Stone-Marchesin Model Equations of Three-Phase Flow in Oil Reservoir Simulation

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1. Introduction

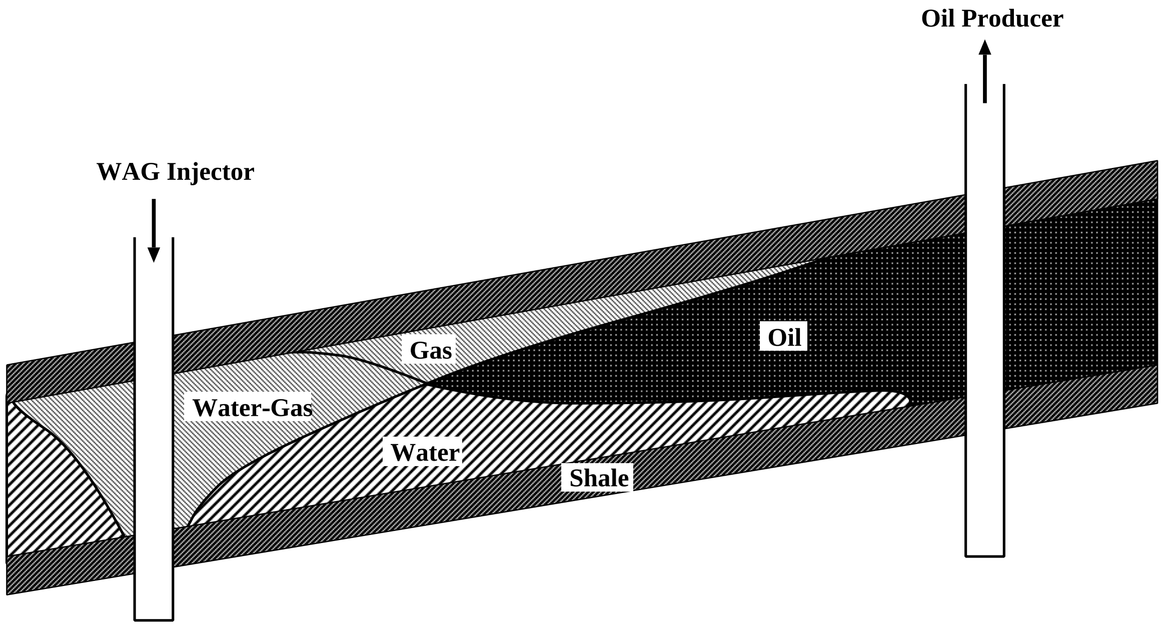


Figure 1: WAG Enhanced Oil Recovery (schematic picture)

Primary Recovery By the underground pressure, usually about 20% of the oil in an oil reservoir can be extracted.

Secondary recovery By injecting water and gas (air or CO_2), generally 25% to 35% of the oil can be extracted.

Water-Alternating-Gas (WAG) Enhanced Oil Recovery:

Water Injection Good sweep efficiency, but 40 to 60% of the original oil on-site is left behind.

Gas Injection Good displacement efficiency, but an expensive operation.

WAG Injection More efficient than injection of water or gas alone.

Overview: Marchesin D. & Plohr B. (2001), Theory of Three-Phase Flow Applied to Water-Alternating-Gas Enhanced Oil Recovery, *Proceedings of the 8th International Conference in Magdeburg*, Vol.II, Birkhäuser Verlag, 693–702.

Plan of this Talk:

- Model equations
- Hyperbolicity, elliptic region
- Geometry of characteristic field, 2-phase like flow curves
- Compressive, under and overcompressive shock waves
- Geometry of Hugoniot curves
- Entropy functions

Stone's Model [3]: Neglect the gravity. Assume: the medium is homogeneous and the flow is incompressible and immiscible.

	water	gas	oil
Volume Fractions:	$s_w = u$	$s_g = v$	$s_o = 1 - u - v$
Permeability Functions:	k_w	k_g	k_o
Fluid Viscosity:	μ_w	μ_g	μ_o
Fluid Velocity:	v_w	v_g	v_o
Pressure:	p_w	p_g	p_o

Capillary pressure:

$$p_c = p_{non\ wetting\ phase} - p_{wetting\ phase}.$$

Water-oil interface: water is the wetting phase.
Gas-oil interface: oil is the wetting phase.

$$p_{ow} = p_o - p_w, \quad p_{go} = p_g - p_o.$$

Leverett's assumption: p_{ow} is a decreasing function of $u = s_w$, and p_{go} is an increasing function of $v = s_g$.

Darcy's Law

$$v_i = -\frac{k_i}{\mu_i}\nabla p_i, \quad i = w,\ g,\ o.$$

Mass Conservation Laws:

$$\frac{\partial s_i}{\partial t} + \nabla \cdot v_i = 0, \quad i = w,\ g,\ o.$$

By eliminating $\frac{\partial p_o}{\partial x}$ and denoting

$$\mathcal{D} = \sum_{j=w,\,g,\,o} \frac{k_j}{\mu_j}$$

$$\begin{cases} \dfrac{\partial s_w}{\partial t} + \dfrac{\partial}{\partial x}\Big(\dfrac{k_w}{\mu_w \mathcal{D}}\Big) &= \dfrac{\partial}{\partial x}\Big[\dfrac{k_w}{\mu_w}\Big\{\Big(1 - \dfrac{k_w}{\mu_w \mathcal{D}}\Big)\dfrac{\partial p_{wo}}{\partial x} - \dfrac{k_g}{\mu_g \mathcal{D}}\dfrac{\partial p_{go}}{\partial x}\Big\}\Big], \\ \dfrac{\partial s_g}{\partial t} + \dfrac{\partial}{\partial x}\Big(\dfrac{k_g}{\mu_g \mathcal{D}}\Big) &= \dfrac{\partial}{\partial x}\Big[\dfrac{k_g}{\mu_g}\Big\{-\dfrac{k_w}{\mu_w \mathcal{D}}\ \dfrac{\partial p_{wo}}{\partial x} + \Big(1 - \dfrac{k_g}{\mu_g \mathcal{D}}\Big)\dfrac{\partial p_{go}}{\partial x}\Big\}\Big] \end{cases}$$

Stone's assumption: The water and gas permeability functions depend only on the water and gas volume fraction

$$k_w = k_w(u), \quad k_g = k_g(v).$$

If the capillary pressure is negligible, by using *relative permeability functions*

$$f(u) = \frac{k_w(u)}{\mu_w}, \quad g(v) = \frac{k_g(v)}{\mu_g} \quad h(u, v) = \frac{k_o(u, v)}{\mu_o}$$

$$\text{Water:} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{f(u)}{f(u)+g(v)+h(u,v)}\right] \;=\; 0, \tag{1}$$

$$\text{Gas:} \quad \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left[\frac{g(v)}{f(u)+g(v)+h(u,v)}\right] \;=\; 0 \tag{2}$$

for $(u,v) \in \Omega : \; 0 < u + v < 1, \; u, \; v > 0$.

