A Finite Volume Evolution Galerkin Scheme for Wave Propagation in Heterogeneous Media

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Outline

1. Introduction
   - Aim of the present work
   - Governing Equations

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   - Aim of the present work
   - Governing Equations
2. Bicharacteristics of Multi-dimensional Hyperbolic Systems
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Aim

- To model the propagation of acoustic waves in heterogeneous media.
- To extend the Finite Volume Evolution Galerkin (FVEG) scheme for linear hyperbolic systems with spatially varying flux functions.
- To derive a genuinely multi-dimensional finite volume scheme for the acoustic wave equation system.

Governing Equations

Propagation of acoustic waves in an ideal gas at rest initially is given by,

$$\begin{pmatrix} p \\ \rho_0 u \\ \rho_0 v \end{pmatrix}_t + \begin{pmatrix} \gamma p_0 u \\ p \\ 0 \end{pmatrix}_x + \begin{pmatrix} \gamma p_0 v \\ p \\ 0 \end{pmatrix}_y = 0, \quad (1)$$

$\rho_0 = \rho_0(x, y)$, $p_0 =$ const are the ambient density and pressure. In non-conservation form

$$\mathbf{v}_t + \mathbf{A}_1 \mathbf{v}_x + \mathbf{A}_2 \mathbf{v}_y = 0, \quad (2)$$

where $\mathbf{v} = \begin{pmatrix} p \\ u \\ v \end{pmatrix}$, $\mathbf{A}_1 = \begin{pmatrix} 0 & \gamma p_0 & 0 \\ \frac{1}{\rho_0} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$\mathbf{A}_2 = \begin{pmatrix} 0 & 0 & \gamma p_0 \\ 0 & 0 & 0 \\ \frac{1}{\rho_0} & 0 & 0 \end{pmatrix}$. Note that (2) is a linear system with spatially varying coefficients.

Definition of a characteristic surface

Definition

A characteristic surface $\Omega\colon \varphi(x, y, t) = 0$ of (1) is a surface of discontinuity of the first derivatives.

The one parameter family of characteristic surfaces $\varphi(x, y, t) = \text{const}$ is govered by the characteristic partial differential equation

$$Q(\mathbf{x}, \varphi_t, \nabla\varphi) \equiv \det\left(\varphi_t \mathbf{I} + \varphi_x \mathbf{A}_1 + \varphi_y \mathbf{A}_2\right) = 0, \tag{3}$$

$\mathbf{A}_1$ and $\mathbf{A}_2$ are the flux Jacobian matrices. Note that (3) is a nonlinear first order PDE for φ, of the Hamilton-Jacobi type. This is a generalization of the eikonal equation in optics.

Definition of a bicharacteristic curve

Definition

The characteristic curves of (3) are called bicharacteristic curves

- These are curves in (x, y, t)-space.
- The generators of characteristic surfaces.
- Advection curves, stream lines for Euler equations.
- A hyperbolic system of m equations has m families of bicharacteristic curves.

[Courant-Hilbert 1962, Prasad 2001]

