A relaxation framework for morphodynamics modelling

E. Audusse

LAGA, UMR 7569, Univ. Paris 13
BANG project-team, INRIA Paris-Rocquencourt

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Joint work with

- BANG project - ANGE group
  M.O. Bristeau, J. Sainte-Marie
- EDF LNHE – Saint-Venant Lab.
  N. Goutal, M. Jodeau
- C. Berthon, C. Chalons, O. Delestre, S. Cordier

Shallow water flows

- Shallow water equations

$$\begin{aligned} \partial_t h + \nabla \cdot (h\mathbf{u}) &= 0, \\ \partial_t (h\mathbf{u}) + \nabla \cdot \left(h\mathbf{u} \otimes \mathbf{u} + \frac{gh^2}{2} I \right) &= -gh\nabla z - 2\Omega \times h\mathbf{u} - \kappa(h,\mathbf{u})\mathbf{u} \end{aligned}$$

- Applications



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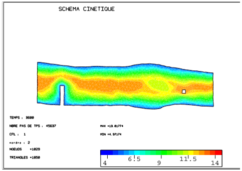
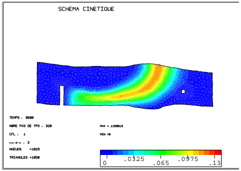
- Positive and well-balanced numerical schemes
  - Extended Godunov schemes (Greenberg-Leroux)
  - Kinetic interpretation of source terms (Pertame-Simeoni)
  - Extended Suliciu relaxation schemes (Bouchut)
  - Hydrostatic reconstruction (ABBKP)
  - Hydrostatic upwind (Berthon-Foucher)
  - ...

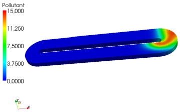
From SW flows to coupled problems

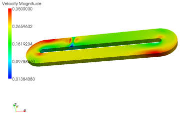
- Pollutant processes

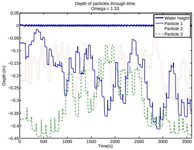




- Hydrobiological processes (A.C. Boulanger)







Morphodynamic processes

- Dune formation
- Coastal erosion
- Impact on industrial building (harbour, dam, nuclear plant...)
- River morphodynamic
- Strong events (tsunami, dam drain or break...)

Hyperbolic models in Morphodynamics

- Suspended sediment model
  - Applications : High coupling and light sediments
  - SW equations + Transport + Bottom evolution (ODE)
  - Closure : Erosion and deposition source terms
- Bedload transport model
  - Applications : Low coupling or heavy sediments
  - SW equations + Bottom evolution
  - Closure : Sediment flux
  - Empirical formula : Grass, Meyer-Peter-Müller, Einstein...

Numerical strategies for bedload transport

- Steady state strategy
  - Hydrodynamic computation on fixed topography
    $\rightsquigarrow$ Steady state
  - Evolution of topography forced by hydrodynamic steady state
  - Efficient for low coupling and different time scales (dune formation)
- External coupling
  - Use of two different softwares for hydro- and morphodynamics
  - Allow to use existing solvers and different numerical strategies
  - Actual strategy at EDF (MASCARET-COURLIS)
  - Efficient for low coupling (slow river morphodynamics)
- Internal coupling
  - Solution of the whole system at once
  - Need for a new solver
  - Efficient for high coupling (dam drain, tsunami)

Saint-Venant – Exner model

$$\begin{aligned}
\frac{\partial H}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{H} + \frac{g}{2}H^2\right) &= -gH\frac{\partial Z}{\partial x}, \\
\rho_s(1-p)\frac{\partial Z}{\partial t} + \frac{\partial Q_s}{\partial x} &= 0,
\end{aligned}$$

- Hyperbolic for classical choices of $Q_s(h, u)$
- Eigenvalues hard to compute except for special choices of Q_s
- No dynamic effects in the solid phase
- No transport in the fluid phase
- Numerical strategies : Hudson, Nieto, Morales, Benkhaldoun...

