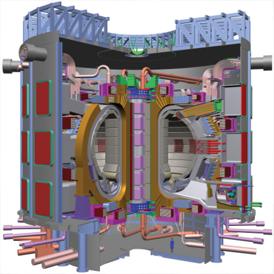


In collaboration with Ph. Angot and O. Guès

Financial support: FR-FCM and ANR ESPOIR

Picture: XKCD

The ITER tokamak



- Plasma : Ion and electrons soup.
- Magnetic confinement.
- Heating.
- Goal: Perform the fusion reaction as a reliable source of energy.

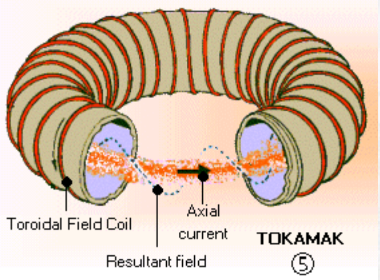
Key figures:

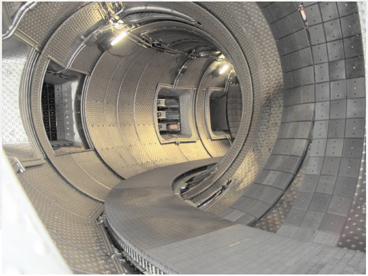
Fusion power $\approx 500 MW$

$$\frac{\text{Fusion power}}{\text{Power consumption}} \geq 10$$

Plasma duration $\geq 300\,s$

Limiter configuration





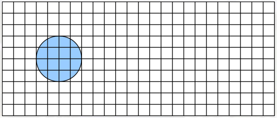
TORE SUPRA, Cadarache (source: CEA)

Wall-plasma interaction

TORE SUPRA, Cadarache
From ccd camera (visible)
(Source: CEA)

- Magnetic confinement not perfect $\Rightarrow$ Control the interactions (limiter, divertor).
- ANR ESPOIR: Numerical simulation of the edge plasma using penalization methods.

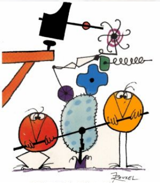
Why penalty methods ?



- Non-body-fitted Cartesian-mesh.
- Possible use of efficient solver : pseudo-spectral, multiscale grids....

A few references for applications :

- Incompressible flows [ANGOT, Math. Meth. Appl. Sci. , 1999]
- Compressible flows [LIU, VASILYEV, JCP, 2007]
- Pseudo spectral methods for edge plasma [ISOARDI *et al.*, JCP, 2010]



Why do simple when one can do complicated ?

Shadocks, from I. Ramière's thesis.

The 1D hyperbolic system (along a magnetic field line)

$$(t,x) \in \mathbb{R}^+ \times]-L, L[$$

$$\partial_t N + \partial_x \Gamma = S$$

$$\partial_t \Gamma + \partial_x \left(\frac{\Gamma^2}{N} + N \right) = 0$$

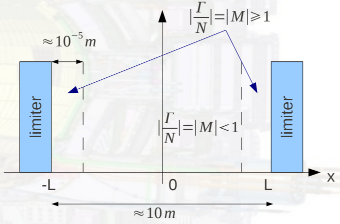
Boundary conditions:

$M(., -L) = -1 + \eta$

and $M(., L) = 1 - \eta$

Initial: $N(0, .)$ and $\Gamma(0, .)$

N = plasma density
Γ = plasma momentum
$M = \frac{\Gamma}{N}$ = "velocity"



- Strictly hyperbolic 1D.
- Eigenvalues : $M - 1$ and $M + 1$.
- One incoming wave : one boundary condition admissible on each boundary.

A first approach [Isoardi *et al.*, JCP, 2010]

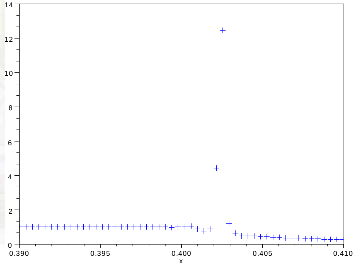
$$\partial_t N + \partial_x \Gamma + \frac{\chi}{\varepsilon} N = (1-\chi) S \qquad 0 < \varepsilon \ll 1 \qquad M = \frac{\Gamma}{N}$$

$$\partial_t \Gamma + (1-\chi)\partial_x \left(\frac{\Gamma^2}{N} + N \right) + \frac{\chi}{\varepsilon}(\Gamma - M_0 N) = 0$$

$$\chi(x) = \begin{cases} 0 \text{ in the plasma} \\ 1 \text{ in the limiter} \end{cases}$$

Numerical test :

$\varepsilon = 10^{-3}$, $\delta x \approx 1 \cdot 10^{-3}$, $t \approx 8.8 \cdot 10^{-3}$ (stop : $|M_i^n| > 10$)



M versus x

Two problems:

- 2 fields penalized.
- Sense of $(1-\chi)\partial_x \left(\frac{\Gamma^2}{N} + N \right)$?

