Well posedness for Schrödinger type equations

Joint work

A. Ascanelli, C. Boiti, L. Zanghirati

University of Ferrara, Italy

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The Cauchy problem

$$(CP)\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) & (t,x) \in [0,T]\times\mathbb{R} \\ u(0,x) = g(x) & x \in \mathbb{R} \end{cases}$$

$p \geq 2$, $D = -i\partial$

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t,x)D_x^j \quad (1)$$

- $a_p \in C([0,T];\mathbb{R})$,
- $a_j \in C([0,T];\mathcal{B}^\infty)$, $a_j(t,x) \in \mathbb{C}$, $0 \leq j \leq p-1$.

Aim: Sufficient conditions for $H^\infty = \cap_s H^s(\mathbb{R})$ well-posedness of (CP):
$\forall f \in C([0,T];H^\infty)$, $g \in H^\infty$, $\exists!\ u \in C([0,T];H^\infty)$.

Rem. and Ex.:

- $a_p(t) \in \mathbb{R} \Leftrightarrow$ real characteristic $\tau = -a_p(t)\xi^p$ (nec. by Lax-Mizohata)
- ($p = 1 \Rightarrow$ Hyperbolic equation)
- $p = 2 \Rightarrow$ Schrödinger equation: $(i\partial_t + \partial_x^2)u = f$
- $p = 3 \Rightarrow$ Korteweg De Vries equation: $\partial_t u + \partial_x^3 u + \partial_x(A(u)) = 0$

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Literature

$$P(t, x, D_t, D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t, x)D_x^j$$

- $a_j \in \mathbb{R}$ for all $0 \leq j \leq p \Rightarrow$ many results: linear/quasilinear arbitrary order equations, also with low regularity in time of the coefficients [Agliardi, Zanghirati, Cicognani, Colombini, A., Hirosawa, Reissig...]
- $a_j \in \mathbb{C}$ for $0 \leq j \leq p - 1 \Rightarrow$ results only for $p = 2, 3$. [Mizohata, Ichinose, Doi, Takeuchi, Kajitani, Baba, A., Cicognani, Colombini, Reissig ...]

Known results for $p = 2$

$$P(t, x, D_t, D_x) = D_t + a_2(t)D_x^2 + a_1(t, x)D_x + a_0(t, x)$$

- Ichinose (1984): $a_2(t) \equiv 1$, $a_1 = a_1(x)$
  sufficient and necessary condition for H^∞ well posedness of (CP):

$$\exists M, N \geq 0 \quad \text{s.t. } \forall \varrho \geq 0$$
$$\sup_{\mathbb{R}} \left| \int_0^\rho \mathrm{Im}\, a_1(x \pm \theta) d\theta \right| \leq M \log(1 + \rho) + N \qquad (2)$$

Rem:

- (2) is necessary also for $x \in \mathbb{R}^n$, sufficient only if $n = 1$
- If $M = 0$: L^2 well posedness of (CP)

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where $\langle x \rangle = \sqrt{1 + x^2}$.

Rem:

  - (3) is sufficient also for $x \in \mathbb{R}^n$
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Known results for $p = 3$

$$P(t, x, D_t, D_x) = D_t + a_3(t)D_x^3 + \sum_{j=0}^{2} a_j(t, x)D_x^j$$

- Cicognani-Colombini (2010): $a_3(t) \in \mathbb{R}$, $a_3(t) \geq 0$
  sufficient condition for H^∞ well posedness of (CP):

$$|\operatorname{Im} a_2(t, x)| \leq \frac{C a_3(t)}{\langle x \rangle} \quad \forall (t, x) \in [0, T] \times \mathbb{R} \tag{4}$$

$$|\operatorname{Im} a_1(t, x)| + |\operatorname{Im} D_x a_2(t, x)| \leq \frac{C a_3(t)}{\langle x \rangle^{1/2}} \quad \forall (t, x) \in [0, T] \times \mathbb{R} \tag{5}$$

Rem:

- if $a_3(t) \leq 0$ $\forall t \in [0, T]$: same result
- Results also for $p = 2$, generalizing [KB] to the case $a_2(t) \neq$ constant

Known results for $p = 3$

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Theorem 1 (A. Ascanelli, C. Boiti, L. Zanghirati): $p \geq 2$

