On nonlinear conservation laws with nonlocal diffusion term

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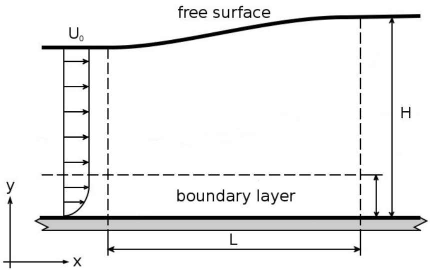
outline

shallow water flow



boundary conditions at free surface

incompressible Navier-Stokes equations

no-slip boundary conditions at rigid bottom

Assumptions (Kluwick, Cox, Exner and Grinschgl (2010))

1. Froude number $1 < \frac{U_0}{\sqrt{gH}} << 2$
2. length scales $H << L$
3. Reynolds number $1 << Re = \frac{L\sqrt{gH}}{\nu}\frac{H^2}{L^2}$
   $\Rightarrow$ viscous effects only important in boundary layer

incompressible Navier-Stokes equation

$$
\begin{aligned}
u_x + v_y &= 0\,, \\
u_t + uu_x + vu_y &= -p_x + \frac{1}{Re}\left(\frac{H^2}{L^2}u_{xx} + u_{yy}\right), \\
v_t + uv_x + vv_y &= -\frac{L^2}{H^2}(p_y + 1) + \frac{1}{Re}\left(\frac{H^2}{L^2}v_{xx} + v_{yy}\right).
\end{aligned}
$$

x, y ...coordinates, u, v ...velocity components, t ...time, p ...pressure

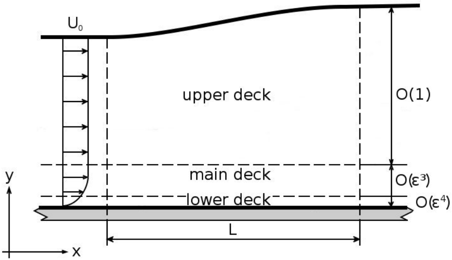
no-slip boundary condition

$$
y = s(x,t)\ :\quad u = 0\,,\quad v = s_t\,.
$$

boundary conditions at free surface

$$
y = h(x,t)\ :\quad h_t + uh_x - v = 0\,,\qquad p = -Th_{xx}\left(1 + \frac{H^2}{L^2}(h_x)^2\right)^{-3/2}.
$$

triple-deck structure: lower deck



matching conditions

$$\lim_{Y\to\infty} U(X,Y) = Y + A\,,$$
$$\lim_{X\to-\infty} U(X,Y) = Y\,.$$

no-slip b.c. at $Y=0$:

$$U=0\,,\ \ V=0\,.$$

governing equations for (U,V,P) with $X\in\mathbb{R}$, $Y\in\mathbb{R}_+$ and $t\in\mathbb{R}_+$

$$\partial_X U + \partial_Y V = 0\,,$$
$$U\partial_X U + V\partial_Y U = -\partial_X P + \partial_Y^2 U\,,$$
$$\partial_t P + \partial_X P - P\partial_X P = K_1\partial_X A + K_2\partial_X^3 P\,.$$

linear flow response

interaction equation for pressure P

$$\partial_t P + \partial_X P = K_1 \partial_X \mathcal{D}^{1/3} P + K_2 \partial_X^3 P, \quad X \in \mathbb{R}, \quad t \in \mathbb{R}_+,$$

with constants K_1 and K_2 related to

- K_1 streamline curvature and boundary displacement effects
- K_2 detuning and surface tension

and a nonlocal operator $\mathcal{D}^\alpha$ with $0 < \alpha < 1$ defined as

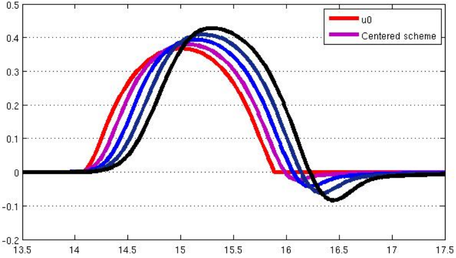
$$(\mathcal{D}^\alpha P)(X) := \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{X} \frac{P'(\xi)}{(X-\xi)^\alpha} \, d\xi \, .$$

