A dispersive property of the Euler-Korteweg system

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The Euler-Korteweg system consists of the following equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + (u \cdot \nabla) u + \nabla g_0(\rho) = \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right), \end{cases} \tag{EK}$$

It is a perturbation of the classical Euler equations, that takes into account the capillarity effects. The quantities involved are the density ρ and the velocity u, K is the so called Korteweg stress tensor.

On the opposite of the Navier-Stokes equations, the perturbation is *dispersive*.

The linearized system near a constant state $(\underline{u}, \underline{\rho})$ admits the following dispersion relation

$$(\tau + i\underline{u} \cdot \xi)^2 + \underline{\rho}(\underline{g}'|\xi|^2 + \underline{K}|\xi|^4) = 0.$$

At high frequencies it amounts to $\tau \sim \pm i\sqrt{\underline{K}\underline{\rho}}|\xi|^2$ and thus bears some similarity with the usual Schrödinger equation

$$i\partial_t u + a\Delta u = 0,$$

whose dispersion relation is $\tau + ia|\xi|^2 = 0$.

Local well-posedness of the Euler-Korteweg system in any dimension was obtained in 2007.

Theorem (Benzoni-Danchin-Descombes '07)

Given $s > d/2+1$, $u_0 \in H^s(\mathbb{R}^d)$, $\rho_0 \in C_b^0(\mathbb{R}^d)$ such that $\nabla\rho \in H^s$, there exists $T > 0$ and a unique solution $(\rho, u) \in C_T C_b \times C_T H^s,\ \nabla\rho \in C_T H^s$ of the Cauchy problem

$$\begin{cases} \partial_t \rho + div(\rho u) = 0, \\ \partial_t u + (u\cdot\nabla)u + \nabla g_0(\rho) = \nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right), \\ (\rho, u)|_{t=0} = (\rho_0, u_0) \end{cases} \tag{1}$$

The proof relied on rather involved *a priori* estimates, for an extended (formally equivalent) system displaying a better structure :

$$\begin{cases} \partial_t \zeta + u \cdot \nabla \zeta + a(\zeta)\mathrm{div} u = 0, \\ \partial_t z + (u \cdot \nabla) z + i(\nabla z) \cdot w + i\nabla(a\mathrm{div} z) = -g'(\zeta)\mathrm{Re}(z). \end{cases} \quad (2)$$

where $\zeta = R(\rho)$, R is a primitive of the application $\rho \to \sqrt{K(\rho)/\rho}$, $w = \nabla \zeta$, $z = u + iw$, $a(\zeta) = \sqrt{R^{-1}(\zeta) K(R^{-1}(\zeta))}$.
The second equation actually looks like a *quasi-linear degenerate Schrödinger equation*.

Opposedly to the Schrödinger equation, no dispersive estimate was proved yet for the Euler-Korteweg system. Our main result is a local smoothing property.

Theorem

Under the assumptions of the local well-posedness theorem, if moreover u_0 is irrotational, the curves associated to the hamiltonian $a(x,0)|\xi|^2$ are unbounded and $\nabla_{x,t}a(0,x) \leq C/(1+|x|^2)$, then any solution $(u,\nabla\rho) \in (C_T H^s)^2$ additionally satisfies for some $\widetilde{T}$

$$(u,\nabla\rho)/(1+|x|) \in L^2([0,\widetilde{T}];\, H^{s+1/2}(\mathbb{R}^d)).$$

A few comments on the assumptions :

- the irrotationality seems natural since $\partial_t z + i\underline{a}\nabla \text{div} z = 0$ admits trivial stationary solution, which precisely correspond to "purely rotational" initial data, $\text{div}(z_0) = 0$.
- The second assumption means that the solutions of the differential equation

$$\begin{cases} X(t)' &= \nabla_\xi (a(X(t),0)|\Xi(t)|^2) = 2a(X(t),0)\Xi(t), \\ \Xi(t)' &= -\nabla_x (a(X(t),0)|\Xi|^2), \end{cases}$$

  satisfy $\lim_{t\to\infty} \|X(t)\| = +\infty$. It is standard in the frame of linear Schrödinger equations with variable coefficients.

Some elements of proof :

- Since the local gain of regularity is only of $1/2$ derivative, it can hardly be obtained by basic multiplier techniques. It is necessary to use slightly more sophisticated tools.
- Nonlinearities appear even in the highest order derivatives, thus the pseudo-differential calculus is not well-suited as it usually requires a lot of smoothness from the coefficients. A more fitted tool would be Bony's paradifferential calculus.

Para-differential calculus allows to replace a product uv by $T_u v + R(u,v)$, where R is (hopefully) smooth, and T_u acts $H^s \to H^s$. It is even possible for a class of symbol $s(x,\xi)$ that satisfy minimal regularity assumptions to define the paradifferential operators T_s.