If $w = 0$, 2-phase flow is governed by the Buckley-Leverett equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{f(u)}{f(u)+g(1-u)}\right) = 0.$$

N.B. 1 *An example of a single conservation law with a* non-convex *flux function.*

2. Geometry of Characteristic Vector Field

Hyperbolicity: Denote

$$\mathcal{D} = f(u) + g(u) + h(u,v), \quad F(U) = {}^t\!\left(\frac{f}{\mathcal{D}}, \frac{g}{\mathcal{D}}\right).$$

Hyperbolic $F'(U)$ has *real* eigenvalues $\lambda_1(U), \lambda_2(U)$ for any $U \in \Omega$.

Strictly Hyperbolic Eigenvalues are *distinct*: $\lambda_1(U) < \lambda_2(U)$.

Umbilical Point U^* $\lambda_1(U^*) = \lambda_2(U^*)$ and $F'(U)$ is diagonalizable, hence a scalar matrix.

The eigenvalue $\lambda_j(U)$ is called the jth *characteristic speed* and corresponding right eigenvector $R_j(U)$ is called the jth *characteristic vector field*.

2-Phase Like Flow Curves (Medeiros [16]):

$$F'(U) = \frac{1}{\mathcal{D}^2} \begin{pmatrix} f'(g+h) - fh_u & -f(g'+h_v) \\ -g(f'+h_u) & g'(f+h) - gh_v \end{pmatrix}$$

$R_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, R_v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, R_w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively, are characteristic vectors at $v = 0, u = 0, w = 0$, respectively.

2-phase like flow curves $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are defined by

$$\mathcal{L}_1 : \quad (f' - g')h + (f+g)(h_u - h_v) = 0,$$
$$\mathcal{L}_2 : \quad g' + h_v = 0, \quad \mathcal{L}_3 : \quad f' + h_u = 0.$$

where R_w, R_v, R_u, respectively are characteristic vectors on $\mathcal{L}_1, \mathcal{L}_2$, $\mathcal{L}_3$, respectively.

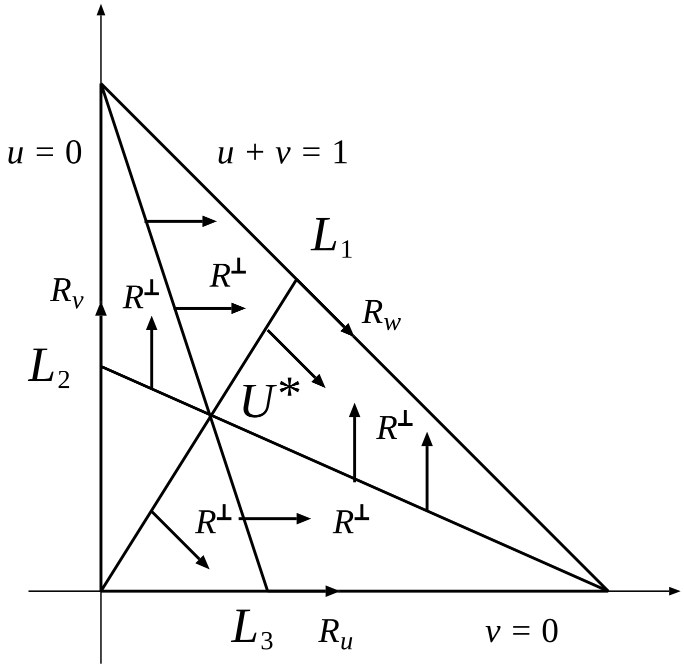


Figure 2: 2-Phase Like Curves (Quadratic Marchesin's Model)

Introduce the 2-phase like variable:

$$\xi = (f' - g')h - (f + g)(h_u - h_v), \quad \eta = g' + h_v, \quad \zeta = f' + h_u.$$

Lemma 1 *The discriminant has the form:*

$$\begin{aligned} D_{\text{char}} &= \frac{1}{\mathcal{D}^4} \Big[\{f'(g + h) - g'(f + h) - fh_u + gh_v\}^2 \\ &\qquad + 4fg(f' + h_u)(g' + h_v)] \\ &= \frac{1}{\mathcal{D}^4} \left\{ (\xi - f\eta + g\zeta)^2 + 4fg\eta\zeta \right\} \\ &= \frac{1}{\mathcal{D}^4} \left\{ \xi^2 - 2\xi(f\eta - g\zeta) + (f\eta + g\zeta)^2 \right\}. \end{aligned}$$

Theorem 1 1. *The system is* hyperbolic *in the following 3 regions:*

$$(1)\ \eta\zeta > 0,\ (2)\ \xi > 0,\ \eta > 0,\ \zeta < 0,\ (3)\ \xi < 0,\ \eta < 0,\ \zeta > 0$$

2. Elliptic regions *appear in the following 2 regions:*

$$(1)\ \xi < 0,\ \eta > 0,\ \zeta < 0,\ (2)\ \xi > 0,\ \eta < 0,\ \zeta > 0$$

3. *If* $\xi = \eta = \zeta = 0$ *at* U^*, *then* U^* *is an* umbilical point.

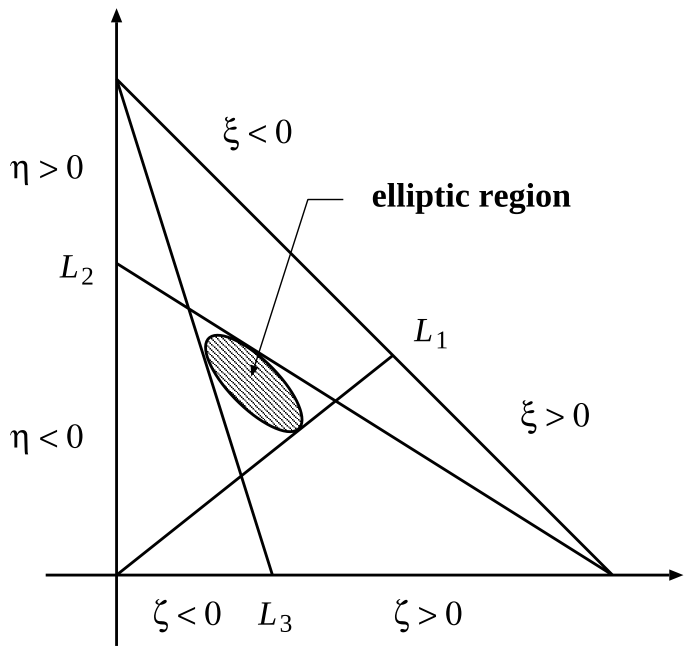


Figure 3: Existence of Elliptic Region ($\xi > 0, \eta < 0, \zeta > 0$)