Fowler equation: model for dune formation

$$\partial_t u + \frac{1}{2}\partial_x u^2 = -\partial_x \mathcal{D}^{1/3} u + \partial_x^2 u, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+,$$

where $u(x,t)$ represents dune amplitude.

no maximum principle (Alibaud, Azerad and Isèbe (2010))



bore-like traveling wave solutions

nonlocal operator $\partial_x \mathcal{D}^\alpha$

For $0 < \alpha < 1$, the operator

$$(\partial_x \mathcal{D}^\alpha u)(x) = \partial_x \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha} \, dy$$

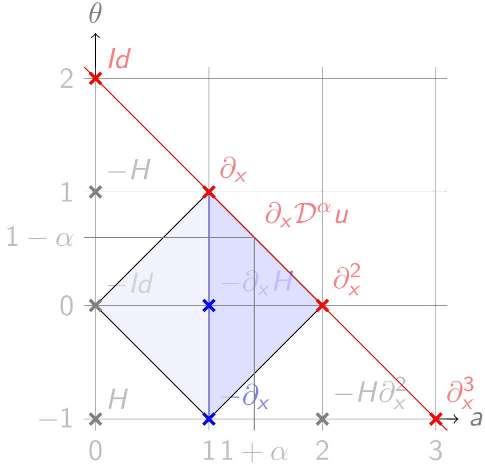
is a Fourier multiplier operator

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha u)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} (\partial_x \mathcal{D}^\alpha u)(x) \, dx = \Lambda(\xi) \mathcal{F} u(\xi)$$

with $u \in \mathcal{S}(\mathbb{R})$ and symbol

$$\Lambda(\xi) = -\left(\sin\left(\alpha \frac{\pi}{2}\right) - i \, \mathrm{sgn}(\xi) \cos\left(\alpha \frac{\pi}{2}\right)\right) |\xi|^{\alpha+1}, \quad \xi \in \mathbb{R} .$$

Fourier multiplier operator $(\mathcal{F}Tf)(\xi) = -\psi(-\xi)(\mathcal{F}f)(\xi)$



Riesz-Feller operator T

$$\psi(\xi) = |\xi|^a \exp\left(i\,\mathrm{sgn}(\xi)\theta\frac{\pi}{2}\right)$$

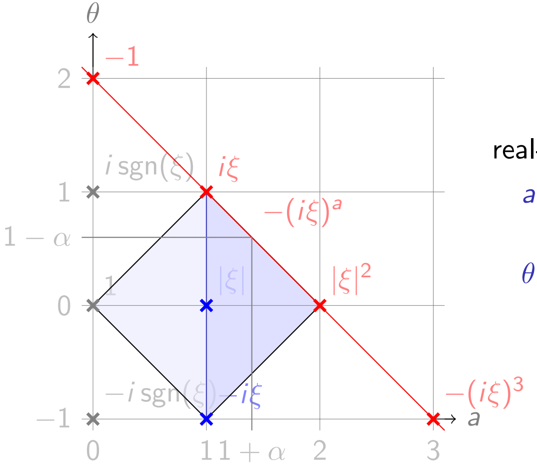
real-valued parameters

- a index of stability $0 < a \leq 2$
- θ skewness parameter $|\theta| \leq \min(a, 2-a)$

Hilbert transform

$$Hf := p.v.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy$$

Fourier multiplier $\psi_{a,\theta}(\xi) := |\xi|^a \exp\left(i\,\mathrm{sgn}(\xi)\theta\frac{\pi}{2}\right)$



real-valued parameters

- a index of stability
  $0 < a \leq 2$
- θ skewness parameter
  $|\theta| \leq \min(a, 2-a)$

fractional diffusion equation

$$\partial_t u = \partial_x \mathcal{D}^\alpha u\,, \quad x \in \mathbb{R}\,, \quad t \in \mathbb{R}_+\,,$$

for some fixed α with $0 < \alpha < 1$.

