Asymptotical Solutions in a Reactive non-ideal Hydrodynamic medium

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Basic Equations

The equations of one-dimensional unsteady planar and non-planar motion in a reactive non-ideal hydrodynamic medium are

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \frac{m \rho u}{x} = 0,$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} + \frac{m \gamma p u}{x} - (\gamma - 1) \rho q Q = 0, \qquad (1)$$

$$\frac{\partial \lambda}{\partial t} + u \frac{\partial \lambda}{\partial x} - Q = 0.$$

The equation of state, for motion in a non-ideal gas, is of the form:

$$p = \rho R T (1 + b\rho),$$

where b is the internal volume of the gas molecules with $b\rho \ll 1$.

First we put the system of equations (1) in the matrix form as follows:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B = 0, \qquad (2)$$

where U and B are column vectors defined by

$$U = (\rho, u, p, \lambda)^{tr}, \quad \text{and}$$

$$B = \left(\frac{m\rho u}{x}, 0, \frac{m\gamma(1+b\rho)pu}{x} - (\gamma-1)\rho q Q, -Q \right)^{tr}, \qquad (3)$$

and A is the 4 x 4 matrix the components

$$A^{11} = A^{22} = A^{33} = A^{44} = u,$$

$$A^{12} = \rho, \qquad A^{23} = \frac{1}{\rho}, \tag{4}$$

$$A^{32} = \rho a^2 = \gamma p(1 + b\rho).$$

Also the eigenvectors satisfying the normalization condition $L^{(i)} R^{(j)} = \delta_{ij}, \ \ 1 \le i, j \le 4$, where δ_{ij} represents Kronecker delta. These normalized eigenvectors are obtained as

$$L^{(1)} = \left(1, \quad 0, \quad -\frac{1}{a_0^2}, 0\right), \quad R^{(1)} = (1, \ 0, \ 0, \ 0)^{tr},$$

$$L^{(2)} = \left(0, \quad 0, \ \ 0, \ \ 1\right), \quad R^{(2)} = (0, \ 0, \ 0, \ 1)^{tr}, \tag{5}$$

$$L^{(3)} = \left(0, \quad \frac{\rho_0}{2a_0}, \frac{1}{2a_0^2}, 0\right), \quad R^{(3)} = (1, \frac{a_0}{\rho_0}, a_0^2, 0)^{tr},$$

$$L^{(4)} = \left(0, \quad -\frac{\rho_0}{2a_0}, \frac{1}{2a_0^2}, 0\right), \quad R^{(4)} = (1, -\frac{a_0}{\rho_0}, a_0^2, 0)^{tr}$$

We look for the asymptotic solution of the form:

$$U \sim U_0 + \varepsilon U_1(x, t, \bar{\theta}) + \varepsilon^2 U_2(x, t, \bar{\theta}) + O(\varepsilon^3), \qquad (6)$$

where $\tilde{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ represent the "fast variable" defined as $\theta_i = \frac{\phi_i}{\varepsilon}$, where ϕ_i is the phase function

Now we use (6) in (2), expand *A* and *B* in Taylor's series in powers of ε about $U = U_0$, replace the partial derivatives $\frac{\partial}{\partial X}$ by $\frac{\partial}{\partial X} + \varepsilon^{-1} \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial X} \frac{\partial}{\partial \theta_i}$, and equate to zero the coefficients of ε^0 and ε^1 in the resulting expansions, to obtain

$$O(\varepsilon^0): \sum_{i=1}^{4} \left(I \frac{\partial \phi_i}{\partial t} + A_0 \frac{\partial \phi_i}{\partial x} \right) \frac{\partial U_1}{\partial \theta_i} = 0 \tag{7}$$

$$O(\varepsilon^1): \sum_{i=1}^{4} \left(I \frac{\partial \phi_i}{\partial t} + A_0 \frac{\partial \phi_i}{\partial x} \right) \frac{\partial U_2}{\partial \theta_i} = -\frac{\partial U_1}{\partial t} - A_0 \frac{\partial U_1}{\partial x} - (U_1.\Delta B)_0 - \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial x} (U_1.\Delta A)_0 \frac{\partial U_1}{\partial \theta_i}. \tag{8}$$

The phase functions ϕ_i $(1 \le i \le 4)$ satisfy the eikonal equation

$$\mathrm{Det} \left(I \frac{\partial \phi_i}{\partial t} + A_0 \frac{\partial \phi_i}{\partial x} \right) = 0. \tag{9}$$