Integral Curves of Characteristic Vector Fields: Let A be a 2×2 matrix. Note that

$$\boldsymbol{x} : \text{ an eigenvectors of } A \iff {}^t\boldsymbol{x}^{\perp} A \boldsymbol{x} = 0, \quad ({}^t\boldsymbol{x}^{\perp}\boldsymbol{x} = 0).$$

The integral curve of the characteristic vector fields: solutions to the differential equation

$$^t\dot{U}^{\perp} F'(U)\dot{U} = 0.$$

The equation of the trajectory:

$$g\zeta du^2 + (\xi - f\eta + g\zeta)dudv - f\eta dv^2 = 0. \tag{3}$$

Equivalently

$$\frac{dv}{du} = \frac{1}{2f\eta}\left(\xi - f\eta + g\zeta \pm \sqrt{\Delta}\right)$$

or

$$\frac{du}{dv} = -\frac{1}{2g\zeta}\left(\xi - f\eta + g\zeta \pm \sqrt{\Delta}\right)$$

where

$$\Delta = \left\{\xi^2 - 2\xi(f\eta - g\zeta) + (f\eta + g\zeta)^2\right\}.$$

Notice that

$$|\xi - f\eta + g\zeta| < \sqrt{\Delta},\ if\ \ \eta\zeta > 0,$$
$$|\xi - f\eta + g\zeta| > \sqrt{\Delta},\ if\ \ \eta\zeta < 0.$$

Marchesin's model: Marchesin, Paes-Leme, Schaeffer & Shearer [17].

Theorem 2 (Existence of Umbilical Point) *Assume:*

$$h(u, v) = h(1 - u - v),$$

$$f(0) = g(0) = h(0) = 0,\ f''(u), g''(v), h''(w) > 0.$$

Then the system of equations (1), (2) *is* hyperbolic *and has a* unique umbilical point *in* Ω.

The integral curves of the characteristic vector fields sketched:

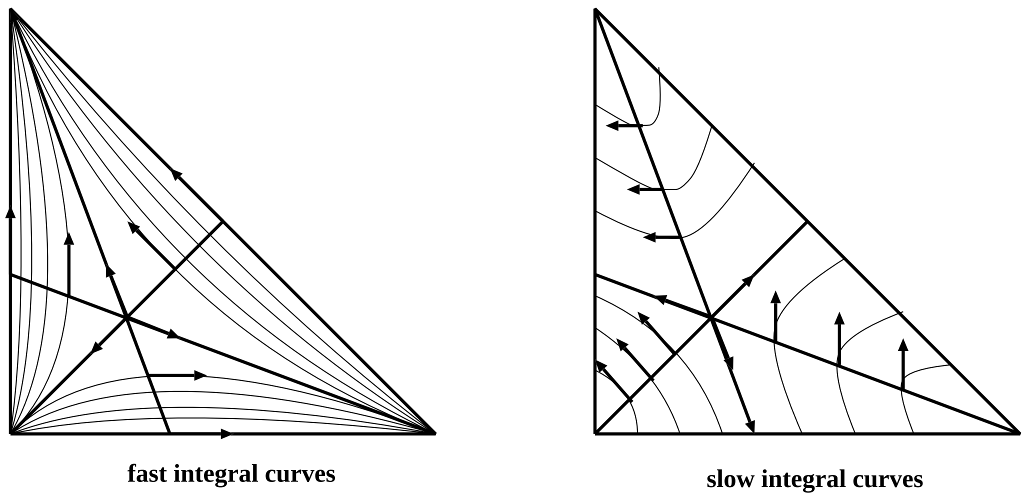


Figure 4: Integral Curves

N.B. 2 *Impossible to construct globally in Ω fast or slow characteristic fields.*

3. Quadratic Marchesin Model

Quadratic Relative Permeability Functions:

$f(u) = \alpha u^2,\ g(v) = \beta v^2,\ h(u,v) = \gamma(1-u-v)^2, \quad \alpha,\ \beta,\ \gamma > 0$

The model equations:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\alpha u^2}{\alpha u^2 + \beta v^2 + \gamma(1-u-v)^2}\right] = 0, \qquad (4)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\beta v^2}{\alpha u^2 + \beta v^2 + \gamma(1-u-v)^2}\right] = 0. \qquad (5)$$

A unique umbilical point and the coincident characteristic speed:

$$U^* = \frac{\gamma}{\beta\gamma + \gamma\alpha + \alpha\beta}\begin{pmatrix}\beta \\ \alpha\end{pmatrix}, \quad \lambda^* = \frac{2\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)}{\alpha\beta^2\gamma^2 + \beta\gamma^2\alpha^2 + \gamma\alpha^2\beta^2}.$$

2-phase like curves:

$$\mathcal{L}_1: \quad \alpha u - \beta v = 0$$

$$\lambda = \frac{2\gamma\alpha w u}{\mathcal{D}^2},\ R = \begin{pmatrix} \beta \\ \alpha \end{pmatrix},\ \lambda^{\perp} = \frac{2\alpha u}{\mathcal{D}}, \quad R^{\perp} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\mathcal{L}_2: \quad (\beta + \gamma)v - \gamma(1-u) = 0$$

$$\lambda = \frac{2\alpha\beta u v}{\mathcal{D}^2},\ R = \begin{pmatrix} \beta + \gamma \\ -\gamma \end{pmatrix},\ \lambda^{\perp} = \frac{2\beta v}{\mathcal{D}},\ R^{\perp} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\mathcal{L}_3: \quad (\alpha + \gamma)u - \gamma(1-v) = 0$$

$$\lambda = \frac{2\alpha\beta u v}{\mathcal{D}^2},\ R = \begin{pmatrix} -\gamma \\ \alpha + \beta \end{pmatrix},\ \lambda^{\perp} = \frac{2\alpha u}{\mathcal{D}},\ R^{\perp} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

Theorem 3 *Each 2-phase like flow curve $\mathcal{L}_j$, $j = 1, 2, 3$ is a line and the characteristic vector field R is parallel to the direction of the line.*

Useful lemmas: Along the 2-phase like flow curve $\mathcal{L}_1 : u = \beta\tau$, $v = \alpha\tau$,

$$\mathcal{D} = (\alpha + \beta)(\beta\gamma + \gamma\alpha + \alpha\beta)\tau^2 - 2\gamma(\alpha + \beta)\tau + \gamma.$$

Hence $\mathcal{D}_\tau = 0$, if and only if $(\beta\gamma + \gamma\alpha + \alpha\beta)\tau = \gamma$.

Lemma 2 *The quantity $\mathcal{D}$ attains its minimum at the umbilical point.*

Confine our attention to the 2-phase like flow curve $\mathcal{L}_1$. At $u = v = 0$, we have $\lambda = \lambda^\perp = 0$ and at $u + v = 1\,(w = 0)$, $\lambda = 0$, $\lambda^\perp > 0$.