strongly continuous, convolution semigroup

$$T_t : L^p(\mathbb{R}) \to L^p(\mathbb{R})\,, \quad u_0 \mapsto T_t u_0 = u(t,x) = K(t,\cdot) * u_0\,,$$

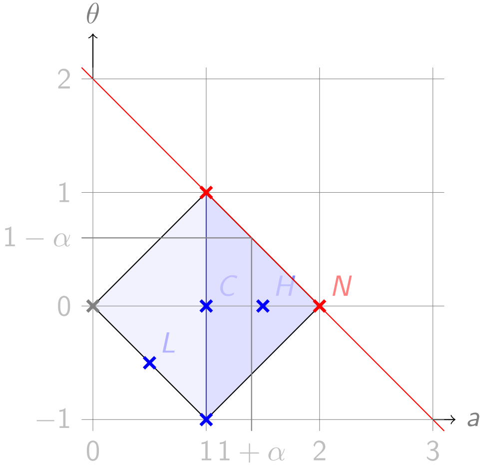
with $1 \le p < \infty$ and kernel $K(t,x) = \mathcal{F}^{-1}(\exp(\Lambda(.)t))(x)$.

Properties of $K(t,x)$: for all $x \in \mathbb{R}$, $t > 0$ and $m \in \mathbb{N}$,

- non-negative $K(t,x) \ge 0$
- integrable $\|K(t,.)\|_{L^1(\mathbb{R})} = 1$
- scaling $K(t,x) = t^{-\frac{1}{1+\alpha}} K(1, xt^{-\frac{1}{1+\alpha}})$
- smooth $K(t,x)$ is $\mathcal{C}^\infty$ smooth
- bounded there exists $B_m \in \mathbb{R}_+$ such that

$$|\partial_x^m K|(t,x) \le t^{-\frac{1+m}{1+\alpha}} \frac{B_m}{1 + t^{-\frac{2}{1+\alpha}}|x|^2}$$

Lévy strictly stable distributions on $\mathbb{R}$



random variable X

$E[\exp(i\xi X)] = \exp(-\psi(\xi))$

$\psi(\xi) = |\xi|^a \exp\left(i \operatorname{sgn}(\xi)\theta\frac{\pi}{2}\right)$

distributions

- L Lévy-Smirnov
  PDF $\frac{x^{-3/2}}{2\sqrt{\pi}} \exp\left(-\frac{1}{4x}\right)$, $x > 0$.
- C Cauchy(-Lorentz)
  PDF $\frac{1}{\pi}\frac{1}{1+x^2}$
- H Holtsmark
- N Normal (Gaussian)
  PDF $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$

approximate identity

Theorem (Stein, Weiss)

Suppose $\phi \in L^1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \phi(x)dx = 1$ *and for* $\epsilon > 0$ *let* $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. *If* $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, *or* $f \in C_0 \subset L^\infty(\mathbb{R}^n)$, *then* $\|f * \phi_\epsilon - f\|_p \to 0$ *as* $\epsilon \to 0$.

Theorem (Lieb, Loss)

Let j *be in* $L^1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} j = 1$. *For* $\epsilon > 0$, *define* $j_\epsilon(x) := \epsilon^{-n} j(x/\epsilon)$, *so that* $\int_{\mathbb{R}^n} j_\epsilon = 1$ *and* $\|j_\epsilon\|_1 = \|j\|_1$. *Let* $f \in L^p(\mathbb{R}^n)$ *for some* $1 \le p < \infty$ *and define the convolution* $f_\epsilon := j_\epsilon * f$. *Then*

$$f_\epsilon \in L^p(\mathbb{R}^n) \quad \text{and} \quad \|f_\epsilon\|_p \le \|j\|_1 \|f\|_p .$$
$$f_\epsilon \to f \text{ strongly in } L^p(\mathbb{R}^n) \text{ as } \epsilon \to 0 .$$

If $j \in C_c^\infty(\mathbb{R}^n)$, *then* $f_\epsilon \in C^\infty(\mathbb{R}^n)$ *and* $D^\alpha f_\epsilon = (D^\alpha j_\epsilon) * f$.

