Generalizing the Bardos-LeRoux-Nédélec boundary condition for scalar conservation laws

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Plan of the talk

THE PROBLEM

Problem considered.

Our problem is:

$$(H)\begin{cases} u_t + \operatorname{div} \varphi(u) = 0 & \text{in } \ Q := (0,T)\times\Omega \\ u(0,\cdot) = u_0 & \text{on } \ \Omega \\ \varphi_\nu(u) := \varphi(u)\cdot\nu \in \beta_{(t,x)(u)} & \text{on } \ \Sigma := (0,T)\times\partial\Omega, \end{cases}$$

- Ω : domain of $\mathbb{R}^N$ with Lipschitz boundary; $T > 0$
- $\varphi : z \in \mathbb{R} \mapsto (\varphi_1(z), \varphi_2(z), \cdots, \varphi_N(z)) \in \mathbb{R}^N$ is Lipschitz, normalized by $\varphi(0) = 0$
- $u_0 \in L^\infty(\Omega)$

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- $u_0 \in L^\infty(\Omega)$
- ν : the unit outward normal vector on $\partial\Omega$
- $\beta_{(t,x)}(.)$: a “Caratheodory” family of maximal monotone graphs on $\bar{\mathbb{R}}$.

Important particular cases

Dissipative boundary conditions $\varphi_\nu(u) \in \beta_{(t,x)}(u)$ include:

- the Dirichlet condition $u = u^D(t,x)$ on Σ :

$$\beta_{(t,x)} = \{u^D(t,x)\} \times \mathbb{R},$$

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- Mixed Dirichlet-Neumann boundary conditions,
Robin boundary conditions,...
- obstacle boundary conditions
- ...and many other boundary conditions (BC),
less practical but still interesting, mathematically.

The BLN condition
and alternative formulations

The BLN condition...

Let us recall the Bardos-Le Roux-Nédélec result in the case of homogenous Dirichlet condition ($u^D \equiv 0$, $\beta = \{0\} \times \mathbb{R}$);

For BV (bounded variation) data u_0 there exists a unique function $u \in L^\infty \cap \mathrm{BV}((0,T) \times \Omega)$ such that

- $\forall k \in \mathbb{R}, \forall \xi \in C_c^\infty([0,T) \times \Omega)$

$$\int_Q |u-k|\,\xi_t + \int_\Omega |u_0 - k|\,\xi(0)$$
$$+ \int_Q \operatorname{sign}(u-k)(\varphi(u) - \varphi(k)) \cdot \nabla\xi \geq 0$$

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- on the boundary: u has a strong trace γu such that

$$(BLN) \quad \left| \begin{array}{l} \text{for all } k \in [\min(0,\gamma u), \max(0,\gamma u)], \\ \operatorname{sign}(\gamma u)(\varphi(\gamma u) \cdot \nu - \varphi(k) \cdot \nu) \geq 0 \ \text{ a.e. on } \Sigma. \end{array} \right.$$

...the BLN condition and its justification...

Example. Dimension one, $\Omega = [0,1]$, the linear case :

we consider $\varphi(z) := z$ and the homogeneous Dirichlet datum $u^D := 0$.

In this case, we have the problem $u_t + u_x = 0, \ u|_{t=0} = u_0$ and condition (BLN) reads :

- at the point $x = 0$, $\gamma u = 0$;
- at the point $x = 1$, γu is arbitrary.

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Solutions are limits of vanishing viscosity approximation:

$$u = \lim_{\varepsilon \downarrow 0} u^\varepsilon, \quad u_t^\varepsilon + u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon,\ u^\varepsilon|_{t=0} = u_0 \text{ and } u^\varepsilon|_{x=0} = 0 = u^\varepsilon|_{x=1}.$$

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But the sequence $(u^\varepsilon)_\varepsilon$ develops a boundary layer as $\varepsilon \downarrow 0$: in a layer of thickness $\overline{\overline{o}}_{\varepsilon\downarrow 0}(1)$ near the boundary point $x = 1$, u^ε undergoes a change of order $\overline{\overline{O}}(1)$ and passes from the prescribed value zero to some value $\widetilde{u}^\varepsilon$. The sequence $\widetilde{u}^\varepsilon$ does converge to a value γu satisfying condition (BLN).