We choose the simplest phase function of this equation, namely

$$\phi_i(x,t) = x - \lambda_i\ t, \quad 1 \le i \le 4. \tag{10}$$

For each phase ϕ_i, $\dfrac{\partial U_1}{\partial \theta_i}$ is parallel to the right eigenvector $R^{(i)}$ of A_0, and thus

$$U_1 = \sum_{i=1}^{4} \sigma_i(x,t,\theta_i)\, R^{(i)}. \tag{11}$$

where $\sigma_i = (L^{(i)}.U_1)$ is a scalar function called the wave amplitude. We assume that $\sigma_i(x,t,\theta_i)$ has zero mean value with respect to the fast variable θ_i, that is,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_i(x,t,\theta_i)\, d\theta_i = 0. \tag{12}$$

We then use (11) in (8) and solve for U_2. To begin with we write

$$U_2 = \sum_{j=1}^{4} m_j \, R^{(j)} ,$$

substitute this value in (8), and premultiply the resulting equation by $L^{(i)}$ to obtain the system of decoupled inhomogeneous first order PDEs:

$$\sum_{j=1}^{4} (\lambda_i - \lambda_j) \frac{\partial m_i}{\partial \theta_j} = -\frac{\partial \sigma_i}{\partial t} - \lambda_i \frac{\partial \sigma_i}{\partial x} - L^{(i)} (U_1 . \Delta B)_0 - \sum_{i=1}^{4} L^{(i)} (U_1 . \Delta A)_0 \frac{\partial U_1}{\partial \theta_j}. \tag{13}$$

The characteristic ODEs for the i-th equation in (13) are given by

$$\dot{\theta}_j = \lambda_i - \lambda_j \quad \text{for} \quad j \neq i, \quad \dot{\theta}_i = 0, \quad \dot{m}_i = H_i, \tag{14}$$

where

$$H_i = (x, t, \theta_1, \theta_2, \theta_3, \theta_4) = -\frac{\partial \sigma_i}{\partial t} - \lambda_i \frac{\partial \sigma_i}{\partial x} - L^{(i)}(U_1.\Delta B)_0 - \sum_{i=1}^{4} L^{(i)}(U_1.\Delta A)_0 \frac{\partial U_1}{\partial \theta_j}.$$

The wave amplitude σ_i, $1 \leq i \leq 4$, satisfy the following system of coupled integro-differential equations

$$\frac{\partial \sigma_i}{\partial t} + \lambda_i \frac{\partial \sigma_i}{\partial x} + a_i \sigma_i + \Gamma^i_{ii} \sigma_i \frac{\partial \sigma_i}{\partial \theta_i} + \sum_{i \neq j \neq k} \Gamma^i_{jk} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sigma_j (\theta_i + (\lambda_i - \lambda_j) s) \sigma'_k (\theta_i + (\lambda_i - \lambda_k) s) ds = 0, \tag{15}$$

where $\sigma'_k = \frac{\partial \sigma_k}{\partial \theta_k}$ and the coefficients a_i and Γ^i_{jk}

$$a_i = L^{(i)} (R^{(i)} . \Delta B)_0, \qquad \Gamma^i_{jk} = L^{(i)} (R^{(j)} . \Delta A)_0 R^{(k)}. \tag{16}$$

The interaction coefficients Γ^i_{jk} denote the strength of coupling between the *j*-th and *k*-th wave modes ($j \neq k$).

The coefficients Γ^i_{ii} referring to the non-linear self-interaction, are non-zero for genuinely non-linear waves and zero for linearly degenerate waves. The coefficients a_i , Γ^i_{jk} and Γ^i_{ii} provide a picture of the non-linear interaction process, the non-zero being determined by

$$a_1 = \frac{(\gamma - 1) q \left(Q + \rho Q_\rho\right)_0}{a_0^2}, \qquad a_2 = -\left.\frac{\partial Q}{\partial \lambda}\right|_0,$$

$$a_3 = \frac{m a_0}{2x} - \frac{(\gamma - 1) q \left(Q + \rho Q_\rho + a_0 Q_u + \rho_0 a_0^2 Q_p\right)_0}{2 a_0^2}, \tag{17}$$

$$a_4 = -\frac{m a_0}{2x} - \frac{(\gamma - 1) q \left(Q + \rho Q_\rho - a_0 Q_u + \rho_0 a_0^2 Q_p\right)_0}{2 a_0^2},$$

$$\Gamma_{13}^1 = \frac{a_0}{\rho_0 (1 + b\rho_0)} = -\Gamma_{14}^1,$$

$$\Gamma_{34}^1 = -\Gamma_{43}^1 = \frac{a_0 \left[\gamma (1 + b\rho_0)^2 - 1\right]}{\rho_0 (1 + b\rho_0)},$$

$$\Gamma_{14}^3 = -\Gamma_{13}^4 = -\frac{a_0 (1 + 2b\rho_0)}{2\rho_0 (1 + b\rho_0)}, \tag{18}$$

$$\Gamma_{33}^3 = -\Gamma_{44}^4 = \frac{\left[\gamma (1 + b\rho_0)^2 + 2b\rho_0 + 1\right] a_0}{2\rho_0 (1 + b\rho_0)}.$$