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t,x)D_x^j \qquad (t,x) \in [0,T] \times \mathbb{R}$$

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Sufficient condition for H^∞ well posedness of (CP):

$$|\operatorname{Im} a_j(t,x)| \leq \frac{Ca_p(t)}{\langle x \rangle^{\frac{j}{p-1}}},\ 1 \leq j \leq p-1 \tag{6}$$

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$$|\operatorname{Im} D_x{}^\beta a_j(t,x)| \leq \frac{Ca_p(t)}{\langle x \rangle^{\frac{j-[\beta/2]}{p-1}}} \quad 1 \leq [\beta/2] \leq j-1,\ 3 \leq j \leq p-1 \tag{8}$$

$$|\operatorname{Re} D_x^\beta a_j(t,x)| \leq Ca_p(t) \quad 0 \leq \beta \leq j-1,\ 3 \leq j \leq p-1 \tag{9}$$

with $C > 0$, $\forall (t,x) \in [0,T] \times \mathbb{R}$.

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More precisely: under conditions (6)-(9) the Cauchy problem

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is H^∞ well-posed with loss of derivatives:
$\exists \sigma > 0$ such that $\forall f \in C([0,T]; H^s)$, $g \in H^s$, $\exists! u \in C([0,T]; H^{s-\sigma})$ which satisfies the following energy estimate:

$$\|u(t,\cdot)\|^2_{s-\sigma} \leq C_s \left(\|g\|_s^2 + \int_0^t \|f(\tau,\cdot)\|_s^2 d\tau \right), \ \forall t \in [0,T].$$

Rem:

- If $p = 2$ we recapture [KB], if $p = 3$ we recapture [CC]
- If $a_p(t) \leq 0$ $\forall t \in [0,T]$: same result
- If $a_p(t) \geq C > 0$ $\forall t \in [0,T]$: C instead of $Ca_p(t)$ in (6)-(9)
- (slightly faster decay) if $|\operatorname{Im} a_{p-1}(t,x)| \leq Ca_p(t)\langle x\rangle^{-(1+\varepsilon)}$, $\varepsilon > 0$ $\forall (t,x) \in [0,T] \times \mathbb{R}$ we obtain H^s well posedness (without loss of derivatives).

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Theorem 2 (A. Ascanelli, C. Boiti): $m \geq 1$

$$P(t,x,D_t,D_x) = D_t^m + \sum_{j=1}^{m} \sum_{\alpha \leq jp} a_\alpha^{(j)}(t,x) D_x^\alpha D_t^{m-j} \quad (t,x) \in [0,T] \times \mathbb{R}$$

$$a_{jp}^{(j)}(t) \in C^{m-2}([0,T];\mathbb{R}), \quad 1 \leq j \leq m$$

$$a_\alpha^{(j)} \in C([0,T];\mathcal{B}^\infty), \quad a_\alpha^{(j)}(t,x) \in \mathbb{C}, \quad 0 \leq \alpha \leq jp-1,\ 0 \leq j \leq m$$

- $\tau^m + \sum_{j=1}^{m} a_{jp}^{(j)}(t)\xi^\alpha \tau^{m-j} = \prod_{j=1}^{m}(\tau - \lambda_j^{(p)}(t,\xi)),\ (\lambda_j^{(p)} \in \mathbb{R}!)$
- $|\lambda_j(t,\xi) - \lambda_k(t,\xi))| \geq C\langle\xi\rangle^p > 0, \quad j \neq k, |\xi| >> 1, \quad C > 0$

Sufficient condition for H^∞ well posedness of Cauchy problem:

$$|\operatorname{Im} a_{jp-k}{}^{(j)}(t,x)| \leq C\langle x\rangle^{-\frac{p-k}{p-1}},\ 1 \leq k \leq p-1$$

$$|\operatorname{Im} D_x a_{jp-k}{}^{(j)}(t,x)| \leq C\langle x\rangle^{-\frac{p-k-1}{p-1}},\ 1 \leq k \leq p-2$$

$$|\operatorname{Im} D_x{}^\beta a_{jp-k}{}^{(j)}(t,x)| \leq \frac{C}{\langle x\rangle^{\frac{p-k-[\beta/2]}{p-1}}} \quad 1 \leq [\beta/2] \leq p-k-1,\ 1 \leq k \leq p-3$$

for $1 \leq j \leq m,\ C > 0,\ \forall (t,x) \in [0,T] \times \mathbb{R}$ (with loss of derivatives).