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Thus, the "formal BC" $u|_\Sigma = 0$ is transformed into an "effective BC" expressed by the Bardos-LeRoux-Nédélec condition.

Alteratives to the BLN approach...

Essential feature of the Bardos-LeRoux-Nédélec framework: existence of strong traces of u on the boundary Σ.

This is achieved by ensuring that u belongs to the space BV. This is natural for the Dirichlet BC but BV is not a natural space e.g. for the zero-flux BC. Yet the BV framework can be bypassed in many ways.

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- (F. Otto ('96)) notion of a boundary entropy-entropy flux pair and use of weak traces (they always exist) to give L^∞ theory. In the present work, we will not pursue this line.
- (J. Carrillo ('99)) (for general degenerate parabolic eqns) for the homogeneous Dirichlet BC only, a subtle choice of up-to-the boundary entropy inequalities with standard entropy-flux pairs. Indeed, "semi-Kruzhkov" (or Serre) entropies $(u-k)^\pm$ are used, test functions do not vanish on the boundary but
  - while dealing with $(u-k)^+$, one takes $k \in \mathcal{K}_+ := \{k \in \mathbb{R},\, k \geq 0\} \equiv \left\{k \in \mathbb{R} \,\middle|\, \varphi_\nu(k) \leq \sup \beta(k)\right\}$;
  - while dealing with $(u-k)^-$, one takes $k \in \mathcal{K}_- := \{k \in \mathbb{R},\, k \leq 0\} \equiv \left\{k \in \mathbb{R} \,\middle|\, \varphi_\nu(k) \geq \inf \beta(k)\right\}$.

  $\Rightarrow$ our (second) definition is similar; sets $\mathcal{K}_\pm$ are crucial.

...Alternatives to BLN approach, boundary traces...

- (A. Vasseur ('01), E.Yu. Panov ('07)) revival of the original BLN strong-trace formulation: the strong trace of a merely L^∞ entropy solution u does exist !!! (a bit less than this, but still sufficient for our needs...)

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  This subtle "regularity" result for entropy solutions comes along with compactifying effects of the non-linearity φ (P.L. Lions, B. Perthame, and E. Tadmor ('94), E.Yu. Panov ('94)).

Effective BC graph

The BLN condition and its extrapolation.

Our goal is to generalize condition (BLN) by replacing $\beta = \{0\} \times \mathbb{R}$ with a general maximal monotone graph.

Let us first reformulate the boundary condition as :

$$(\widetilde{u}, \varphi_\nu(\widetilde{u})) \in \widetilde{\beta}_{(t,x)} \quad (\ \text{i.e., } \varphi_\nu(u) \in \widetilde{\beta}_{(t,x)}(u)\ ),$$

where $\widetilde{\beta}_{(t,x)}$ is the following maximal monotone subgraph of $\varphi_\nu(.)$: (Dubois,LeFloch):

$$\widetilde{\beta}_{(t,x)} := \left\{ (z, \varphi_\nu(z)) \;\middle|\; \begin{array}{l} \text{sign}(z)(\varphi_\nu(z) - \varphi_\nu(k)) \geq 0 \\ \text{for all } k \in [\min(0,z), \max(0,z)] \end{array} \right\}$$