The resonance equations (15) can now be written as

$$\frac{\partial \sigma_1}{\partial t} + a_1\sigma_1 = 0, \qquad \frac{\partial \sigma_2}{\partial t} + a_2\sigma_2 = 0,$$

$$\frac{\partial \sigma_3}{\partial t} + a_0\frac{\partial \sigma_3}{\partial x} + a_3\sigma_3 + \Gamma_{33}^3\,\sigma_3\frac{\partial \sigma_3}{\partial \theta_3} - \lim_{P\to\infty}\frac{1}{2P}\int_{-P}^{P} K\left(x, t, \frac{\theta_3+\phi}{2}\right)\sigma_4(x, t, \phi)\,d\phi = 0,$$

$$\frac{\partial \sigma_4}{\partial t} - a_0\frac{\partial \sigma_4}{\partial x} + a_4\sigma_4 + \Gamma_{44}^4\,\sigma_4\frac{\partial \sigma_4}{\partial \theta_4} + \lim_{P\to\infty}\frac{1}{2P}\int_{-P}^{P} K\left(x, t, \frac{\theta_4+\phi}{2}\right)\sigma_3(x, t, \phi)\,d\phi = 0, \qquad (19)$$

where, the kernel K is defined as

$$K\left(x, t, \frac{\theta+\phi}{2}\right) = \frac{\Gamma_{14}^3}{2}\frac{\partial \sigma_1}{\partial \theta_1}\left(x, t, \frac{\theta+\phi}{2}\right). \qquad (20)$$

Let the initial value of σ_j be $\sigma_j\big|_{t=0} = \sigma_j^0(x, \theta_j)$. Hence $(19)_{1,2}$ gives

$$\sigma_1(x, t, \theta_1) = \sigma_1^0(x, \theta_1)\,e^{-a_1 t}$$

and $\sigma_2(x,t,\theta_2)=\sigma_2^0(x,\theta_2)\,e^{-a_2 t}$, and then the kernel *K* is given by

$$K(x,t,\theta)=\frac{\Gamma_{14}^3}{2}\,e^{-a_1 t}\,\frac{\partial\sigma_1^0}{\partial\theta_1}(x,\theta_1). \qquad (21)$$

If the initial data $\sigma_j^0(x,\theta)$ are 2π periodic functions , then (19) becomes

$$\frac{\partial\sigma_3}{\partial t}+a_0\frac{\partial\sigma_3}{\partial x}+a_3\sigma_3+\Gamma_{33}^3\,\sigma_3\frac{\partial\sigma_3}{\partial\theta_3}-\frac{1}{2\pi}\int_{-\pi}^{\pi}K\left(x,t,\frac{\theta_3+\phi}{2}\right)\sigma_4(x,t,\phi)\,d\phi=0, \qquad (22)$$

$$\frac{\partial\sigma_4}{\partial t}-a_0\frac{\partial\sigma_4}{\partial x}+a_4\sigma_4+\Gamma_{44}^4\,\sigma_4\frac{\partial\sigma_4}{\partial\theta_4}+\frac{1}{2\pi}\int_{-\pi}^{\pi}K\left(x,t,\frac{\theta_4+\phi}{2}\right)\sigma_3(x,t,\phi)\,d\phi=0.$$

The asymptotic solution (6) of hyperbolic system (2) satisfies the initial data

$$U(x,0)=U_0+\varepsilon\,U_1^0(x,x/\varepsilon)+O(\varepsilon^2), \qquad (23)$$

Using the Lagrange's subsidiary equations for $(19)_3$ and $(19)_4$, we obtain

$$\frac{d\theta_j}{dx} = \frac{e_j \, \Gamma^j_{jj} \, \sigma_j}{a_0}, \qquad \frac{dt}{dx} = \frac{e_j}{a_0}, \qquad j = 3, 4 \tag{24}$$

where

$$e_j = \begin{cases} +1, \text{ if } \; j = 3, \\ -1, \text{ if } \; j = 4. \end{cases}$$