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for $1 \leq j \leq m,\ C > 0,\ \forall (t,x) \in [0,T] \times \mathbb{R}$ (with loss of derivatives).

Theorem 2 (A. Ascanelli, C. Boiti): $m \geq 1$

$$P(t,x,D_t,D_x) = D_t^m + \sum_{j=1}^{m} \sum_{\alpha \leq jp} a_\alpha^{(j)}(t,x) D_x^\alpha D_t^{m-j} \quad (t,x) \in [0,T] \times \mathbb{R}$$

$$a_{jp}^{(j)}(t) \in C^{m-2}([0,T];\mathbb{R}), \quad 1 \leq j \leq m$$

$$a_\alpha^{(j)} \in C([0,T];\mathcal{B}^\infty), \quad a_\alpha^{(j)}(t,x) \in \mathbb{C}, \quad 0 \leq \alpha \leq jp-1,\ 0 \leq j \leq m$$

- $\tau^m + \sum_{j=1}^{m} a_{jp}^{(j)}(t)\xi^\alpha \tau^{m-j} = \prod_{j=1}^{m} (\tau - \lambda_j^{(p)}(t,\xi))$, $(\lambda_j^{(p)} \in \mathbb{R}!)$
- $|\lambda_j(t,\xi) - \lambda_k(t,\xi))| \geq C\langle\xi\rangle^p > 0, \quad j \neq k, |\xi| >> 1, \quad C > 0$

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Theorem 1 (A. Ascanelli, C. Boiti, L. Zanghirati):

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t,x)D_x^j$$

Sufficient condition for H^∞ well posedness of (CP):

(6) $|\operatorname{Im} a_j(t,x)| \leq \dfrac{Ca_p(t)}{\langle x\rangle^{\frac{j}{p-1}}}, \ 1 \leq j \leq p-1$

(7) $|\operatorname{Im} D_x a_j(t,x)| \leq \dfrac{Ca_p(t)}{\langle x\rangle^{\frac{j-1}{p-1}}}, \ 2 \leq j \leq p-1$

(8) $|\operatorname{Im} D_x{}^\beta a_j(t,x)| \leq \dfrac{Ca_p(t)}{\langle x\rangle^{\frac{j-[\beta/2]}{p-1}}} \quad 1 \leq [\beta/2] \leq j-1, \ 3 \leq j \leq p-1$

(9) $|\operatorname{Re} D_x^\beta a_j(t,x)| \leq Ca_p(t) \quad 0 \leq \beta \leq j-1, \ 3 \leq j \leq p-1$

with $C > 0$, $\forall (t,x) \in [0,T] \times \mathbb{R}$.

Energy estimate: $\exists \sigma > 0$ s.t. $\forall f \in C([0,T]; H^s)$, $g \in H^s$

$$\|u(t,\cdot)\|^2_{s-\sigma} \leq C_s \left(\|g\|^2_s + \int_0^t \|f(\tau,\cdot)\|^2_s d\tau \right), \ \forall t \in [0,T].$$

Sketch of the Proof of Theorem 1.

Energy method:

$$iP = \partial_t + ia_p(t)D_x^p + \sum_{j=0}^{p-1} ia_j(t,x)D_x^j := \partial_t + A$$

$$\Rightarrow \frac{d}{dt}\|u\|_0^2 = 2\,\mathrm{Re}\langle \partial_t u, u\rangle = 2\,\mathrm{Re}\langle iPu, u\rangle - 2\mathrm{Re}\langle Au, u\rangle$$
$$= 2\,\mathrm{Re}\langle if, u\rangle - 2\mathrm{Re}\langle Au, u\rangle$$
$$\leq \|f\|_0^2 + \|u\|_0^2 - 2\mathrm{Re}\langle Au, u\rangle.$$

We would like to obtain

$$\mathrm{Re}\langle Au, u\rangle \geq -C\|u\|_0^2 \tag{10}$$

Tools: sharp-GårdingTheorem + Fefferman-Phong inequality for PDOs
Recall: $A(x, D_x)$ PDO of order m has symbol $A(x,\xi)$ in the standard class S^m defined by

$$|\partial_\xi^\alpha \partial_x^\beta A(x,\xi)| \leq C_{\alpha,\beta,h}\langle\xi\rangle_h^{m-\alpha} \qquad \alpha,\beta \in \mathbb{N},\ h \geq 1,$$