$\Rightarrow$ Dirac measure next to the interface.

An optimal penalty method

Penalization of a single field such that $M \to M_0$.

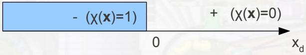
$$\partial_t N + \partial_x \Gamma = S_N$$

$$\partial_t \Gamma + \partial_x \left(\frac{\Gamma^2}{N} + N \right) + \frac{\chi}{\varepsilon} \left(\frac{\Gamma}{M_0} - N \right) = S_\Gamma$$

Initial conditions: $N(0,.)$ and $\Gamma(0,.)$known

- M_0 is a constant such that $0 < M_0 = 1 - \eta < 1$.
- Also obtained by a method inspired from [Fornet and Guès, DCDS, 2009].
- Does not generates boundary layers.

Convergence analysis theorem I



$$\begin{cases} \partial_t \mathbf{v} + \sum_{j=1}^{d} \mathbf{A}_j(\mathbf{v})\partial_j \mathbf{v} = \mathbf{f}(\mathbf{v}) & \text{in }]-T_0, T[\times \mathbb{R}_+^d \\ \mathbf{P}\mathbf{v}_{|x_d=0} = \mathbf{0} & \text{on }]-T_0, T[\times \mathbb{R}^{d-1} \end{cases} \quad (1)$$

- $\mathbf{A}_j$:matrices, symmetric, $\mathcal{C}^\infty$, independant from $(t, \mathbf{x})$ outside a compact set.
- $\mathbf{P}$ = orthogonal projection matrix.
- Maximal strictly dissipative and non characteristic boundary conditions.

Convergence analysis theorem II

Penalized system :

$$\partial_t \mathbf{v}_\varepsilon + \sum_{j=1}^{d} \mathbf{A}_j(\mathbf{v}_\varepsilon)\partial_j \mathbf{v}_\varepsilon + \frac{\chi}{\varepsilon}\mathbf{P}\mathbf{v}_\varepsilon = \mathbf{f}(\mathbf{v}_\varepsilon) \text{ in }]-T_0, T[\times\mathbb{R}^d \quad (2)$$

Theorem ($T_0 > 0$)

Consider, $\mathbf{v}^{0,+}_{|]-T_0,0[} \in H^\infty \cap Lip$ solution of (1) on $]-T_0, 0[$. There exists $T > 0$ and $\varepsilon_0 > 0$ such that both the penalized ($\forall\varepsilon \in]0,\varepsilon_0[$) and the BVP (1) has a smooth solution (resp. $\mathbf{v}_\varepsilon$ on $]-T_0, T[\times\mathbb{R}^d$ and $\mathbf{v}^{0,+}$ on $]-T_0, T[\times\mathbb{R}^d_+$) such that :

$$\forall s \in \mathbb{N}, \quad \|\mathbf{v}_\varepsilon - \mathbf{v}^{0,+}\|_{H^s(]-T_0,T[\times\mathbb{R}^d_+)} = \mathcal{O}(\varepsilon)$$

Convergence analysis theorem: Sketch of proof I

Formal asymptotic expansion of a continuous solution :
$\mathbf{v}_\varepsilon(t,x) \sim \mathbf{U}^{\pm}_{\varepsilon}(t,x) = \sum_{n=0}^{+\infty} \varepsilon^n \mathbf{U}^{n,\pm}(t,x)$
Substituting the expansion and classifying :

- Inside the physical domain : $\sum_{n=0}^{\infty} \varepsilon^n \left(\partial_t \mathbf{U}^{n,+} + \dots\right) = \mathbf{S}$
- In the obstacle :
$\frac{\varepsilon^{-1}}{M_0} P\mathbf{U}^{0,-} + \sum_{n=0}^{\infty} \varepsilon^n \left(\partial_t \mathbf{U}^{n,-} + \dots + \frac{1}{M_0} P\mathbf{U}^{n+1,-}\right) = \mathbf{S}$

Computations of the terms $\mathbf{U}^{n,\pm}$: by induction.