Theorem (Lévy process)

For every $0 < \alpha < 1$, there exists a Lévy process $\{X_t \mid t \geq 0\}$ such that the probability distribution μ^t of X_t has probability density function $K(t,x)$. Moreover, the associated transition semigroup $\{P_t\}$ is a strongly continuous semigroup on $C_0(\mathbb{R})$ with $\|P_t\| = 1$. The infinitesimal generator L_α of $\{P_t\}$ is given for $f \in C_0^2(\mathbb{R})$ by

$$L_\alpha f(x) = \cos\left((1-\alpha)\frac{\pi}{2}\right) \int_0^\infty \frac{f(x+y) - f(x) - yf'(x)}{y^{2+\alpha}} \, \mathrm{d}y$$

and has core $C_c^\infty(\mathbb{R})$.

transition function

$$P_t(x, B) = \mu^t(B - x) \quad \text{for } t \geq 0,\, x \in \mathbb{R},\, B \in \mathcal{B}(\mathbb{R})$$

transition semigroup Define for $f \in C_0(\mathbb{R})$

$$(P_t f)(x) = \int_{\mathbb{R}} P_t(x, \mathrm{d}y) f(y) = \int_{\mathbb{R}} f(x+y) K(t,y) \, \mathrm{d}y = E[f(x + X_t)],$$

then $P_t f \in C_0(\mathbb{R})$ by the Lebesgue convergence theorem.

scalar conservation law with nonlocal diffusion

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \tag{1}$$

where $f(u)$ is a smooth flux function.

Theorem (Droniou, Gallouët and Vovelle (2003))

The Cauchy problem of (1) *with initial datum* $u_0 \in L^\infty(\mathbb{R})$ *has a unique global solution* $u(t,x)$, *in the sense that* $u \in L^\infty((0,\infty) \times \mathbb{R})$ *satisfies*

$$u(t,x) = (K(t,.) * u_0)(x) - \int_0^t \big(K(t-\tau,.) * \partial_x f(u(\tau,.))\big)(x) d\tau \tag{2}$$

almost everywhere. Moreover,

1. $u \in C^\infty((0,\infty) \times \mathbb{R})$ *and* $u \in C_b^\infty((t_0,\infty) \times \mathbb{R})$ *for all* $t_0 > 0$.
2. u *satisfies equation* (1) *in the classical sense.*
3. *for all* $t > 0$, $\|u(t,.)\|_\infty \le \|u_0\|_\infty$ *and, in fact,* u *takes its values between the essential lower and upper bounds of* u_0.
4. $u(t) \xrightarrow{t \to 0} u_0$ *in* $L^\infty(\mathbb{R})$ *weak-*$*$ *and in* $L^p_{loc}(\mathbb{R})$ *for all* $p \in [1,\infty)$.

sketch of proof

Droniou, Gallouët and Vovelle established the result in case of a nonlocal diffusion operator with symbol $-|\xi|^{1+\alpha}$ for $0 < \alpha < 1$.

existence Suppose $u_0 \in C_c^\infty(\mathbb{R})$.
Construct approximate solution u^δ, $\delta > 0$, by a splitting method:

- Define $u^\delta(0, .) = u_0$
- Define $u^\delta(t, x)$ on $(t, x) \in (2n\delta, (2n+1)\delta] \times \mathbb{R}$, $n \in \mathbb{N}_0$, as the solution of $\partial_t u = 2\partial_x \mathcal{D}^\alpha u$ with initial condition $u^\delta(2n\delta, .)$.
- Define $u^\delta(t, x)$ on $(t, x) \in ((2n+1)\delta, (2n+2)\delta] \times \mathbb{R}$, $n \in \mathbb{N}_0$, as the solution of $\partial_t u + 2\partial_x f(u) = 0$ with initial condition $u^\delta((2n+1)\delta, .)$.

For $\delta_0 > 0$ sufficiently small, any compact set $Q \subset \mathbb{R}$ and $T > 0$, $\{u^\delta \mid \delta \in (0, \delta_0]\}$ is relatively compact in $C([0, T]; L^1(Q))$.

limit function $u \in C([0, T]; L^1(\mathbb{R}))$ satisfies mild formulation (2).

scalar conservation law with vanishing nonlocal diffusion

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_x \mathcal{D}^\alpha u^\epsilon, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \tag{3}$$

where $f(u)$ is a smooth flux function.