We integrate $(24)_2$ to obtain

$$x - e_j \, a_0 \, t = s_j = \text{constant.}$$

In terms of the characteristic equations, $(19)_3$ and $(19)_4$ can be written as

$$\frac{d\sigma_j}{dt} = -a_j \, \sigma_j \,, \quad j = 3,4 \tag{25}$$

which yields after integration

$$\sigma_j = \sigma_j^0(s_j, \xi_j) e^{-a_j t}, \tag{26}$$

along the rays $s_j = x - e_j a_0 t = \text{constant}$, where the function σ_j^0 is obtained from the initial condition (23).

Thus, we obtain from (24)

$$\frac{d\theta_j}{dt} = \Gamma_{jj}^j \sigma_j = \Gamma_{jj}^j \sigma_j^0(s_j, \xi_j) e^{-a_j t}, \tag{27}$$

which yields after integration

$$\xi_j = \theta_j + \frac{\Gamma_{jj}^j}{a_j}\left(\sigma_j - \sigma_j^0\right). \tag{28}$$

The solution of (2), satisfying (23), where $U_1^0(x, x/t)$ has compact support, is obtained as

$$\rho(x,t) = \rho_0 + \varepsilon\,\sigma_1^0(x, x/\varepsilon)e^{-a_1 t} + \varepsilon\left(\sigma_3^0(x - a_0 t, \xi_3)e^{-a_3 t} + \sigma_4^0(x + a_0 t, \xi_4)e^{-a_4 t}\right) + O\left(\varepsilon^2\right),$$

$$u(x,t) = \varepsilon\,\frac{a_0}{\rho_0}\left(\sigma_3^0(x - a_0 t, \xi_3)e^{-a_3 t} - \sigma_4^0(x + a_0 t, \xi_4)e^{-a_4 t}\right) + O\left(\varepsilon^2\right),$$

(29)

$$p(x,t) = p_0 + \varepsilon\,a_0^2\left(\sigma_3^0(x - a_0 t, \xi_3)e^{-a_3 t} + \sigma_4^0(x + a_0 t, \xi_4)e^{-a_4 t}\right) + O\left(\varepsilon^2\right),$$

$$\lambda(x,t) = \lambda_0 + \varepsilon\,\sigma_2^0(x, x/\varepsilon)e^{-a_2 t} + O\left(\varepsilon^2\right).$$

and the initial values for σ_i $(1 \le i \le 4)$ are obtained from (29) specified at *t=0* as

$$\sigma_1^0(x, x/\varepsilon) = \rho_1^0(x, x/\varepsilon) - \frac{1}{a_0^2}\, p_1^0(x, x/\varepsilon),$$

$$\sigma_2^0(x, x/\varepsilon) = \lambda_1^0(x, x/\varepsilon),$$

$$\sigma_3^0(x, \xi_3) = \frac{1}{2a_0}\left(\rho_0\, u_1^0(x, \xi_3) + \frac{1}{a_0}\, p_1^0(x, \xi_3)\right), \qquad (30)$$

$$\sigma_4^0(x, \xi_4) = -\frac{1}{2a_0}\left(\rho_0\, u_1^0(x, \xi_4) - \frac{1}{a_0}\, p_1^0(x, \xi_4)\right).$$

This is the complete solution of (2) and (23); any multivalued overlap in this solution is resolved by introducing shock into the solution.

The shock location θ_j^s satisfies the relation

$$\frac{d\theta_j^s}{dt} = \frac{1}{2}\Gamma_{jj}^j\left(\sigma_j^{(-)} + \sigma_j^{(+)}\right), \quad j = 3,4 \qquad (31)$$

Here, $\sigma_j^{(-)}$ and $\sigma_j^{(+)}$ are the values of σ_j just ahead and behind the shock, respectively. We have $\sigma_j^{(-)} = 0$ for the undisturbed region ahead of the shock. Now we use (26) to obtain the differential equation of the location of the shock

$$\frac{d\theta_j}{dt} = \frac{\Gamma_{jj}^{j}}{2}\sigma_j^0\left(s_j, \xi_j\right)e^{-a_j t} \qquad (32)$$

Conclusion

The method of weakly non-linear geometrical acoustics is used to obtain the small amplitude high frequency asymptotic solution to the basic equations governing one dimensional unsteady planar, cylindrically and spherically symmetric flow in a reactive non-ideal hydrodynamic medium in which an irreversible chemical reaction takes places.

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Thank You!