Convergence analysis theorem: Sketch of proof II

- $\mathbf{v}_\varepsilon(t,x) = \sum_{n=0}^{K} \varepsilon^n \mathbf{U}^{n,\pm}(t,x) + \varepsilon \mathbf{w}_\varepsilon(t,x)$.
- Equation for $\mathbf{w}_\varepsilon$.
- Approximation of $\mathbf{w}_\varepsilon$ by an iterative scheme $(\mathbf{w}^k)$.
- Energy estimates:

Lemma

Weighted norm : $\|\mathbf{w}\|_{2,\lambda} = \|e^{-\lambda t}\mathbf{w}\|_2$
Assumptions : $\|\mathbf{w}^k\|_\infty < R$ *and* $\|\partial_j \mathbf{w}^k\|_\infty < R$ $(j \in \{0,\ldots,d-1\})$

$$\forall \lambda > \lambda_0(R), \sqrt{\lambda}\|\mathbf{w}^{k+1}\|_{2,\lambda} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}\mathbf{w}^{k,-}\|_{2,\lambda} \le \frac{C(R)}{\sqrt{\lambda}}\|\mathbf{g}\|_{2,\lambda}$$

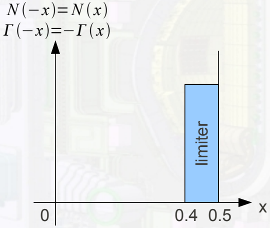
- $(\mathbf{w}^k)$ bounded sequence (for some norms).
- Existence of $\mathbf{w}_\varepsilon = \lim_{k\to\infty} \mathbf{w}^k$.

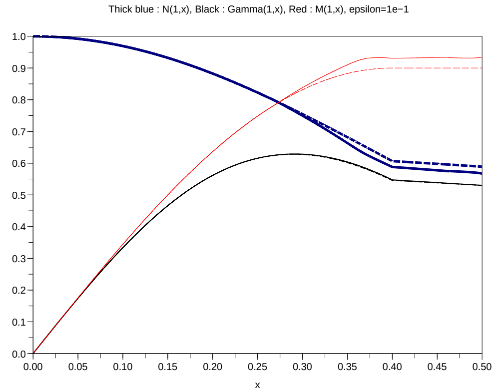
Numerical test (2nd order FV scheme)

Manufactured solution :

$$N(t,x) = \exp\left(\frac{-x^2}{0.16(t+1)}\right)$$

$$\Gamma(t,x) = \sin\left(\frac{\pi x}{0.8}\right)\exp\left(\frac{-x^2}{0.16(t+1)}\right)$$





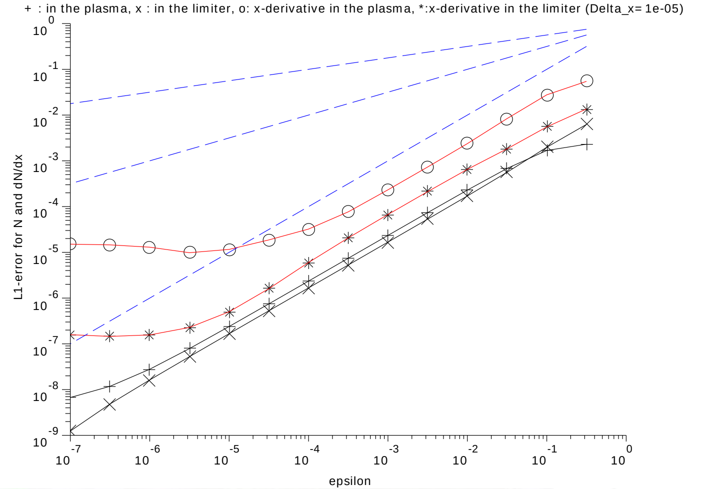
Continuous lines : Approximate solution ($\varepsilon = 0.1$)

Dashed lines : exact solution.

Computations up to $t = 1$.

Mesh step: $\delta x = 10^{-5}$

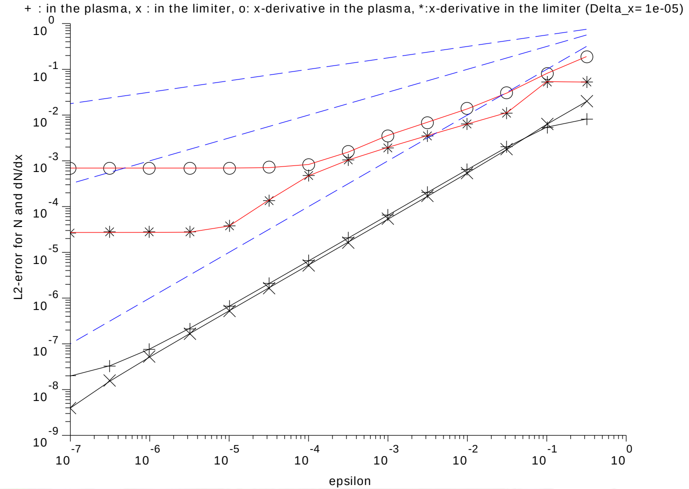
Numerical tests



L^1 error for N and $\partial_x N$ as a function of ε.

