The Riemann problem for a full-wave Maxwell system modeling electromagnetic propagation in a nonlinear Kerr medium

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Kerr's model

Maxwell's equations:

$$\begin{array}{rcl} \partial_t D - \mathrm{curl} H & = & 0 \\ \partial_t B + \mathrm{curl} E & = & 0 \end{array}$$

$$\mathrm{div} D = \mathrm{div} \mathrm{B} = 0$$

Constitutive laws:

$$\left\{ \begin{array}{rcl} B & = & \mu_0 H \\ D & = & \epsilon_0 (1 + \epsilon_r |E|^2) E \end{array} \right.$$

Kerr model

$$\begin{cases} \partial_t D - \mathrm{curl} H & = & 0 \\ \partial_t H + \mu_0^{-1} \mathrm{curl} P(D) & = & 0 \end{cases}$$

where $P = Q^{-1}$ and $Q(E) = \epsilon_0(1 + \epsilon_r |E|^2)E$.

$$\mathrm{div} D = \mathrm{div} H = 0.$$

Mathematical entropy : electromagnetic energy.
Hyperbolic symmetrizable system of conservation laws.

Notation:

$$E = P(D).$$

Properties of the Kerr model

- Eigenvalues in a given direction $\omega \in \mathbb{R}^3$, $|\omega| = 1$:

$$\lambda_1 \leq \lambda_2 < \lambda_3 = \lambda_4 = 0 < \lambda_5 = -\lambda_2 \leq \lambda_6 = -\lambda_1$$

  with

$$\lambda_1^2 = \frac{c^2}{1 + \epsilon_r |E|^2}, \quad \lambda_2^2 = c^2 \frac{1 + \epsilon_r(|E|^2 + 2(E \cdot \omega)^2)}{(1 + \epsilon_r |E|^2)(1 + 3\epsilon_r |E|^2)}.$$

  Moreover $\lambda_1^2 = \lambda_2^2$ if and only if $D \times \omega = 0$.
- The characteristic fields 1, 3, 4, 6 are linearly degenerate.
- In the open domain $\{D \in \mathbb{R}^3; D \times \omega \neq 0\}$ the characteristic fields 2, 5 are genuinely nonlinear.

2D reduced models with spatial variable $x = (x_1, x_2)$

Transverse Magnetic (TM):

$$H = (0, 0, h), \quad D = (D_1, D_2, 0).$$

Transverse Electric (TE):

$$D = (0, 0, d), \quad H = (H_1, H_2, 0).$$

In each case : 3×3 strictly hyperbolic system of conservation laws with 0 as simple eigenvalue.

Eigenvalues 1 and 6 of the 6×6 system are lost.

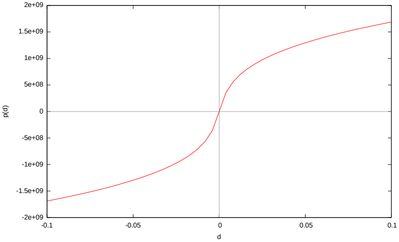
A 1D reduced model with spatial variable $x = x_1$

$$D = (O, d, 0), \quad H = (0, 0, h).$$

$$\mathrm{div} D = \mathrm{div} H = 0 \quad \text{always.}$$

Kerr system reads as a p-system:

$$\begin{cases} \partial_t d + \partial_x h = 0, \\ \partial_t h + \mu_0^{-1} \partial_x p(d) = 0. \end{cases}$$



A relaxation system for Kerr model: Kerr-Debye system

$$\begin{cases} \partial_t D - \mathrm{curl} H & = & 0 \\ \partial_t H + \mu_0^{-1} \mathrm{curl} E & = & 0, \qquad\qquad E = \dfrac{D}{\epsilon_0(1+\chi)} \\ \partial_t \chi & = & -\dfrac{1}{\tau}\left(\chi - \epsilon_r |E|^2\right) \end{cases}$$

In situations of physical interest τ is very small.

Hyperbolic, partially dissipative system, relaxation of Kerr model in the sense of Chen-Levermore-Liu (CPAM 1994).

R.W. Ziolkowski. IEEE Transactions on Antennas and Propagation 45(3):375-391, 1997.

P. Huynh. PhD thesis, 1999.

Goal and motivations

- Relaxation and existence for strong solutions: see Carbou-Hanouzet (JHDE 2009), Kanso (PhD thesis 2012).
- Study of shocks and related Kerr-Debye shock profiles: AD-Hanouzet, CMS 2011
- Numerical approximation of Kerr system by a Kerr-Debye relaxation finite volumes scheme: AD-Berthon 2009 in 1D, Kanso PhD thesis in 2D.
- Here, we study the Riemann problem for the 6×6 system and relate its solutions with those of the reduced models in order to
  - Perform numerical approximation by Godunov scheme
  - Understand better the weak solutions

SEE ALSO A. DE LA BOURDONNAYE JCP 2000, FOR GODUNOV TYPE SCHEMES FOR REDUCED KERR SYSTEMS.

Notation: $u = (D, H)$.