Relaxation model

$$
\begin{aligned}
\partial_t H + \partial_x Hu &= 0 \\
\partial_t Hu + \partial_x \left(Hu^2 + \Pi\right) &= -gH\partial_x Z \\
\partial_t \Pi + u\partial_x \Pi + \frac{a^2}{H}\partial_x u &= \frac{1}{\lambda}\left(\frac{gH^2}{2} - \Pi\right) \\
\partial_t Z + \partial_x \Omega &= 0 \\
\partial_t \Omega + \left(\frac{b^2}{H^2} - u^2\right)\partial_x Z + 2u\partial_x \Omega &= \frac{1}{\lambda}(Q_s - \Omega)
\end{aligned}
$$

- (Π, Ω) : Auxiliary variables (fluid pressure, sediment flux)
- $\lambda > 0$: (Small) relaxation parameter
- $(a, b) > 0$: Have to be fixed to ensure stability

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Main properties

- Formally tends to SW-Exner model when λ tends to 0
- No explicit dependency on sediment flux Q_S
- Always hyperbolic ($H \neq 0$)
- Eigenvalues easy to compute (case $a < b$)

$$u - \frac{b}{H} < u - \frac{a}{H} < u < u + \frac{a}{H} < u + \frac{b}{H}$$

- Linearly degenerate system

$\rightsquigarrow$ Exact (homogeneous) Riemann problem "easy" to solve

Stability of the relaxation model

- Chapman-Enskog expansion

$$\Pi = p(h) + \lambda \Pi_1 + \ldots \qquad \Omega = Q_s(h,u) + \lambda \Omega_1 + \ldots$$

- Insert in the auxiliary equations

$$\begin{aligned} -\Pi_1 &= \frac{1}{h}\left(a^2 - h^2 p'(h)\right)\partial_x u + O(\lambda) \\ -\Omega_1 &= \left(u\partial_h Q_s - \frac{p'(h)}{h}\partial_u Q_s\right)\partial_x h + \left(-h\partial_h Q_s + u\partial_u Q_s\right)\partial_x u \\ &\quad + \left(\frac{b^2}{h^2} - u^2 - g\partial_u Q_s\right)\partial_x z + O(\lambda) \end{aligned}$$

- Insert in the physical equations

Stability of the relaxation model

- Diffusive physical system $W = (h, u, Z)^T$

$$\partial_t W + A(W)\partial_x(W) = \lambda\partial_x\left(D(W)\partial_x W\right) + O(\lambda^2)$$

- Diffusion matrix

$$D(W) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{h}\left(a^2 - h^2 p'(h)\right) & 0 \\ \times & \times & \left(\frac{b^2}{h^2} - u^2 - g\partial_u Q_s\right) \end{pmatrix}$$

- Stability requirement

$$a^2 > h^2 p'(h), \qquad b^2 > (hu)^2 + gh^2\partial_u Q_s$$

Relaxation scheme with time splitting

- Start from (H^n, u^n, Z^n)
- Computation of auxiliary variables

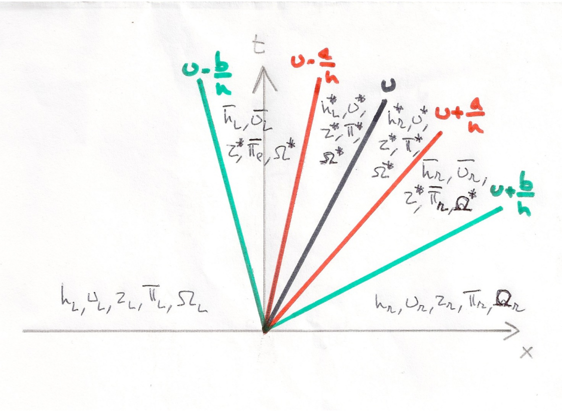
$$\begin{pmatrix} \Pi^n \\ \Omega^n \end{pmatrix} = \begin{pmatrix} p(h^n) \\ Q_s(h^n, u^n) \end{pmatrix} \Leftrightarrow \begin{cases} \partial_t \Pi &= \frac{1}{\lambda}\left(\frac{gH^2}{2} - \Pi\right) \\ \partial_t \Omega &= \frac{1}{\lambda}(Q_s - \Omega) \end{cases} \quad "\lambda = 0"$$

- Solution of homogeneous Riemann problems

$$X_R = X(X_l, X_r, x, t)$$

- Computation of new physical variables $(H^{n+1}, u^{n+1}, Z^{n+1})$

Solution of Riemann problem



Solution of Riemann problem

- Continuity of the Riemann invariants (k-wave)
- Computation of the intermediate states
- Computation of the new variables

$$W_i^{n+1} = \int_{x_{i-1/2}}^{x_i} W_R(U_{i-1}, U_i, x, \Delta t^n) + \int_{x_i}^{x_{i+1/2}} W_R(U_i, U_{i+1}, x, \Delta t^n)$$

- CFL condition ensures that Riemann pbs do not interact
- Definition of intermediate states
  Positivity of the intermediate water heights