Optimal convergence rate for N and $\partial_x N$: $\mathcal{O}(\varepsilon)$

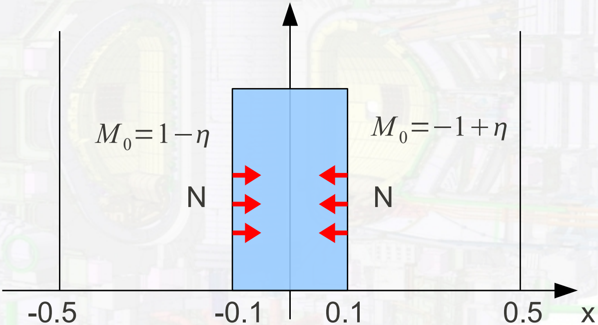
Numerical tests



L^2 error for N and $\partial_x N$ as a function of ε.

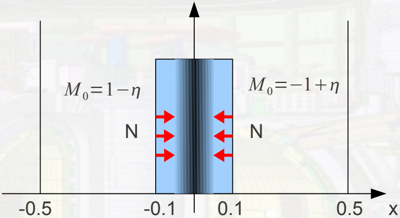
Non optimal rate for $\partial_x N$ in L^2 norm : Artefact ?

Two interfaces and transport of N



Concentration of N at the center !

Prevent information from crossing the limiter

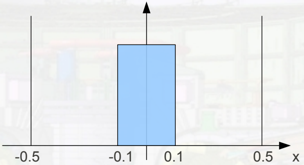


$$\partial_t \begin{pmatrix} N \\ \Gamma \end{pmatrix} + \partial_x \left(\alpha \mathbf{f} \begin{pmatrix} N \\ \Gamma \end{pmatrix} \right) + \frac{\chi}{\varepsilon} \begin{pmatrix} 0 \\ \frac{\Gamma}{M_0} - N \end{pmatrix} = \mathbf{S}$$

$\alpha(x)$ is :

- Smooth.
- $= 1$ inside the plasma area and in a neighbourhood of the interface.
- $= 0$ in the central area of the limiter.

Implementation



Idea from Greenberg-Le Roux (α = unknown):

$$\partial_t \begin{pmatrix} N \\ \Gamma \\ \alpha \end{pmatrix} + \nabla \mathbf{F} \begin{pmatrix} N \\ \Gamma \\ \alpha \end{pmatrix} \partial_x \begin{pmatrix} N \\ \Gamma \\ \alpha \end{pmatrix} + \frac{\chi}{\varepsilon} \begin{pmatrix} 0 \\ \frac{\Gamma}{M_0} - N \\ 0 \end{pmatrix} = \begin{pmatrix} S_N \\ S_\Gamma \\ 0 \end{pmatrix}$$

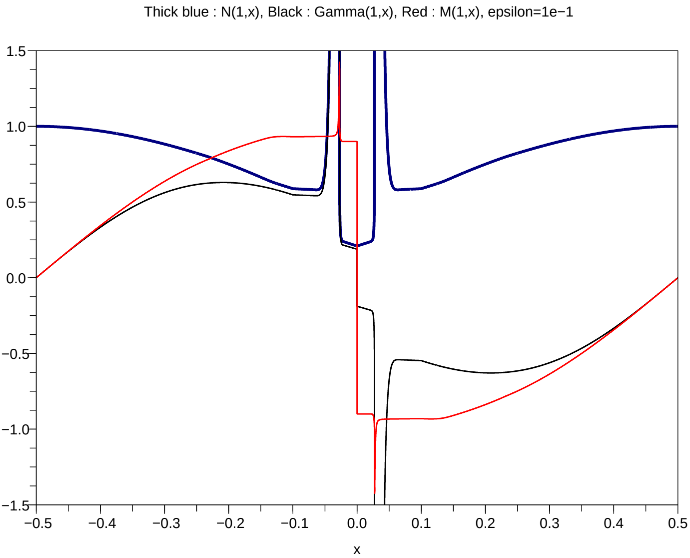
Scheme VFRoe ncv + 2$^{\text{nd}}$ order extensions

Periodic boundary conditions at $x = \pm 0.5$.

Step : $\delta x = 10^{-5}$.

Computations up to $t = 1$.

Numerical tests for the two faces limiter



Continuous lines : Approximate solution ($\varepsilon = 0.1$).

Conclusions and perspectives

- Penalization of a single field.
- Well-defined terms.
- No boundary layer and optimal convergence rate.
- Penalization of the two sides limiter.
- More equations to model : the energy, the current.

The End

Thank you for your attention !