Lemma 3 *The characteristic speed λ attains its maximum in the interior of each 2-phase like flow curve.*

Lemma 4

$$\left.\frac{\partial\lambda}{\partial w}\right|_{U=U_*} = \frac{2\alpha\beta\gamma(\alpha\beta-\beta\gamma-\gamma\alpha)}{(\alpha+\beta)(\beta\gamma+\gamma\alpha+\alpha\beta)\mathfrak{D}} \quad on \quad \mathfrak{L}_1$$

$$\left.\frac{\partial\lambda}{\partial v}\right|_{U=U_*} = \frac{2\alpha\beta(\beta\gamma-\gamma\alpha-\alpha\beta)}{(\beta\gamma+\gamma\alpha+\alpha\beta)\mathfrak{D}} \quad on \quad \mathfrak{L}_2$$

$$\left.\frac{\partial\lambda}{\partial u}\right|_{U=U_*} = \frac{2\alpha\beta(\gamma\alpha-\alpha\beta-\beta\gamma)}{(\beta\gamma+\gamma\alpha+\alpha\beta)\mathfrak{D}} \quad on \quad \mathfrak{L}_3$$

Study $\mathfrak{L}_1$ as a rarefaction or shock curve:

1. If $\alpha\beta > \beta\gamma+\gamma\alpha$, then $\mathfrak{L}_1$ consists of slow rarefaction curve and fast shock curve. The slow rarefaction curve ends in the interior of the 2-phase like flow curve.
2. If $\alpha\beta = \beta\gamma+\gamma\alpha$, then $\mathfrak{L}_1$ consists of slow and fast shock curves.

3. If $\alpha\beta < \beta\gamma + \gamma\alpha$, then $\mathfrak{L}_1$ consists of slow shock curve and fast rarefaction curve. The fast rarefaction curve ends in the interior of the 2-phase like flow curve.

N.B. 3 *Note that: if $\alpha\beta \geq \beta\gamma + \gamma\alpha$, then*

$$\beta\gamma < \gamma\alpha + \alpha\beta, \quad \gamma\alpha < \alpha\beta + \beta\gamma.$$

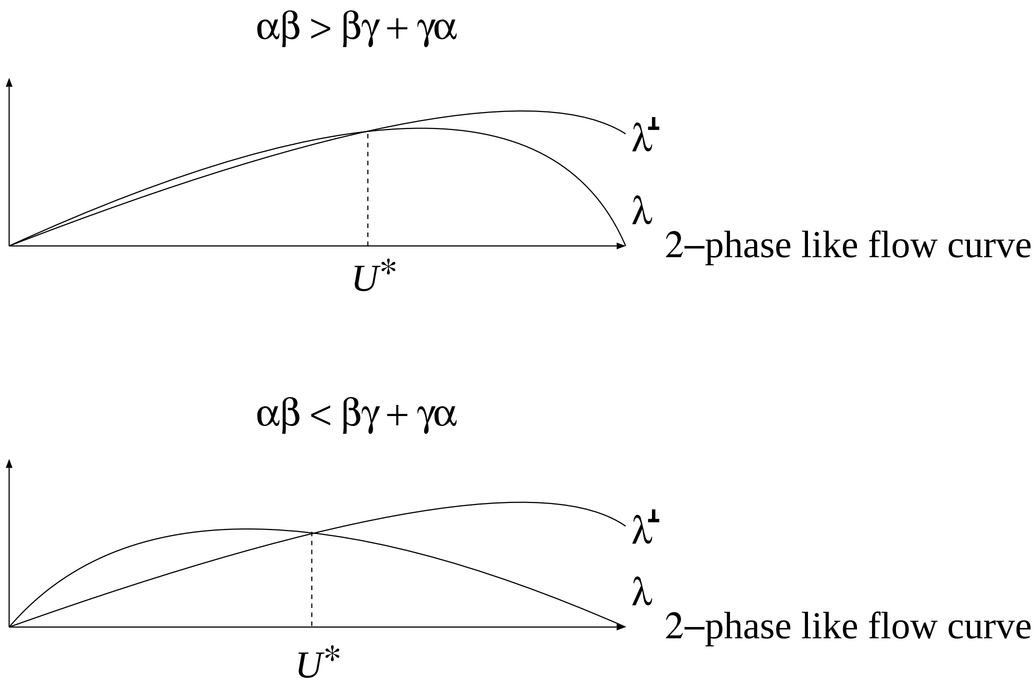
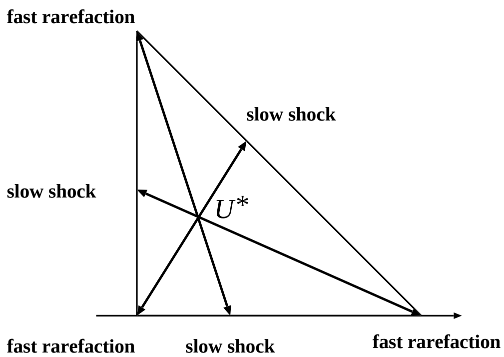
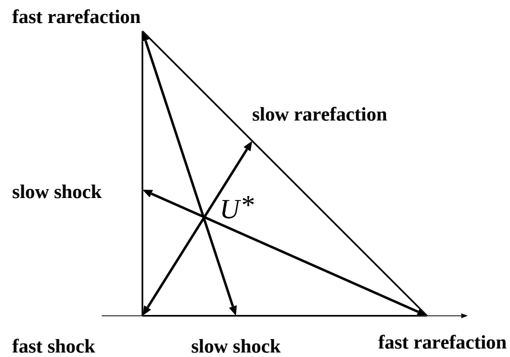


Figure 5: Characteristic Speeds



Case I



Case II ($\alpha\beta > \beta\gamma + \gamma\alpha$)

Theorem 4 (Case I) *Suppose that the following (all) three inequalities hold*

$$\alpha\beta < \beta\gamma + \gamma\alpha, \;\; \beta\gamma < \gamma\alpha + \alpha\beta, \;\; \gamma\alpha < \alpha\beta + \beta\gamma. \tag{6}$$

Then each 2-phase like flow curve in a neighbourhood of the umbilical point consists of slow shock curve and fast rarefaction curve. The fast rarefaction curves end in the interior of 2-phase like flow curves.

Theorem 5 (Case II) *Suppose that one of the following three inequalities hold*

$$\alpha\beta > \beta\gamma + \gamma\alpha, \;\; \beta\gamma > \gamma\alpha + \alpha\beta, \quad \gamma\alpha > \alpha\beta + \beta\gamma. \qquad (7)$$

Then then one of the three 2-phase like flow curve in a neighbourhood of the umbilical point consists of slow rarefaction curve and fast shock curve. Each of the other two 2-phase like flow curves has the property of Case I. The fast and slow rarefaction curves end in the interior of 2-phase like flow curves.