Examples

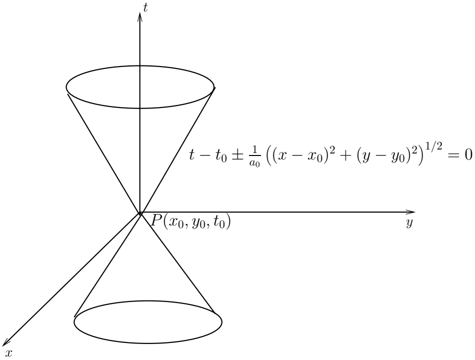
For the wave equation

$$u_{tt} - a_0^2 \left(u_{xx} + u_{yy}\right) = 0, \tag{4}$$

the eikonal is

$$\varphi_t - a_0 \left(\varphi_x^2 + \varphi_y^2\right)^{1/2} = 0. \tag{5}$$

An important solution is the characterestic conoid



Examples contd

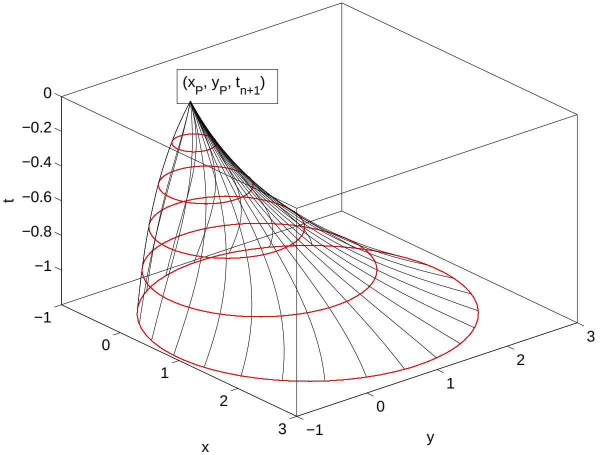


Figure: Characteristic conoid for the acoustic wave equation system

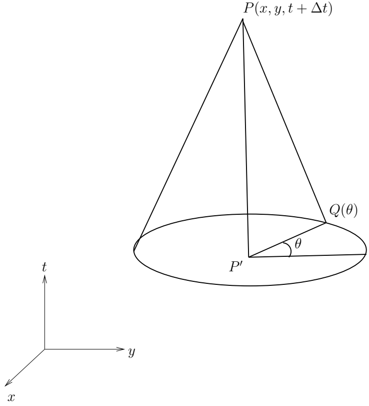


Figure: Bicharacteristics along the Mach cone through P and $Q(\theta)$

- Can we get the solution at P using the values at $Q(\theta)$?
- As $Q(\theta)$ moves along the circle we get contributions from infinitely many directions!

Result

Lemma (Extended lemma on bicharacteristics, Prasad, 1993)

For a hyperbolic system

$$\mathbf{u}_t + \sum_{j=1}^{d} \left(\mathbf{f}_j(\mathbf{u})\right)_{x_j} = 0, \tag{6}$$

the evolution of the p-th bicharacteristic family is given by

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \mathbf{l}^{(p)} \mathbf{A}_j \mathbf{r}^{(p)}, \tag{7}$$

$$\frac{\mathrm{d}n_j}{\mathrm{d}t} = \mathbf{l}^{(p)} \left\{ \sum_{k=1}^{d} n_k \left(\lambda^{(p)} \sum_{s=1}^{d} n_s \frac{\partial \mathbf{A}_s}{\partial \eta_k^j} \right) \right\} \mathbf{r}^{(p)}, \tag{8}$$

where $\mathbf{l}^{(p)}$ and $\mathbf{r}^{(p)}$ are left and right eigenvectors corresponding to the eigenvalue $\lambda^{(p)}$ of the matrix pencil $A := \sum_{j=1}^{d} n_j \mathbf{A}_j$.

Compatibility conditions

Result (Prasad and Ravindran, 1984)

For a hyperbolic system of quasilinear equations

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} \mathbf{A}_j(\mathbf{u}) \partial x_j \mathbf{u} = 0, \tag{9}$$

The transport equation along the p^{th} family of bicharacteristics is given by

$$\mathbf{l}^{(p)} \frac{d\mathbf{u}}{dt} + \sum_{j=1}^{d} \mathbf{l}^{(p)} \left(\mathbf{A}_j - \chi_j^{(p)} \mathbf{I}_m \right) \partial x_j \mathbf{u} = 0, \tag{10}$$

where $\chi_j^{(p)} = \mathbf{l}^{(p)} \mathbf{A}_j \mathbf{r}^{(p)}$ is the ray velocity vector and
$\frac{d}{dt} \equiv \partial_t + \sum_{j=1}^{d} \chi_j^{(p)} \partial x_j$ is the derivative along the p^{th} bicharacteristic.

Exact integral representation

Result (Ostkamp, 1995)

For a linear hyperbolic system an exact integral representation of the solution is given by

$$
\begin{aligned}
\mathbf{u}(P) = {} & \frac{1}{|S^{d-1}|}\int_{S^{d-1}} \sum_{k=1}^{m} \mathbf{r}^{(k)}(P)\mathbf{l}^{(k)}(Q_k)\mathbf{u}(Q_k)\mathrm{d}S \\
& + \frac{1}{|S^{d-1}|}\int_{S^{d-1}} \int_{t_n}^{t_{n+1}} \sum_{k=1}^{m} \mathbf{r}^{(k)}(P)\frac{\mathrm{d}\mathbf{l}^{(k)}}{dt}(\tilde{Q}_k)\mathbf{u}(\tilde{Q}_k)\mathrm{d}\tau\mathrm{d}S \\
& - \frac{1}{|S^{d-1}|}\int_{S^{d-1}} \int_{t_n}^{t_{n+1}} \sum_{k=1}^{m}\sum_{j=1}^{d} \mathbf{r}^{(k)}(P)\mathbf{l}^{(k)}(\tilde{Q}_k)\left(\mathbf{A}_j - \chi_j^{(k)}\mathbf{I}\right)\frac{\partial \mathbf{u}}{\partial x_j}\mathrm{d}\tau\mathrm{d}
\end{aligned}
\tag{11}
$$