Theorem (Droniou (2003))

Suppose $0 < \alpha \leq 1$ and $u_0 \in L^\infty(\mathbb{R})$. The solution u^ϵ of

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_x \mathcal{D}^\alpha u^\epsilon \ , \qquad u^\epsilon(0,x) = u_0(x) \ ,$$

converges as $\epsilon \to 0$ in $C([0,T]; L^1_{loc}(\mathbb{R}^n))$ for all $T > 0$ to the entropy solution of the Cauchy problem

$$\partial_t u + \partial_x f(u) = 0 \ , \qquad u(0,x) = u_0(x) \ .$$

Moreover, if $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then

$$\|u^\epsilon - u\|_{C([0,T];L^1(\mathbb{R}))} = O\left(\epsilon^{1/(1+\alpha)}\right) \ .$$

traveling wave solutions of equation (1)

Consider wave speed $s \in \mathbb{R}$ and traveling wave variable $\xi := x - st$.

Definition

A traveling wave solution of (1) is a solution of the form $u(t,x) = \bar{u}(\xi)$, for some function $\bar{u}$ that connects different endstates $\lim_{\xi \to \pm\infty} \bar{u}(\xi) = u_\pm$.

traveling wave equation

$$h(u) := f(u) - su - \big(f(u_-) - su_-\big) = \mathcal{D}^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha}\, dy \tag{4}$$

properties of specific traveling wave solutions

- Rankine-Hugoniot condition $f(u_+) - f(u_-) = s(u_+ - u_-)$
- For convex flux function $f(u)$ and a monotone solution, standard entropy condition $u_- > u_+$.

existence of traveling wave solution

Theorem (A., Hittmeir and Schmeiser (2011))

Suppose $f \in C^\infty(\mathbb{R})$ *is a convex function and* (u_-, u_+, s) *satisfy the Rankine-Hugoniot condition as well as the entropy condition* $u_- > u_+$*. Then there exists a traveling wave solution of* (1)*, which is decreasing and unique (up to translations) among all functions* $u \in \{u_- + v \mid v \in H^2(-\infty, 0) \cap C_b^1(\mathbb{R})\}$*.*

traveling wave equation linearized at $u = u_-$

$$h'(u_-)v(\xi) = \mathcal{D}^\alpha v(\xi) := \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\xi} \frac{v'(y)}{(\xi - y)^\alpha}\, dy, \quad \xi \in \mathbb{R}_-,$$

has solutions of the form

$$v(\xi) := b \exp(\lambda \xi) \text{ with } \lambda := h'(u_-)^{1/\alpha} \text{ and } b \in \mathbb{R}.$$

local existence + uniqueness

Lemma

For every sufficiently small $\epsilon > 0$, $\xi_\epsilon := \frac{\log \epsilon}{\lambda}$ and $I_\epsilon = (-\infty, \xi_\epsilon]$, there exists an ϵ-independent $\delta > 0$, such that the equation (4) has solutions $u_{up,\epsilon}, u_{down,\epsilon} \in u_- + H^2(I_\epsilon)$, which satisfy

$$u_{up,\epsilon}(\xi_\epsilon) = u_- + \epsilon\,, \qquad u_{down,\epsilon}(\xi_\epsilon) = u_- - \epsilon \tag{5}$$

and are unique among all functions $\{u \mid \|u - u_-\|_{H^2(I_\epsilon)} \le \delta\}$. Moreover,

$$\|u_{up,\epsilon} - u_- - e^{\lambda\xi}\|_{H^2(I_\epsilon)} \le C\epsilon^2\,, \qquad \|u_{down,\epsilon} - u_- + e^{\lambda\xi}\|_{H^2(I_\epsilon)} \le C\epsilon^2$$

holds with an ϵ-independent constant C and $\lambda := h'(u_-)^{1/\alpha}$.