Schaeffer-Shearer's Classification [17]: May assume that $U^* = O$ and $F(O) = O$. Taylor expansion of the flux function $F(U)$ near $U = O$:

$$F(U) = \lambda^* U + Q(U) + O(1)|U|^3$$

where $\lambda^* = \lambda_1(U^*) = \lambda_2(U^*)$ and $Q : \mathbb{R}^2 \to \mathbb{R}^2$ is a homogeneous quadratic mapping.

Schaeffer-Shearer [17] shows that every hyperbolic quadratic mapping $Q(U)$ with an isolated umbilical point $U = O$ is equivalent to

$$Q(U) = \frac{1}{2}\begin{pmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{pmatrix} = \frac{1}{2}\nabla C(U), \tag{8}$$

$$C(U) = \frac{1}{3}au^3 + bu^2 v + uv^2. \tag{9}$$

where a and b are two real parameters satisfying $a \neq 1 + b^2$. In their classification:

$$\textbf{Case I}: \ a < \frac{3}{4}b^2 \quad \text{or} \quad \textbf{Case II}: \ \frac{3}{4}b^2 < a < 1 + b^2.$$

4. Undercompressive and Overcompressive Shock Waves

Rankine-Hugoniot condition: A jump discontinuity:

$$U(x,t) = \begin{cases} U_L & \text{for } x < st, \\ U_R & \text{for } x > st, \end{cases} \tag{10}$$

a piecewise constant weak solution to the the conservation laws$\Longleftrightarrow$ the *Rankine-Hugoniot condition*:

$$s(U_R - U_L) = F(U_R) - F(U_L). \tag{11}$$

The weak solution (10) satisfying (11) is often called a *shock wave* of speed s joining the state U_L, on the left, to the state U_R, on the right.

In the quadratic model, each 2-phase like flow curve $\mathcal{L}_j$, $j = 1, 2, 3$ is a line and the characteristic vector field R is parallel to the direction of the line.

Lemma 5 (B. Temple) *Suppose that an integral curve of a characteristic vector field constitutes a line. Then the line is also a Hugoniot curve.*

Compressive shock wave: The shock wave is said to be a *j-compressive* ($j = 1, 2$) if the speed satisfies the *Lax entropy conditions*:

$$\lambda_j(U_R) < s < \lambda_j(U_L),\ \ \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R)$$

Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$.

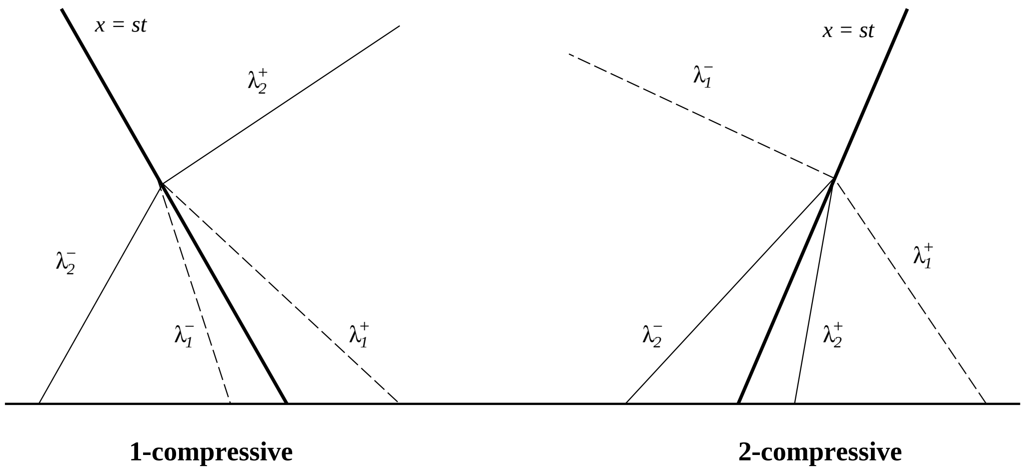


Figure 6: Compressive Shock waves

Undercompressive shock wave: *Undercompressive* if s satisfies

$$\lambda_1(U_R) < s < \lambda_2(U_R),\ \lambda_1(U_L) < s < \lambda_2(U_L)$$

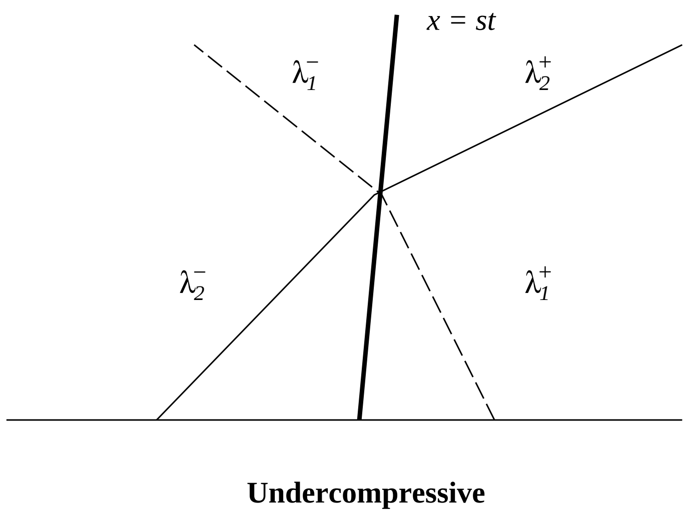


Figure 7: Undercompressive Shock wave

Overcompressive shock wave: *Overcompressive* if s satisfies

$$\lambda_1(U_R) < s < \lambda_1(U_L),\ \lambda_2(U_R) < s < \lambda_2(U_L)$$

Overcompressive shock waves appear only in Case II.

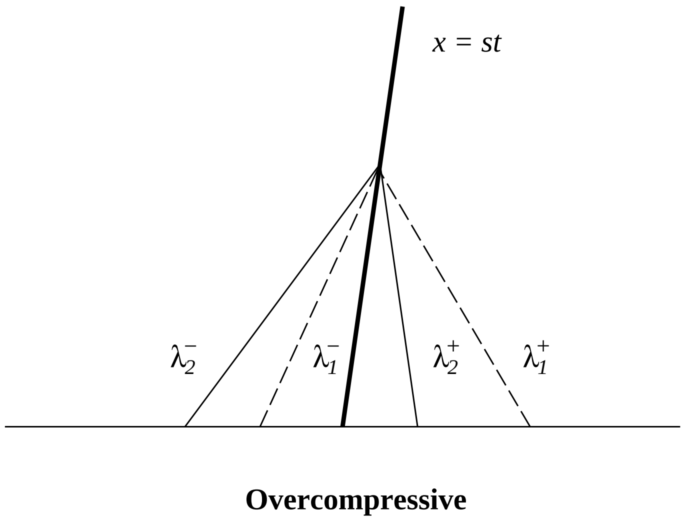


Figure 8: Overcompressive Shock wave

Stability and Admissibility of Shock Waves: It is generally believed

- *Compressive shock waves* are generally stable and admissibility is independent of diffusion matrices in a generic class.
- *Undercompressive shock waves* are stable with additional (kinetic) condition and admissibility depends on diffusion matrices.
- *Overcompressive shock waves* are generally unstable.