Examples

For the acoustic wave equation system,

$$\begin{pmatrix} p \\ \rho_0 u \\ \rho_0 v \end{pmatrix}_t + \begin{pmatrix} \gamma p_0 u \\ p \\ 0 \end{pmatrix}_x + \begin{pmatrix} \gamma p_0 v \\ p \\ 0 \end{pmatrix}_y = 0, \tag{12}$$

$$\begin{aligned} p(P) = & \frac{1}{2\pi}\int_0^{2\pi} \left(p - z_0 u \cos\theta - z_0 v \sin\theta\right)(Q_1)\mathrm{d}\omega \\ & - \frac{1}{2\pi}\int_0^{2\pi}\int_{t_n}^{t_{n+1}} \left(z_0\left(a_{0x}u + a_{0y}v\right)\right)(\tilde{Q}_1)\mathrm{d}\tau\mathrm{d}\omega \\ & - \frac{1}{2\pi}\int_0^{2\pi}\int_{t_n}^{t_{n+1}} (z_0 S)(\tilde{Q}_1)\mathrm{d}\tau\mathrm{d}\omega. \end{aligned} \tag{13}$$

$$
\begin{aligned}
u(P) = & \frac{1}{2\pi z_0(P)} \int_0^{2\pi} \left(-p + z_0 u \cos\theta + z_0 v \sin\theta\right)(Q_1) \cos\omega \mathrm{d}\omega \\
& + \frac{1}{2\pi z_0(P)} \int_0^{2\pi} \int_{t_n}^{t_{n+1}} z_0 \left(a_{0x} u + a_{0y} v\right)(\tilde{Q}_1) \cos\omega \mathrm{d}\tau \mathrm{d}\omega \\
& + \frac{1}{2} u(Q_2) - \frac{1}{2\rho_0(P)} \int_{t_n}^{t_{n+1}} p_x(\tilde{Q}_2) \mathrm{d}\tau \\
& + \frac{1}{2\pi z_0(P)} \int_0^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 S)(\tilde{Q}_1) \cos\omega \mathrm{d}\tau \mathrm{d}\omega.
\end{aligned} \tag{14}
$$

The expression for $v(P)$ is analogous.

$$
S(\tilde{Q}) := a_0 \left\{ u_x(\tilde{Q}) \sin^2\theta - (u_y(\tilde{Q}) + v_x(\tilde{Q})) \sin\theta\cos\theta + v_y(\tilde{Q}) \cos^2\theta \right\}, \tag{15}
$$

is a geometric source term.

Approximation of the exact evolution operators

- The exact evolution operator is an implicit relation.
- It involves the time integrals of the unknown and its derivatives.
- The integrals along the Mach cone are to be simplified.
- We freeze the time integrals at $t = t_n$ to get an explicit relation.
- The geometric source term S contains only tangential derivatives, thanks to this special structure.

Approximate evolution operators

$$
\begin{aligned}
p(P) = \frac{1}{2\pi}\Bigg[& \int_0^{2\pi} (p - z_0(u\cos\omega + v\sin\omega))(\overline{Q})\mathrm{d}\omega \\
& - \Delta t \int_0^{2\pi} (z_0[u\sin\omega - v\cos\omega][-\overline{a}_{0x}\sin\omega + \overline{a}_{0y}\cos\omega])(\overline{Q})\mathrm{d}\omega \\
& - \Delta t \int_0^{2\pi} (z_0(\overline{a}_{0x}u + \overline{a}_{0y}v))(\overline{Q})\mathrm{d}\omega \\
& - \gamma p_0 \sum_{\substack{j=0 \\ \phi_j = j\pi/2}}^{3} \frac{1}{\overline{a}_0^j}\Bigg[\int_{\phi_j}^{\phi_{j+1}} (u\cos\omega + v\sin\omega)(\overline{Q})\mathrm{d}\omega \\
& \qquad + (u\sin\phi_j - v\cos\phi_j)(\overline{Q}(\phi_j^+)) \\
& \qquad - (u\sin\phi_{j+1} - v\cos\phi_{j+1})(\overline{Q}(\phi_{j+1}^-))\Bigg]\Bigg] + \mathcal{O}(\Delta t^2
\end{aligned}
$$