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Sharp-Gårding Theorem:

Let $A(x,\xi) \in S^m$ with $\mathrm{Re}A(x,\xi) \geq 0$.
There exist then pseudo-differential operators $Q(x, D_x)$ and $R(x, D_x)$ with symbols $Q(x,\xi) \in S^m$ and $R(x,\xi) \in S^{m-1}$ such that

$$\begin{aligned} &A(x, D_x) = Q(x, D_x) + R(x, D_x) \\ &\mathrm{Re}\langle Q(x, D_x)v, v\rangle \geq 0 \qquad \forall v \in H^m \\ &R(x,\xi) \sim \psi_1(\xi) D_x A(x,\xi) + \sum_{\alpha+\beta\geq 2} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A(x,\xi), \end{aligned} \tag{11}$$

with $\psi_1 \in S^{-1}, \psi_{\alpha,\beta} \in S^{(\alpha-\beta)/2}$ real valued.

Sharp-Gårding inequality:

$$\mathrm{Re}\langle A(x, D_x)u, u\rangle \geq -C\|u\|^2_{\frac{m-1}{2}}$$

Fefferman-Phong inequality: if $A(x,\xi) \geq 0$

$$\mathrm{Re}\langle A(x, D_x)u, u\rangle \geq -C\|u\|^2_{\frac{m-2}{2}}$$

Construction of A_Λ

We construct a pseudo-differential operator $e^{\Lambda(x,D_x)}$ of order $\delta > 0$ s.t. $A_\Lambda := (e^\Lambda)^{-1} A e^\Lambda$ satisfies

$$\mathrm{Re}\langle A_\Lambda(x, D_x)v, v\rangle \geq -C\|v\|_0^2$$

and consider the Cauchy problem

$$(CP_\Lambda) \begin{cases} P_\Lambda v = f_\Lambda \\ v(0,x) = g_\Lambda \end{cases}$$

for $P_\Lambda := (e^\Lambda)^{-1} P e^\Lambda$, $f_\Lambda := (e^\Lambda)^{-1} f$, $g_\Lambda := (e^\Lambda)^{-1} g$.

Rem: v is a solution of (CP_Λ) if and only if $u = e^\Lambda v$ is a solution of (CP).

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$$(iP)_\Lambda = \partial_t + A + pa_p(t)(\partial_x\Lambda)D_x^{p-1} + R$$
$$\Rightarrow \Lambda(x,\xi) := \lambda_{p-1}(x,\xi) + \lambda_{p-2}(x,\xi) + \ldots + \lambda_1(x,\xi)$$

By asking $pa_p(t)(\partial_x\lambda_{p-j})D_x^{p-1} \sim A|_{ord(p-j)} = ia_{p-j}(t,x)D_x^{p-j}$:

$$\lambda_{p-j} := M_{p-j}\omega\left(\frac{\xi}{h}\right)\int_0^x \langle y\rangle^{-\frac{p-j}{p-1}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy\,\langle\xi\rangle_h^{-j+1}$$

$M_{p-j} > 0$ to be chosen,

$$\omega \in C^\infty(\mathbb{R}): \quad \omega(y) = \begin{cases} 0 & |y| \le 1 \\ |y|^{p-1}/y^{p-1} & |y| \ge 2 \end{cases}$$

$$\psi \in C_0^\infty(\mathbb{R}):\ 0 \le \psi \le 1, \quad \psi(y) = \begin{cases} 1 & |y| \le 1/2 \\ 0 & |y| \ge 1 \end{cases}$$

Then:

- $e^{\Lambda(x,\xi)} \in S^\delta$, $\delta = (p-1)M_{p-1}$,
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Using:
- sharp-Gårding Theorem at levels $p-1,\dots,3$
- Fefferman-Phong Inequality at level 2
- sharp-Gårding Inequality at level 1

we choose $M_{p-1},\dots,M_1$ s.t.

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$$\frac{d}{dt}\|v\|_0^2 \leq C(\|f_\Lambda\|_0^2 + \|v\|_0^2)$$

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Then $u = e^\Lambda v$ satisfies (since $e^\Lambda \in S^\delta$)

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Thank you!