A very basic sketch of proof : fix some symbol $p(x,\xi)$ and consider the derivative

$$\begin{aligned}\frac{d}{dt}\langle T_p z, z\rangle &= \langle T_p(-i\nabla a\mathrm{div} z),\ z\rangle + \langle T_p z,\ -i\nabla a\mathrm{div} z\rangle + \text{l.o.t.}\\ &= \langle\,[T_p, -i\nabla a\mathrm{div}]z, z\rangle + \text{l.o.t.}\end{aligned}$$

use then the rules of para-differential calculus and the irrotationality

$$\begin{aligned}\cdots &= \langle\,[T_p,\, T_{-i|\xi|^2 a}]z, z\rangle + \text{l.o.t.}\\ &= \langle\ T_{\{ip,\,|\xi|^2 a\}} z, z\rangle + \text{l.o.t.}\end{aligned}$$

where $\{ip,\ |\xi|^2 a\}$ is the *Poisson bracket*

$$\sum_{j=1}^{d} \partial_{\xi_j} p \partial_{x_j}(a|\xi|^2) - \partial_{x_j} p \partial_{\xi_j}(a|\xi|^2).$$

We have obtained (roughly)

$$\frac{d}{dt}\langle T_p z, z\rangle \simeq \langle\, T_{\{ip,\,|\xi|^2 a\}} z, z\rangle.$$

If p is a zeroth order operator, and $\{ip,\,|\xi|^2 a\} \geq c|\xi|/(1+|x|^2)$, it is then possible to use a Gärding-like inequality to deduce

$$\frac{d}{dt}\langle T_p z, z\rangle \gtrsim c\|z/\sqrt{1+|x|^2}\|^2_{H^{1/2}(\mathbb{R}^d)}$$
$$\Rightarrow \int_0^{\widetilde{T}} \|z(t)/\sqrt{1+|x|^2}\|^2_{H^{1/2}} dt \lesssim \|z\|_{L^\infty([0,\widetilde{T}],L^2(\mathbb{R}^d))},$$

which is the expected smoothing effect.

Why it is not *that* simple :

- The construction of p is complicated (but similar to the one of Doi for Schrödinger like equations),
- The Gärding inequality is not standard,
- The lower order terms are actually not neglectible,
- Instead of a gain $L^2 \to H^{1/2}$ we want a gain $H^s \to H^{s+1/2}$, thus instead of working on z one has to study the quantity $T_{|\xi|^s} z$, it raises new commutators and more “bad” lower order terms.

Some more details on the gauge method : set $Z_s = T_{\varphi|\xi|^s}z$, the equation satisfied by Z_s is

$$\partial_t Z_s + T_u \cdot \nabla Z_s + i(\nabla Z_s) \cdot w + i[\mathrm{div} a\nabla,\ T_{\varphi|\xi|^s}]z - i\mathrm{div}\, T_a \nabla Z_s = l.o.t.$$

We have $i(\nabla Z_s) \cdot w \simeq T_{-w\cdot\xi|\xi|^s\varphi}z$, and $i[\mathrm{div} a\nabla,\ T_{\varphi|\xi|^s}]z \simeq T_{\{a|\xi^2,\, |\xi|^s\varphi\}}z$. To suppress the bad terms, it is sufficient to have

$$\{a|\xi|^2,\, \varphi|\xi|^s\} = \varphi|\xi|^s\xi \cdot w,$$

and the "miraculous" function $\varphi = \sqrt{\rho}a^{s/2}$ works (as in Benzoni-Danchin-Descombes) !

The Euler-Korteweg system admits traveling waves $(\underline{\rho}, \underline{u})$ that only depend on $x_1 - ct$, the assumptions for the smoothing effect are usually not satisfied even for the linearized equations near such waves :

$$\partial_t z + \underline{u}\nabla z + i\nabla z \cdot \underline{w} + i\nabla a \mathrm{div} z = l.o.t.$$

Nevertheless, it is still possible to prove local smoothing in special cases.

Proposition

Assume that

$$2\sqrt{a_0(x_1)} - \frac{\underline{a_0}'(x_1)}{\sqrt{\underline{a_0}(x_1)}} \geq \alpha > 0,$$

then the same local smoothing property is still true.

Principle of proof : Doi's construction of p does not work, but a simpler one is actually available. Essentially, local smoothing is reduced again to the positivity of $\{\underline{a}|\xi|^2, p\}$, now the choice $p = f(x_1 - ct)\dfrac{x\cdot\xi}{|\xi|}$ gives

$$\{a|\xi|^2, p\} = |\xi|(2\underline{a}f - x_1\underline{a}'f) + \frac{\xi_1\, x\cdot\xi}{|\xi|}(\underline{a}'f + 2f'\underline{a}).$$

For $f = c/\sqrt{\underline{a}}$ the bad term cancels, and the positivity condition becomes

$$\{a|\xi|^2, p\} = |\xi|\left(2\sqrt{\underline{a}} - x_1\frac{\underline{a}'}{\sqrt{\underline{a}}}\right) > 0,$$

which is precisely the assumption.

Thank you for you attention !