$$
\begin{aligned}
u(P) = \frac{1}{\pi z_0(P)} \Bigg[& \int_0^{2\pi} (-p + z_0(u\cos\omega + v\sin\omega))(\overline{Q})\cos\omega \mathrm{d}\omega \\
& + \Delta t \int_0^{2\pi} (z_0[u\sin\omega - v\cos\omega][-\overline{a}_{0x}\sin\omega + \overline{a}_{0y}\cos\omega])(\overline{Q} \\
& + \Delta t \int_0^{2\pi} (z_0(\overline{a}_{0x}u + \overline{a}_{0y}v))(\overline{Q})\cos\omega \mathrm{d}\omega \\
& + \gamma p_0 \sum_{\substack{j=0 \\ \phi_j = j\pi/2}}^{3} \frac{1}{\overline{a}_0^j} \Bigg[\int_{\phi_j}^{\phi_{j+1}} (u(2\cos^2\omega - 1) + 2v\cos\omega\sin\omega)(\\
& \qquad + (u\cos\phi_j \sin\phi_j - v\cos^2\phi_j)(\overline{Q}(\phi_j^+)) \\
& \qquad - (u(\cos\phi_{j+1}\sin\phi_{j+1}) - v\cos^2\phi_{j+1})(\overline{Q}(\phi_j
\end{aligned}
$$

Finite Volume Scheme

Divide a computational domain Ω into a finite number of regular finite volumes $\Omega_{ij} := [i\Delta x, (i+1)\Delta x] \times [j\Delta y, (j+1)\Delta y]$ for $i = 0, \dots, M$, $j = 0, \dots, N$

$$\mathbf{U}_{ij}^{0} = \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \mathrm{U}(\cdot, 0)\mathrm{d}\Omega. \tag{16}$$

The update formula for the finite volume evolution Galerkin scheme is

$$\mathbf{U}_{ij}^{n+1} = \mathbf{U}_{ij}^{n} - \frac{\Delta t}{\Delta x}\delta_x^{ij}\bar{f}_1^{n+1/2} - \frac{\Delta t}{\Delta y}\delta_y^{ij}\bar{f}_2^{n+1/2}. \tag{17}$$

We evolve the cell interface fluxes $\bar{f}_k^{n+1/2}$ to $t_n + 1/2$ using the approximate evolution operator denoted by $E_{\Delta t/2}$ and average them along the cell interface $\mathcal{E}$

$$\bar{f}_k^{n+1/2} := \sum_j \omega_j f_k(E_{\Delta t/2}\mathbf{U}^n(\mathbf{x}^j(\mathcal{E}))), \quad k = 1, 2. \tag{18}$$

Here $\mathbf{x}^j(\mathcal{E})$ are the nodes and ω_j the weights of the quadrature for the flux integration along the edges.

The algorithm

The building blocks of the FVEG scheme are

- Step 1: Polynomial reconstruction of the piecewise constant data using standard recovery procedures.
- Step 2: Discretize the flux integrals in the FV update using either Trapezoidal or Simpson rule.
- Step 3: Construct the local Mach cone at the quadrature nodes.
- Step 4: Evolve the data using the approximate evolution operator and compute fluxes at half time step.
- Step 5: Update the solution using the standard FV scheme.

Remark

The FVEG method is a genuine multi-dimensional generalization of Godunov's REA algorithm.

Smoothly varying wave speed

The computational domain is $[0,1] \times [0,1]$ and the initial conditions are

$$
\begin{aligned}
p(x,y) &= \sin(2\pi x) + \cos(2\pi y), \\
u(x,y) &= 0, \\
v(x,y) &= 0.
\end{aligned}
$$

The initial wave speed is

$$
a_0(x,y) = 1 + \frac{1}{4}\left(\sin(4\pi x) + \cos(4\pi y)\right).
$$

Periodic boundary conditions and final time is $T = 1.0$.