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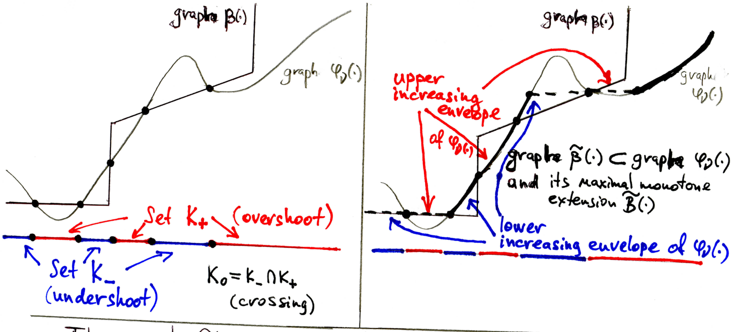
Intuition + heuristics + particular cases (in particular, our previous works A., Sbihi '07,'08) $\Rightarrow$ we associate to a general graph $\beta_{(t,x)}$ the "projected graph" $\widetilde{\beta}_{(t,x)}$ characterized as (wait for pictures)

> $\widetilde{\beta}_{(t,x)}$ is the "closest" to $\beta_{(t,x)}$ maximal monotone subgraph of the graph of the function $\varphi_{\nu(x)} = \varphi \cdot \nu(x)$ that contains the points of crossing of $\beta_{(t,x)}(\cdot)$ with $\varphi_\nu(\cdot)$.



- The graph $\tilde{B}(\cdot)$ is
  - the closest to $\beta(\cdot)$ maximal monotone graph
  - containing all the points of crossing of $\beta(\cdot)$ and $\varphi_p(\cdot)$

  It consists of
  - upper increasing envelopes of the graph $\varphi_p(\cdot)$ over the connected components of K_+
  - lower increasing envelopes — " — over connected components of K_-
- The "effective BC" graph $\tilde\beta(\cdot)$ is the intersection of $\tilde{B}$ and $\varphi_p(\cdot)$

Properties of the “effective BC graph” $\widetilde{\beta}$

The graph $\widetilde{\beta}$ can be characterized in several ways:

– using upper and lower increasing envelopes of $\varphi_\nu(\cdot)$

– using the sets $\mathcal{K}_+ := \left\{k \in \mathbb{R} \,\middle|\, \varphi_\nu(k) \leq \sup \beta(k)\right\}$ and $\mathcal{K}_- := \left\{k \in \mathbb{R} \,\middle|\, \varphi_\nu(k) \geq \inf \beta(k)\right\}$ with semi-Kruzhkov fluxes:

$$z \in Dom\,\widetilde{\beta} \quad \Leftrightarrow \quad \forall k \in \mathcal{K}_\pm \;\; q^\pm(z,k) \geq 0.$$

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Some important properties of $\widetilde{\beta}$:

- $\widetilde{\beta}$ has a unique maximal monotone (on $\mathbb{R}$) extension; denote it $\widetilde{\mathcal{B}}$;
  $\widetilde{\beta}$ is the common part of $\widetilde{\mathcal{B}}$ and the graph of φ_ν

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- Operation $\widetilde{\ }$: $\beta \mapsto \widetilde{\mathcal{B}}$ is a projection
- One can introduce the distance "dist $\left(\widetilde{\mathcal{B}}_1 \, , \, \widetilde{\mathcal{B}}_2\right)$" by taking $\|\widetilde{\mathcal{B}}_1 - \widetilde{\mathcal{B}}_2\|_\infty$. And one can introduce the order relation "$\widetilde{\mathcal{B}}_1 \succeq \widetilde{\mathcal{B}}_2$" by requiring $\widetilde{\mathcal{B}}_1 \geq \widetilde{\mathcal{B}}_2$ pointwise
- One can define distance and order on graphs β
- Then, operation " $\widetilde{\ }$ " is continuous + order-preserving

DEFINITION (PART I)

A first definition of entropy solution.