$\rightsquigarrow$ Additional requirements on parameters a and b

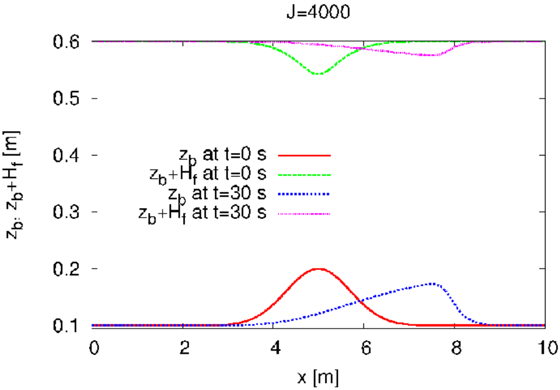
Intermediate states

$$Z^* = \frac{\left(u_r + \frac{b}{H_r}\right) Z_r - \left(u_l - \frac{b}{H_l}\right) Z_l}{\left(u_r + \frac{b}{H_r}\right) - \left(u_l - \frac{b}{H_l}\right)} - \frac{1}{\left(u_r + \frac{b}{H_r}\right) - \left(u_l - \frac{b}{H_l}\right)} \left(\Omega_r - \Omega_l\right),$$

$$\frac{1}{\bar{H}_l} = \left(\frac{1}{H_l^2} - \frac{2g}{b^2 - a^2}(Z^* - Z_l)\right)^{\frac{1}{2}}, \quad \frac{1}{\bar{H}_r} = \left(\frac{1}{H_r^2} - \frac{2g}{b^2 - a^2}(Z^* - Z_r)\right)$$

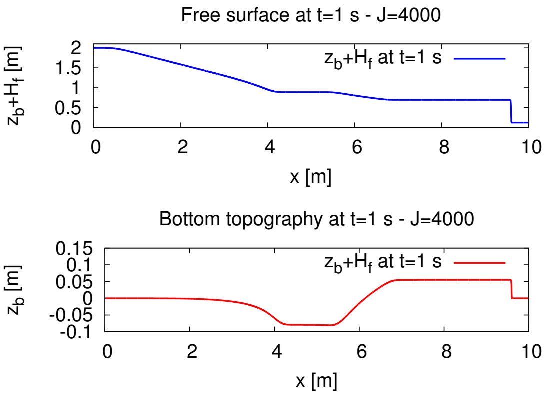
$$\frac{1}{H_l^*} = \frac{1}{\bar{H}_l} - \frac{1}{a^2}(\Pi^* - \bar{\Pi}_l), \quad \frac{1}{H_r^*} = \frac{1}{\bar{H}_r} - \frac{1}{a^2}(\Pi^* - \bar{\Pi}_r)$$

"Steady" flow over a movable bump



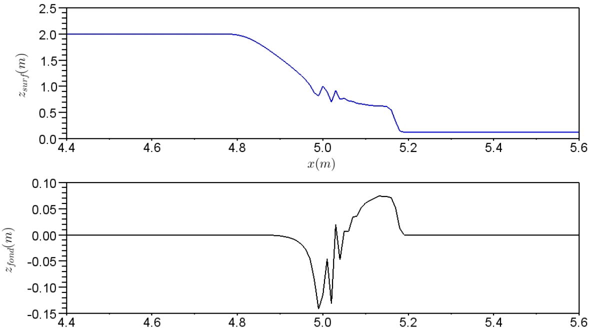
Free surface and bottom evolution (O. Delestre)

Dam break : Comparison for external and internal coupling



Internal coupling (O. Delestre)

Dam break : Comparison for external and internal coupling



External coupling (M. Jodeau)

Three-layer model

- Introduction of three layers in the model
  - NS Fluid layer (density ρ_f)
  - NS Mixed layer (density $\rho_m(C_s)$) + Transport of C_s
  - Solid layer (no transport, but erosion and deposition)
- From Navier-Stokes to Saint-Venant in fluid and mixed layers

$\rightsquigarrow$ Two-layer shallow water flow with variable densities

- Closure laws at the interfaces : model for exchange terms

$\rightsquigarrow$ Solution of a 5×5 non linear system (constant density case)

- Relaxation model

$\rightsquigarrow$ Solution of two "linear" 5×5 uncoupled systems

Perspectives

- Relaxation scheme for SW-Exner model
  - Complete definition of parameters a and b
  - Well-balancing
  - Computation of wet-dry area
  - Second order extension
  - Improved relaxation models
- Modelization for Morphodynamics
  - Closure laws and numerical tests for three-layer model
  - Fluid-structure interaction
  - Multilayer models with transport, erosion and deposition