proof Consider $\bar{u} = u_{down,\epsilon}(\xi) - u_- + \exp(\lambda\xi)$ which satisfies a BVP

$$\big(\mathcal{D}^\alpha - h'(u_-)\big)\bar{u} = h(u_- - \exp(\lambda\xi) + \bar{u}) + h'(u_-)\big(\exp(\lambda\xi) - \bar{u}\big), \quad \bar{u}(\xi_\epsilon) = 0.$$

Cast BVP as a fixed point problem + use Banach's fixed point theorem.

local monotonicity

Lemma

For all $\xi \in I_\epsilon$, the solution $u_{down,\epsilon}(\xi)$ is bounded ($u_{down,\epsilon}(\xi) < u_-$) and monotone (${u'}_{down,\epsilon}(\xi) < 0$).

proof Due to Sobolev imbedding $H^2(I_\epsilon) \hookrightarrow C_b^1(I_\epsilon)$,

$$\left|u_{down,\epsilon} - u_- - e^{\lambda\xi}\right| \leq C\epsilon^2 \quad \text{for all } \xi < \xi_\epsilon \,.$$

There exists $\xi_\star < \xi_\epsilon$, such that $u_{down,\epsilon}(\xi_\star) = u_- - 2C\epsilon^2$ and the solution is bounded from above by u_- for all $\xi \in (\xi_\star, \xi_\epsilon)$. For $\epsilon_2 = 2C\epsilon_1^2$ with $\epsilon_1 = \epsilon$, the translated function $u_{down,\epsilon}(\xi - \xi_{\epsilon_2} + \xi_\star)$ is the unique solution u_{down,ϵ_2}.

Iteration of the argument produces a sequence $\{\epsilon_n\}$, determined by $\epsilon_{n+1} = 2C\epsilon_n^2$, such that the unique solution satisfies $u_{down,\epsilon}(\xi) < u_-$ for all $\xi \in (\xi_{\epsilon_n}, \xi_\epsilon)$ and $n \in \mathbb{N}$.

Similar argument proves monotonicity.

continuation of solution

Lemma

Let $u \in C_b^1(-\infty, \xi_0]$ be a solution of (4). For sufficiently small $\delta > 0$, the solution has a unique continuation in the function space $C_b^1(-\infty, \xi_0 + \delta)$.

proof For monotone functions $u \in C_b^1(\mathbb{R})$, equation (4) is equivalent to

$$u(\xi) - u_- = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\xi} \frac{h(u(y))}{(\xi - y)^{1-\alpha}}\, dy\,. \tag{6}$$

Rewrite equation (6) as a Volterra integral equation on a bounded interval

$$u(\xi) = f(\xi) + \frac{1}{\Gamma(\alpha)} \int_{\xi_0}^{\xi} \frac{h(u(y))}{(\xi - y)^{1-\alpha}}\, dy\,,$$

where

$$f(\xi) = u_- + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\xi_0} \frac{h(u(y))}{(\xi - y)^{1-\alpha}}\, dy\,.$$

Local existence of a smooth solution is a standard result.

global properties

Lemma

Let $u \in C_b^1(-\infty, \xi_0]$ be (a continuation of) the solution $u_{down}(\xi)$ of (4). Then for all $\xi \in (-\infty, \xi_0]$, $u(\xi)$ is nonincreasing and bounded by $u_+ < u(\xi) < u_-$.

Proof of Theorem.

global existence boundedness+local continuation $\Rightarrow$
The solution $u(\xi)$ exists for all $\xi \in \mathbb{R}$ and satisfies $\lim_{\xi \to +\infty} u(\xi) = u_+$.

global uniqueness Let $u \in \{u_- + v \mid v \in H^2(-\infty, 0) \cap C_b^1(\mathbb{R})\}$ be a solution of (4). Then the restriction of the solution to an interval $(-\infty, \xi_0]$ is, up to a shift in ξ, the continuation of u_{up} or u_{down}, or the constant function $u \equiv u_-$. □

asymptotic stability of traveling wave ϕ

Theorem (A., Hittmeir and Schmeiser (2011))