Definition

A function $u \in L^\infty(Q)$ is called entropy solution for Problem (H) if

- $\forall k \in \mathbb{R},\ \forall \xi \in C_c^\infty(Q),\ \xi \geq 0$, the entropy inequalities inside Ω hold :

$$\int_Q (u-k)^\pm \xi_t + \int_Q \operatorname{sign}^\pm(u-k)(\varphi(u)-\varphi(k)) \cdot \nabla\xi \geq 0$$

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• The functions $u, \varphi_\nu(u)$ admit L^1 strong traces[a] on Σ, denoted $\gamma u, \gamma\varphi_\nu(u)$ such that

$$\big(\gamma u\,,\, \gamma\varphi_\nu(u)\big)(t,x) \in \widetilde{\beta}_{(t,x)} \text{ a.e. } (t,x) \in \Sigma.$$

Here the graph $\widetilde{\beta}$ is the projection of β as defined above.

[a]This is ok under additional non-degeneracy assumption on φ. The general case is treated using strong trace of "singular mapping" $V_{\varphi_\nu}(u)$ (it always exists)

Existence ?? Oups..!

UNIQUENESS AND COMPARISON: OK!

Existence..? Uniqueness, comparison, L^1 contraction.

Key drawback of this definition: stability by approximation seems very unlikely (convergence of u_ε to u in $(0, T) \times \Omega$ does not imply anything about convergence of γu_ε...). $\Rightarrow$ pb. for existence and justification[1].
But: it's fully ok for uniqueness!

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Theorem

If $u, \hat{u}$ are entropy solutions for (H) with data $u_0, \hat{u}_0$ respectively, then for all $t \in (0,T)$

$$\int_\Omega (u-\hat{u})^+(t) \le \int_\Omega (u_0-\hat{u}_0)^+. \qquad (L^1C)$$

Remark. Thus we have uniqueness, comparison principle and L^1 continuous dependence on the data u_0 of the entropy solution.

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We can prove a similar inequality for entropy solutions associated with two different "formal BC graphs" $\beta, \hat\beta$ (recall remarks on the distance and the order relation on such graphs). Then (L^1C) still holds if $\beta \succeq \hat\beta$. In general, a distance term can be added to the right-hand side.

Thus we also have a stability result with respect to β!

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...Uniqueness, comparison, L^1 contraction.

For the proof, by the Kruzhkov's doubling of variables argument applied "inside Ω" one deduces the "local Kato inequality"

$$\int_\Omega (u-\hat{u})^+(t)\xi \le \int_\Omega (u_0-\hat{u}_0)^+\xi(0,\cdot) + \int_0^t\int_\Omega q^+(u,\hat{u})\cdot\nabla\xi$$

for all $\xi \in \mathcal{D}([0,t]\times\Omega)$.

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Take for $\xi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^N)$ truncation-near-the-boundary functions ξ_h. We "pay" for this truncation with a new term which is "dissipative". Indeed,

$$(*) \qquad \text{as } h\downarrow 0, \quad \int_0^t\int_\Omega q^+(u,\hat{u})\cdot\nabla\xi_h \longrightarrow -\int_0^t\int_{\partial\Omega}\gamma_w q^+(u,\hat{u}) = -\int_0^t\int_{\partial\Omega} q^+(\gamma u,\gamma\hat{u}) \ \le^{???}\ 0,$$

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By the trace condition of the Definition, both γu and $\gamma\hat{u}$ belong to the domain of a monotone subgraph of φ_ν.
Then the the right-hand side of $(*)$ is non-positive. This yields the global Kato inequality; at the limit, $\xi\equiv 1$ and we conclude.

EQUIVALENT DEFINITION

(STABLE UNDER POINTWISE CONVERGENCE)

Equivalent definition of solution

Proposition (Entropy solution)

Let $u \in L^\infty$. The assertions (i),(ii) are equivalent :

(i) *("def. with traces")* *u is an entropy solution in the above sense*

(ii) *("def. a-la Carrillo" + technicalities)* *The function u verifies:*

$$\forall k \in \mathbb{R} \quad \forall \xi \in \mathcal{D}([0,T) \times \overline{\Omega})^+$$

$$\int_0^T \int_\Omega \left(-(u-k)^\pm \xi_t - q^\pm(u,k) \cdot \nabla \xi\right) - \int_\Omega (u_0 - k)^\pm \xi(0,\cdot)$$
$$\leq \iint_\Sigma C_k \wedge \left(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\right)^\mp \xi(t,x).$$

Here, C_k is a constant that depends on $\|u\|_\infty$ and on k .