Undercompressive shock waves appear in Case I and II.

Theorem 6 (Case I) *Suppose that the (all) three inequalities*

$$\alpha\beta < \beta\gamma + \gamma\alpha, \quad \beta\gamma < \gamma\alpha + \alpha\beta, \quad \gamma\alpha < \alpha\beta + \beta\gamma.$$

hold Then on each 2-phase like flow curve, there exist undercompressive shock waves connecting two states on the curve.

Overcompressive shock waves appear only in Case II.

Theorem 7 (Case II) *Suppose that one of the three inequalities*

$$\alpha\beta > \beta\gamma + \gamma\alpha, \quad \beta\gamma > \gamma\alpha + \alpha\beta, \quad \gamma\alpha > \alpha\beta + \beta\gamma.$$

holds. Then on one of the three 2-phase like flow curve, there exist overcompressive shock waves connecting two states on the curve. Each of the other two 2-phase like flow curves has the property of Case I.

5. Hugoniot Loci

The *Hugoniot locus* of U_0, denoted by $\mathcal{H}(U_0)$: the projection of the set of (U, s) satisfying the Rankine-Hugoniot condition

$$H_{U_0}(U, s) := -s(U - U_0) + F(U) - F(U_0) = O \qquad (12)$$

on to the U plane.

Equivalently, the set of the states U satisfying $F(U) - F(U_0) \propto U - U_0$.

For the Stone-Marchesin model: the plane cubic curve defined by

$$(v - v_0)(\mathcal{D}_0 f(u) - \mathcal{D} f(u_0)) = (u - u_0)(\mathcal{D}_0 g(u) - \mathcal{D} g(v_0)) \quad (13)$$

where $\mathcal{D}_0 = f(u_0) + g(v_0) + h(w_0)$.

Since the above cubic curve has a singularity at $U_0 = {}^t(u_0, v_0)$, it is a *rational curve*. U_0 is the *primary bifurcation point*. By introducing a parameter ξ as

$$v - v_0 = \xi(u - u_0).$$

Lemma 6 *The Hugoniot locus has a rational parametrization*

$$u - u_0 = \frac{2[\alpha u_0\{h_0\xi + (1+\xi)g_0\} - \beta v_0 \xi\{h_0 + (1+\xi)f_0\} + \gamma w_0(\xi f_0 - g_0)(1+\xi)]}{\beta\xi^2\{h_0 + (1+\xi)f_0\} - \alpha\{h_0\xi + (1+\xi)g_0\} + \gamma(\xi f_0 - g_0)(1+\xi)^2}.$$

and $v - v_0 = \xi(u - u_0)$, *where* $f_0 = \alpha u_0^2$, $g_0 = \beta v_0^2$ *and* $h_0 = \gamma w_0^2$.

N.B. 4 *A shock curve is a* C^1 *piece of the Hugoniot locus.*

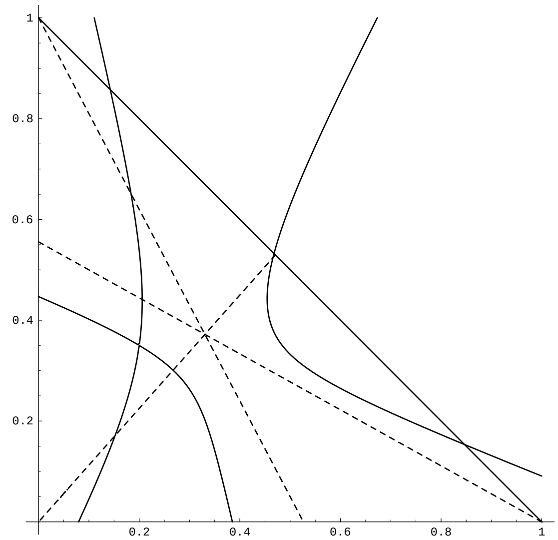


Figure 9: $\alpha = 0.9, \beta = 0.8, \gamma = 1.0, U_L = (0.20,\ 0.35)$

Study degenerate cases.

U_0 on the boundary: The boundaries $u = 0$, $v = 0$ and $w = 0$ are themselves shock curves. Let $w_0 = 0$ and set

$u_0 = z \geq 0,\ v_0 = 1 - z \geq 0$. Then

$$u - u_0 = \frac{2\alpha\beta z(1-z)\left\{1 - z(1+\xi)\right\}}{\alpha(\beta+\gamma)z^2\xi^2 + \gamma\left\{\alpha z^2 - \beta(1-z)^2\right\}\xi - \beta(\gamma+\alpha)(1-z)^2}$$

and $v - v_0 = \xi(u - u_0)$ showing that the Hugoniot locus is composed of a hyperbola plus the boundary $w = 0$. There are no *secondary bifurcation points*.

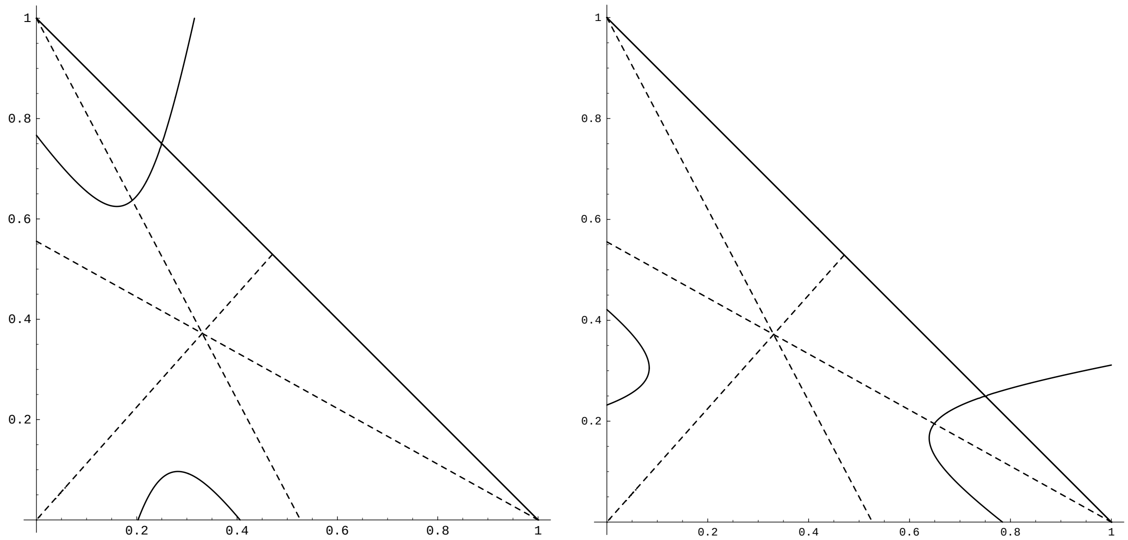


Figure 10: Hugoniot Locus of U_0 : $w_0 = 0$. (left: $0 < z < \frac{1}{2}$, right: $\frac{1}{2} < z < 1$)