We fix $\omega \in \mathbb{R}^3$, $|\omega| = 1$, $u_l \in \mathbb{R}^6$, $u_r \in \mathbb{R}^6$,

$$u(x,0) = \begin{cases} u_l & \text{if} \quad x \cdot \omega < 0, \\ u_r & \text{if} \quad x \cdot \omega > 0. \end{cases}$$

We look for a solution under the form

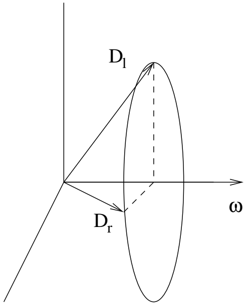
$$u(x,t) = u(x \cdot \omega, t).$$

One-dimensional 6×6 Riemann problem with variable $y = x \cdot \omega$.

Study of the simple waves: contact discontinuities

If (D, H) is a stationary contact discontinuity such that $\mathrm{div} D = 0$ and $\mathrm{div} H = 0$, then it is a constant function.

Others contact discontinuities: related to the eigenvalues 1 and 6. Rotating modes such that $|D|$ and $\mathrm{div} D$, $\mathrm{div} H$ are constant.



Study of the simple waves: shock waves and rarefactions

Eigenvalues 2 and 5.

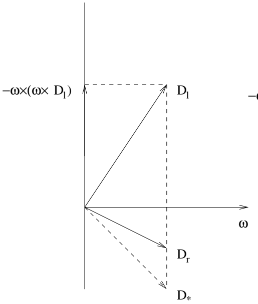
- If $D_l \times \omega \neq 0$ one can define a 2-rarefaction curve, a Lax 2-shock curve and a Liu 2-shock curve.
- If $D_l \times \omega = 0$, one has rarefactions and semi-contact discontinuities.

Lax conditions ensure that the Riemann fan can be constructed for the 6×6 system.

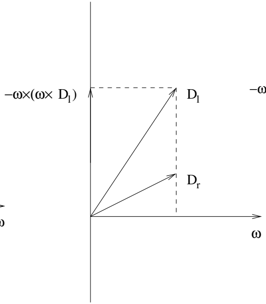
Liu's condition (J. Math. Anal. Appl., 1976): if u_l is a left state which the Hugoniot set $\mathcal{H}(u_l)$ is a union of curves and if $u_r \in \mathcal{H}(u_l)$.

$$\sigma(u_r, u_l) \leq \sigma(u, u_l), \qquad \forall u \in \mathcal{H}(u_l),\ u \text{ between } u_l \text{ and } u_r\,.$$

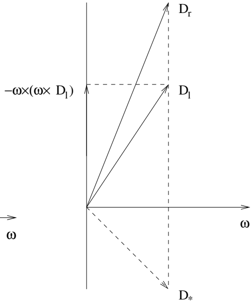
D component for a 2-wave when $D_l \times \omega \neq 0$



Liu 2-shock



Lax 2-shock



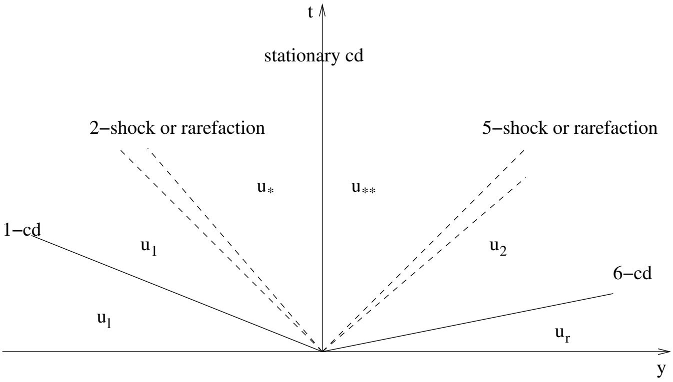
2-rarefaction

Solution of the 6×6 Riemann problem

Initial data:

$$u_0(x) = \begin{cases} u_l & \text{if } x \cdot \omega < 0, \\ u_r & \text{if } x \cdot \omega > 0. \end{cases}$$

We show that for $|u_r - u_l|$ small enough, there exists a unique solution of the form

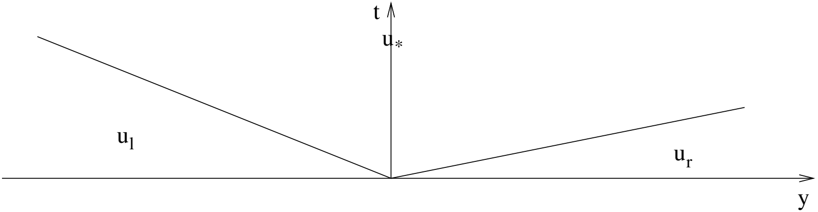


If $\operatorname{div} D_0 = \operatorname{div} H_0 = 0$

- The solution exists without smallness condition.
- $u_* = u_{**}$: no stationary contact discontinuity.
- If moreover $H_r - H_l - \omega \times (\sigma_r D_r - \sigma_l D_l) = 0$

$$D_* \times \omega = 0.$$

1-cd and 2-shock (*resp* 6-cd and 5-shock) merge.