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Here, C_k is a constant that depends on $\|u\|_\infty$ and on k .

And if the sets $\Sigma^\pm(k) := \{(t,x)\in\Sigma \mid k \in \mathcal{K}_\pm(t,x)\}$ are "regular enough" then (i),(ii) are also equivalent to

(ii') *("def. a-la Carrillo")* *The function u verifies*

$$\forall k \in \mathbb{R} \;\; \forall \xi \in \mathcal{D}([0,T)\times\overline{\Omega})^+ \;\; \text{such that } \xi|_{\Sigma\setminus\Sigma^\pm(k)} = 0$$

$$\int_0^T\!\!\int_\Omega \left(-(u-k)^\pm \xi_t - q^\pm(u,k)\cdot\nabla\xi\right) - \int_\Omega (u_0-k)^\pm \xi(0,\cdot) \leq 0.$$

EXISTENCE OF ENTROPY SOLUTIONS.

JUSTIFICATION OF THE SOLUTION NOTION BY CONVERGENCE OF APPROXIMATIONS.

Technique and assumptions to ensure compactness + convergence

All our existence results follow the same scheme:

- approximate (H) by some “simpler” problems (H_ε), solved at previous step (⇒ we will use “multi-layer” approximations[2])

[2] A convincing justification of the solution notion: vanishing viscosity limit ?
Unfortunately, this does not work, e.g., for the zero-flux condition;
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- consequently, get compactness of approximate solutions u_ε
  under non-degeneracy assumption on $\varphi(.)$
- write up-to-the-boundary entropy inequalities for (H_ε)
- finally, pass to the limit in the boundary term of this inequality

In the last steps, the second entropy formulation (ii) is instrumental.

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The key calculation + existence (dishonest proof)

Look at "parabolic up-to-the-boundary entropy inequality"

$$\int_0^T\!\!\int_\Omega \big(-(u^\varepsilon - k)^+\xi_t - q^+(u^\varepsilon, k)\cdot\nabla\xi\big) - \int_\Omega (u_0 - k)^+\xi(0,\cdot)$$

$$\leq -\int_\Sigma \mathrm{sign}^+(u^\varepsilon - k)\big(b^\varepsilon(t,x) - \varphi_{\nu(x)}(k)\big)\xi - \varepsilon\int_0^T\!\!\int_\Omega sign^+(u^\varepsilon - k)\,\nabla u^\varepsilon\cdot\nabla\xi$$

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$\Rightarrow$ first convergence result: OK for $\underline{\beta_{(t,x)}=\widetilde{\mathcal{B}}=}$ subgraph of φ_ν

$\Rightarrow$ existence ("dishonest proof") for almost general graph $\beta_{(t,x)}$ under ad hoc assumptions that ensure L^∞ bound on sols.

Stability results + "honest" existence proof

Goal: prove "honestly" existence of solutions : that is, explain appearance of $\widetilde{\beta}_{(t,x)}$ by passing to the limit from problems set up with graph $\beta_{(t,x)}$.

- Assume existence of sequences of constant sub- and supersolutions:

$$A_m^+ \text{ super-sol.}, \lim_{m\to\infty} A_m^+ = +\infty \quad \text{and} \quad A_m^- \text{ sub-sol.}, \lim_{m\to\infty} A_m^- = -\infty$$

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- For a finer argument, monotone convergence can be used
- Alternative to truncations: (a more classical technique in the world of maximal monotone things): use (adapt) Yosida approximations of $\beta_{(t,x)}$.

References + Thanks

Previous papers available at
lmb.univ-fcomte.fr/Boris-Andreianov
Preprint available on hal.archives-ouvertes.fr

Thank you — Grazie !!!