Suppose f is a convex function, ϕ is a traveling wave solution of (4), and u_0 is such that $W_0(\xi) = \int_{-\infty}^{\xi}(u_0(\eta) - \phi(\eta))\,d\eta$ satisfies $W_0 \in H^2(\mathbb{R})$. If $\|W_0\|_{H^2}$ is small enough, then the Cauchy problem for

$$\partial_t u + \partial_\xi (f(u) - su) = \partial_\xi \mathcal{D}^\alpha u$$

with initial datum u_0 has a unique global solution converging to the traveling wave solution ϕ in the sense that

$$\lim_{t\to\infty} \int_t^\infty \|u(\tau,\cdot) - \phi\|_{H^1}\, d\tau = 0\,.$$

idea of proof

Perturbation $U = u - \phi$ satisfies

$$\partial_t U + \partial_\xi \big(f(u) - f(\phi) - sU \big) = \partial_\xi \mathcal{D}^\alpha U \,.$$

The energy estimate

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} f''(\phi)\phi' U^2 \, d\xi - \frac{1}{2}\int_{\mathbb{R}} f''(\phi + \vartheta U) U^2 \partial_\xi U \, d\xi = -a_\alpha \|U\|_{\dot{H}^{\frac{1+\alpha}{2}}}^2$$

holds for some $0 < \vartheta < 1$ and leads to

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 - C_0\|U\|_{L^2}^2 - L(\|U\|_{L^\infty})\|U\|_{L^\infty}\|U\|_{H^1}^2 \leq -a_\alpha \|U\|_{\dot{H}^{\frac{1+\alpha}{2}}}^2 \,,$$

for a positive nondecreasing function L and positive constants C_0 and a_α.

The primitive of the perturbation

$$W(t,\xi) := \int_{-\infty}^{\xi} U(t,\eta)\, d\eta$$

satisfies for some $0 < \vartheta < 1$

$$\partial_t W + (f'(\phi) - s)\partial_\xi W + \frac{1}{2} f''(\phi + \vartheta U)(\partial_\xi W)^2 = \partial_\xi \mathcal{D}^\alpha W \qquad (7)$$

and the energy estimate

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2} - L(\|U\|_{L^\infty})\|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 \leq -a_\alpha \|W\|^2_{\dot{H}^{\frac{1+\alpha}{2}}} .$$

Lemma

Suppose $W_0 \in H^2(\mathbb{R})$. Then there exists $T > 0$ such that the Cauchy problem for (7) with initial data W_0 has a unique solution $W(t) \in H^2(\mathbb{R})$ for all $t \in [0, T)$.

Lyapunov functional

$$J(t) = \frac{1}{2}(\|W\|^2_{L^2} + \gamma_1\|U\|^2_{L^2} + \gamma_2\|\partial_\xi U\|^2_{L^2})$$

with positive constants γ_1 and γ_2.
Linear combination of energy estimates yields

$$\frac{d}{dt}J + a_\alpha \left(\|W\|^2_{\dot{H}^{\frac{1+\alpha}{2}}} + \gamma_1\|W\|^2_{\dot{H}^{\frac{3+\alpha}{2}}} + \gamma_2\|W\|^2_{\dot{H}^{\frac{5+\alpha}{2}}}\right) - \gamma_1 C_0\|U\|^2_{L^2} - \gamma_2 C_1\|U\|^2_{H^1} - L(\|W\|_{H^2})\|W\|_{H^2}\|U\|^2_{H^{(5+\alpha)/4}} \leq 0$$

Choose $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 C_0\|U\|^2_{L^2} + \gamma_2 C_1\|U\|^2_{H^1} \leq \frac{a_\alpha}{2}\left(\|W\|^2_{\dot{H}^{\frac{1+\alpha}{2}}} + \gamma_1\|W\|^2_{\dot{H}^{\frac{3+\alpha}{2}}} + \gamma_2\|W\|^2_{\dot{H}^{\frac{5+\alpha}{2}}}\right)$$

and get the final estimate

$$\frac{d}{dt}J \leq -\gamma^*\left(\frac{a_\alpha}{2} - \frac{L}{\gamma_*}\|W\|_{H^2}\right)\left(\|W\|^2_{\dot{H}^{\frac{1+\alpha}{2}}} + \gamma_1\|W\|^2_{\dot{H}^{\frac{3+\alpha}{2}}} + \gamma_2\|W\|^2_{\dot{H}^{\frac{5+\alpha}{2}}}\right)$$

fractional Korteweg-de Vries-Burgers equation

$$\partial_t u + u\partial_x u = \epsilon \partial_x \mathcal{D}^\alpha u + \delta \partial_x^3 u, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \tag{8}$$

for some fixed α with $0 < \alpha < 1$ and $\epsilon, \delta \in \mathbb{R}$.