Smoothly varying wave speed

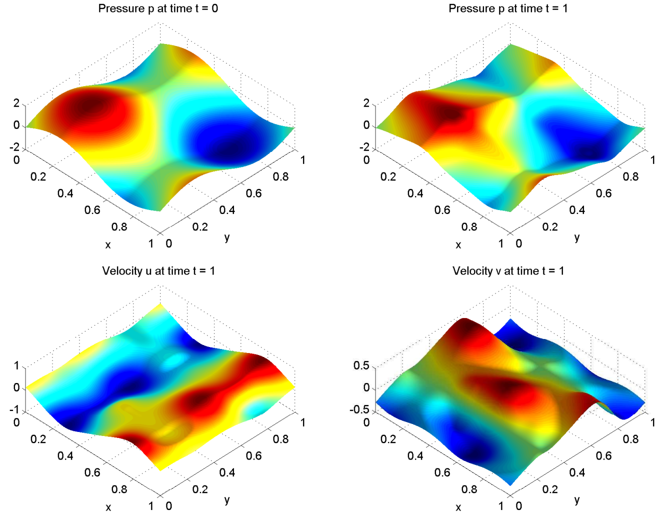


Figure: Results with a smoothly varying wave speed

Radially symmetric wave speed

We model the wave propagation in a radially symmetric medium. The wave speed is

$$a_0(x,y) = \begin{cases} 0.175 & \text{if } r \leq 0.15, \\ 0.350 & \text{if } 0.41 < r \leq 0.59, \\ 0.275 & \text{if } 0.85 < r. \end{cases}$$

The initial pressure is given by

$$p(x,y) = \begin{cases} \bar{p}((r-0.5)/0.18) & \text{if } |r-0.5| < 0.18, \\ 0 & \text{otherwise }. \end{cases}$$

$\bar{p}$ is a suitable polynomial.

Radially symmetric

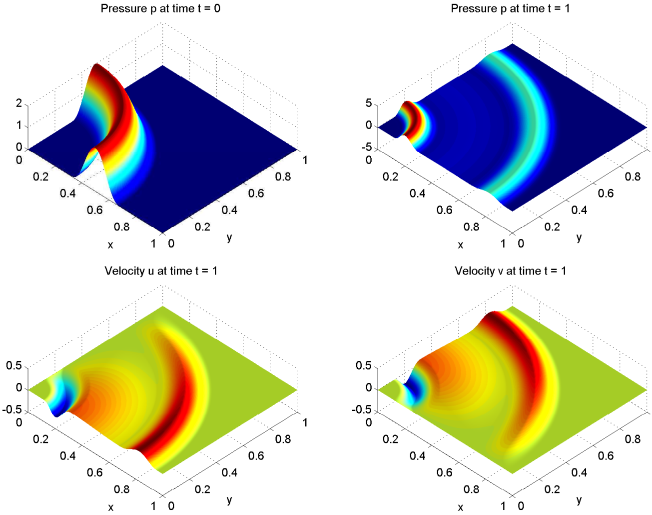


Figure: The solution corresponding to radially symmetric wave speed a_0

Heterogeneous medium with discontinuous wave speed

Propagation of acoustic waves through a layered medium with a single interface. The piecewise constant wave speed is given as

$$a_0(x,y) = \begin{cases} 1.0 & \text{if } x < 0.5, \\ 0.5 & \text{otherwise.} \end{cases}$$

The initial data are

$$p(x,y) = \begin{cases} 1.0 + 0.5(\cos(\pi r/0.1) - 1.0) & \text{if } r < 0.1, \\ 0.0 & \text{otherwise.} \end{cases}$$

$$u(x,y) = v(x,y) = 0.0.$$

Heterogeneous medium

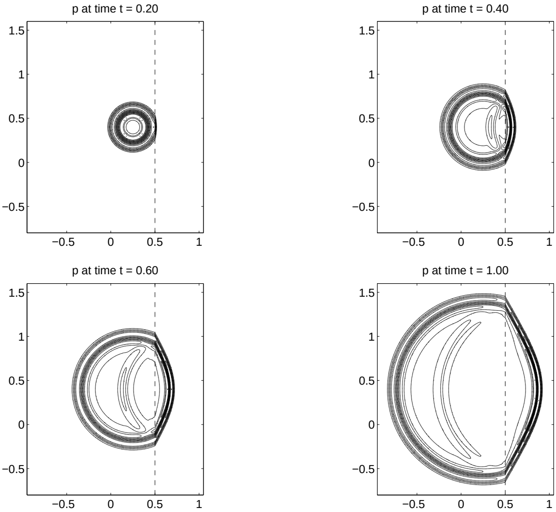


Figure: The pressure isolines for the reflection problem

Thank You for Your Kind Attention!