U_0 on the 2-phase like curves: The 2-phase like curves $\mathcal{L}_j = 0$ $(j = 1,\, 2,\, 3)$are themselves shock curves. Let $U_0 \in \mathcal{L}_1$ and set $u_0 = \dfrac{\beta z}{\alpha + \beta}$, $v_0 = \dfrac{\alpha z}{\alpha + \beta}$ $(0 \leq z \leq 1)$.

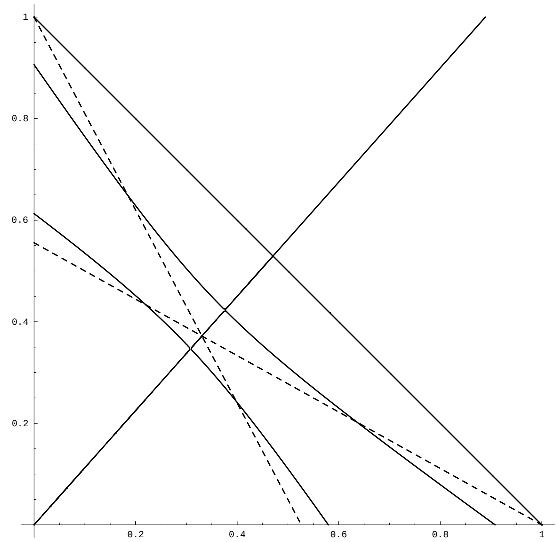


Figure 11: Hugoniot Locus of $U_0 \in \mathcal{L}_1$ and Secondary Bifurcation Point U_{II}

Then

$$u - u_0 = \frac{2\alpha\beta z^2(1+\xi)\left\{\gamma(1-z) - \frac{\alpha\beta}{\alpha+\beta}z\right\}}{\alpha\beta\gamma z^2(1+\xi)^2 + \alpha\beta z^2(1+\xi)(\beta\xi+\alpha) + \gamma(\alpha+\beta)(1-z)^2\xi}$$

and $v - v_0 = \xi(u - u_0)$ showing that the Hugoniot locus is a part of a hyperbola plus the 2-phase like curve $\mathfrak{L}_1$.

By setting $\xi = \dfrac{\alpha}{\beta}$, we have

$$u_{\mathrm{II}} - u_0 = \frac{2\beta z^2 \left\{\gamma(1-z) - \frac{\alpha\beta}{\alpha+\beta} z\right\}}{\gamma(\alpha+\beta)z^2 + 2\alpha\beta z^2 + \gamma(\alpha+\beta)(1-z)^2}.$$

Here $U_{\mathrm{II}} = {}^t(u_{\mathrm{II}}, v_{\mathrm{II}})$ represents the secondary bifurcation point. Notice that the umbilical point is expressed as

$$u^* - u_0 = \frac{\beta \left\{\gamma(1-z) - \frac{\alpha\beta}{\alpha+\beta} z\right\}}{\beta\gamma + \gamma\alpha + \alpha\beta}.$$

By direct computation, we find

$$U_{\mathrm{II}} = U^* \quad \Leftrightarrow \quad z = \frac{1}{2}$$

and

$$u_{\mathrm{II}} < u^* \quad \text{for} \quad z = 0,\ 1.$$

Thus, by setting $U_0^* = U|_{z=\frac{1}{2}} = {}^t\!\left(\frac{\beta}{2(\alpha+\beta)}, \frac{\alpha}{2(\alpha+\beta)}\right)$, we have

$$u_0^* - u^* = \frac{\beta\left\{\alpha\beta - (\beta\gamma + \gamma\alpha)\right\}}{2(\alpha+\beta)(\beta\gamma + \gamma\alpha + \alpha\beta)}.$$

Theorem 8 *If* $\alpha\beta < \beta\gamma + \gamma$ *then*

$$u_0^* < u^*$$

and

$$u_{\mathrm{II}} < u^* \quad \textit{if} \quad U_0 \notin U_0^* U^*, \qquad u_{\mathrm{II}} \geq u^* \quad \textit{if} \quad U_0 \in U_0^* U^*.$$

N.B. 5 *The three 2-phase like curves are Hugoniot loci of the umbilical point. However these curves are* **not** *parallel to asymptotes of generic Hugoniot curves.*

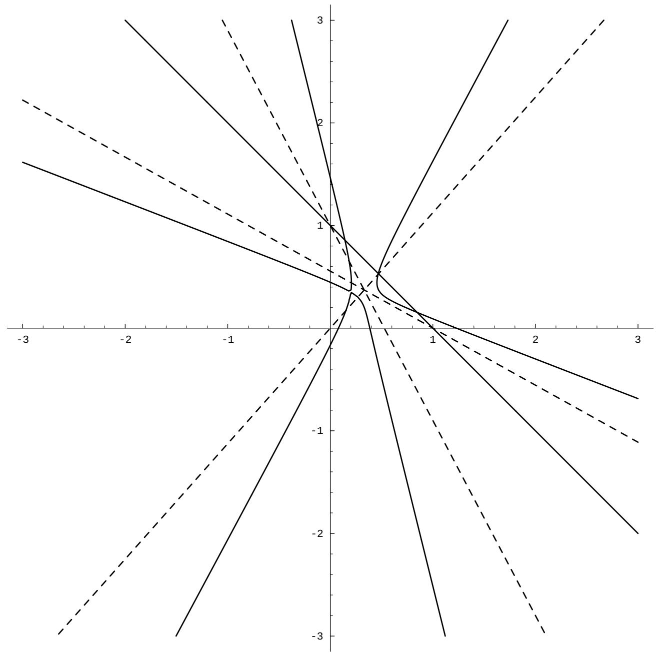


Figure 12: $\alpha = 0.9, \beta = 0.8, \gamma = 1.0, U_L = (0.20,\ 0.35)$

6. Invariant Region

The boundary of Ω consists of 3 segments each of which is a rarefaction curve and at the same time a shock curve. Hoff [8] says that Ω is an *invariant region*.