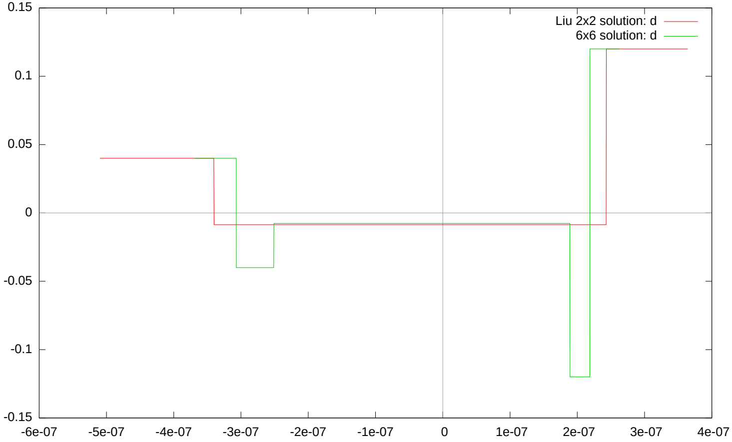


The case of the 2×2 reduced model

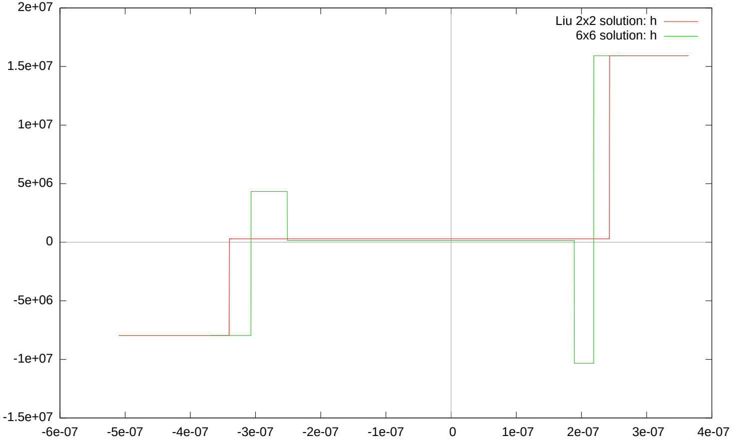
p-system with p convex-concave. Two solutions can be constructed:

- Weak solution as a particular case of the 6×6 system.
- "Liu's solution", see also Wendroff, J. Math. Anal. Appl. 1972.

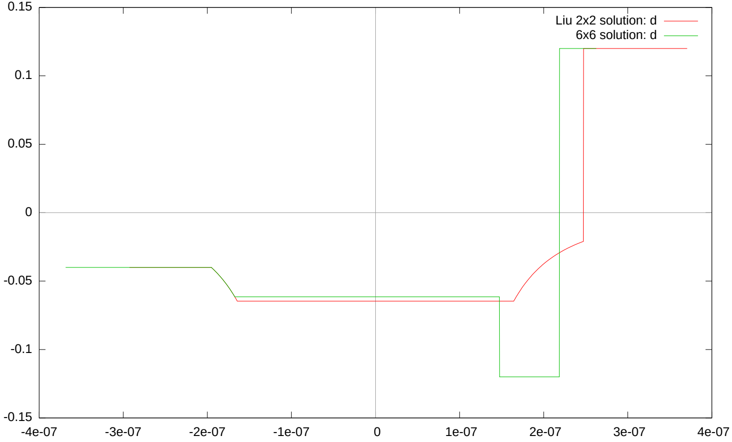
d component at fixed time



h component at fixed time



d component in another case



The case of the 2×2 reduced model

Entropy: electromagnetic energy

$$\eta(d,h) = \mathcal{E}(d) + \frac{1}{2}\mu_0 h^2, \qquad \mathcal{E}(d) = \frac{\epsilon_0}{2}\left(e^2 + \frac{3\epsilon_r}{2}e^4\right)$$

with $e = p(d)$.

Entropy flux: Poynting vector

$$Q(d,h) = eh.$$

Liu's shocks satisfy the following entropy dissipation inequality:

$$[Q(d,h)] - \sigma[\eta(d,h)] = -\frac{\epsilon_0\epsilon_r}{4}\sigma[e]^2\,[e^2] \leq 0.$$

The case of the 2×2 reduced model

- Both solutions are entropic.
- Liu's solution is in general more dissipative than the 6×6 solution because
  1. the entropy dissipation rate is shown to increase with $|[d]|$, which is larger for Liu's shocks than for 6×6 Lax shocks,
  2. for contact discontinuities, entropy is conserved.
- Numerically, the solutions of the relaxation Kerr-Debye system converge towards Liu's solutions.

Conclusion and perspectives

- For $|u_r - u_l|$ small enough we have constructed the solution of Riemann problem for the 3D Kerr system as a composition of simple waves.
- In the divergence free case, the solution exists for any Riemann data.
- For the 2×2 reduced model, we have two entropy solutions.
- Numerical application: Godunov scheme, to be compared with already existing relaxation Kerr-Debye scheme (partially done).