Lax-Friedrichs and Godunov approximations: Lax-Friedrichs and Godunov approximations are based on taking averages and integral averages,respectively, of solution to Riemann problems. Since Ω is *convex*, we have

Theorem 9 *Suppose that all initial all states are sufficiently close and Courant-Friedrichs-Lewy condition holds. Then Ω is an invariant region for Lax-Friedrichs and Godunov approximations.*

7. Entropy Functions

Entropy Equation: The entropy-entropy flux pair (H, Q) satisfies the compatibility condition

$$\nabla Q = F'(U)^t \nabla H.$$

The integrability condition for Q yields the second order linear partial differential equation

$$f\eta H_{uu} + (\xi - f\eta + g\zeta)H_{uv} - g\zeta H_{vv} = 0. \tag{14}$$

Proposition 1 (Lax [11]) *The linear partial differential equation* (14) *is strictly hyperbolic if and only if* (u, v) *belongs to the strictly hyperbolic region of the original equations* (1), (2) The *characteristic condition* is

$$f\eta\phi_u^2 + (\xi - f\eta + g\zeta)\phi_u\phi_v - g\zeta\phi_v^2 = 0 \tag{15}$$

and solutions to the differential equation

$$f\eta\left(\frac{dv}{du}\right)^2 - (\xi - f\eta + g\zeta)\left(\frac{dv}{du}\right) - g\zeta = 0$$

are called *bicharacteristic curves*.

Proposition 2 (Lax [11]) *The bicharacteristic curves are the integral curves of the characteristic vector fields and the Riemann invariants $\phi = w, z$ satisfy the characteristic condition* (15)*.*

Let $\lambda, \mu\,(\lambda < \mu)$ denote the characteristic speeds in a hyperbolic region. By using the Riemann (characteristic) coordinates w, z, the entropy equation (14) has the form

$$H_{wz} + \frac{1}{\lambda - \mu}\{\lambda_z H_w - \mu_w H_z\} = 0. \tag{16}$$

Suppose that the rectangle $[w_0, w_1] \times [z_0, z_1]$ is contained in a strictly hyperbolic region. Entropy functions are constructed by assigning the boundary condition

$$H(w, z_0) = \Phi(w), \quad H(w_0, z) = \Psi(z), \tag{17}$$

which is the *Goursat problem.*

Theorem 10 *Let* $\Phi(w)$, $\Psi(z)$ *are Lipschitz continuous function defined on the interval* $[w_0, w_1]$, $[z_0, z_1]$, *respectively. Then there exists a unique entropy function* $H(w, z)$ *defined on the rectangle* $[w_0, w_1] \times [z_0, z_1]$ *satisfying the boundary condition* (17).

Full proof is found in Sobolev [18].

Covering Regions: It is impossible to construct globally in Ω the fast and slow characteristic fields. Let Ω_ϵ denote Ω with removing small neighbourhoods of three vertices and the umbilical point.

Theorem 11 *There exist a double covering of Ω_ϵ and a triple covering of corresponding region in wz-plane such that the fast and slow characteristic fields are globally well defined. The entropy equation* (14) *and its characteristic form* (16) *are well-deifined on those covering region.*

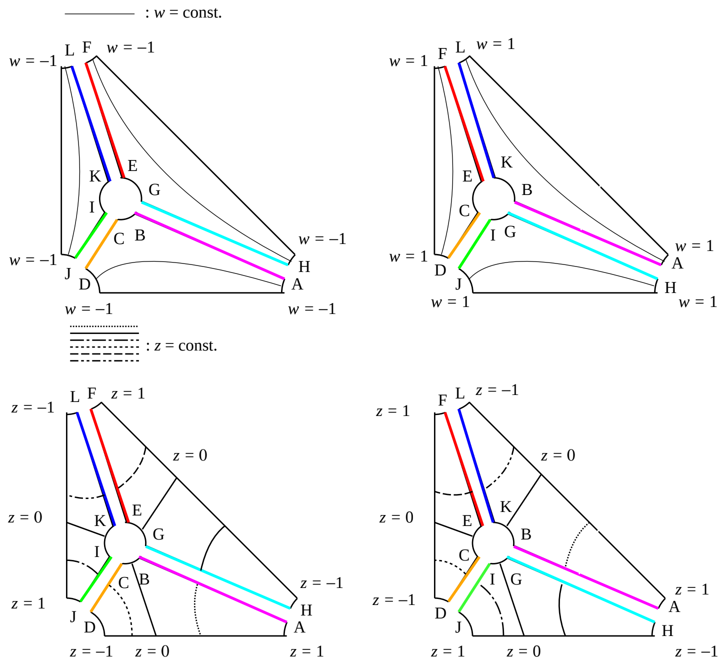


Figure 13: Fast and Slow Integral Curves

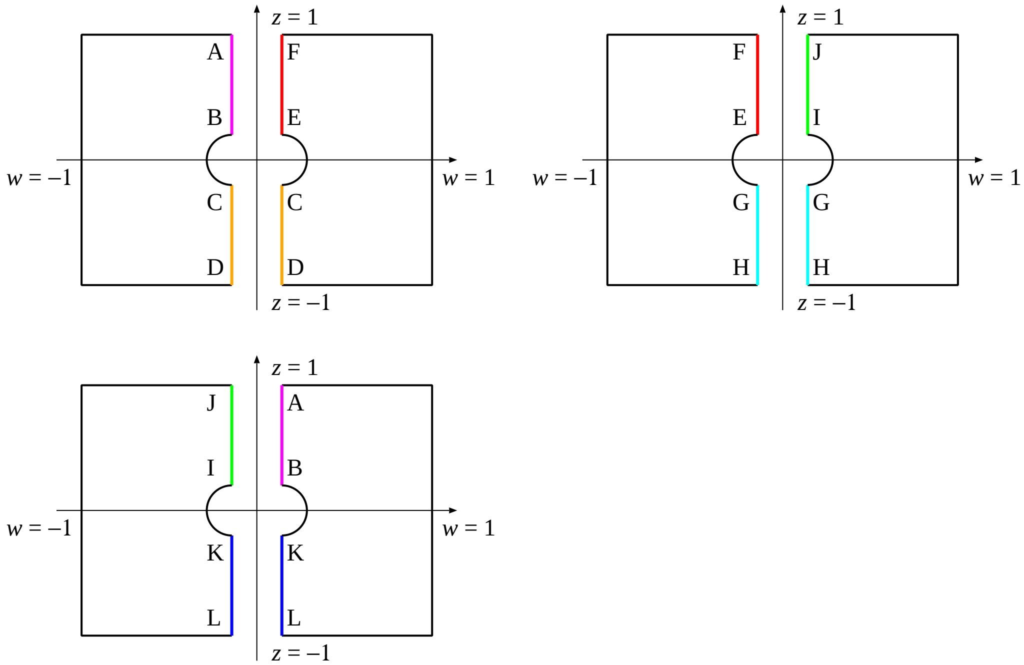


Figure 14: Covering of wz-Plane

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