Theorem (Molinet and Ribaud (2001))

global well-posedness of the Cauchy problem for the Kortweg-de Vries-Burgers equation with fractional Laplacian and initial datum in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$.

existence of traveling wave solutions

Theorem (A., Cuesta and Hittmeir (2012))

Suppose (u_-, u_+, s) satisfy the Rankine-Hugoniot condition as well as the entropy condition $u_- > u_+$. Then there exists a traveling wave solution $\bar{u} \in C_b^2(\mathbb{R})$ of (8), such that $\lim_{\xi \to -\infty} \bar{u}(\xi) = u_-$.

references

F. Achleitner, S. Hittmeir and Ch. Schmeiser. On nonlinear conservation laws with nonlocal diffusion term *J. Diff. Equ.*, 250: 2177–2196, 2011.

J. Droniou, T. Gallouët and J. Vovelle. Global solution and smoothing effect for a non-local regularization of a hyperbolic equation. *J. Evol. Equ.*, 3: 499–521, 2003.

J. Droniou Vanishing non-local regularization of a scalar conservation law. *Electronic J. Diff. Equ.*, 117: 1–20, 2003.

A. Kluwick, E. A. Cox, A. Exner and C. Grinschgl On the internal structure of weakly nonlinear bores in laminar high Reynolds number flow. *Acta Mech.*, 210: 135–157, 2010.

Thank you for your attention.

Riemann-Liouville fractional derivative

For a finite interval $[a, b] \subset \mathbb{R}$, $\alpha \in \mathbb{C} \backslash \mathbb{N}_0$ with $\Re\alpha \geq 0$ and $n = \lfloor \Re\alpha \rfloor + 1$

$$(D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(y)}{(x-y)^{\alpha-n+1}}\, dy\,, \qquad x \in [a, b]$$

properties:

- For $\alpha = n \in \mathbb{N}_0$
$$(D^{n}_{a+} f)(x) = f^{(n)}(x)$$
- For $\alpha, \beta \in \mathbb{C}$ with $\Re\alpha \geq 0$ and $\Re\beta > 0$
$$(D^{\alpha}_{a+}(. - a)^{\beta-1})(x) = \begin{cases} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} & \text{for } \alpha - \beta \notin \mathbb{N}_0 \\ 0 & \text{for } \alpha - \beta \in \mathbb{N} \end{cases}$$
- For $\alpha \in \mathbb{C} \backslash \mathbb{N}_0$ with $\Re\alpha \geq 0$ and $f \in AC^n[a, b]$
$$(D^{\alpha}_{a+} f)(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(y)\, dy}{(x-y)^{\alpha-n+1}}$$

Caputo fractional derivative on finite interval

For a finite interval $[a,b] \subset \mathbb{R}$, $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$ with $\Re\alpha \geq 0$ and $n = \lfloor \Re\alpha \rfloor + 1$

$$({}^C D^{\alpha}_{a+} f)(x) = D^{\alpha}_{a+}\left(f(.) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(. - a)^k \right)(x)\,, \qquad x \in [a,b]$$

properties:

- alternative representation

$$({}^C D^{\alpha}_{a+} f)(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(y)}{(x-y)^{\alpha-n+1}}\, dy & \text{for } \alpha \notin \mathbb{N}_0 \\ f^{(n)}(x) & \text{for } \alpha \in \mathbb{N}_0 \end{cases}$$

- For $\alpha, \beta \in \mathbb{C}$ with $\Re\alpha \geq 0$ and $\Re\beta > 0$

$$({}^C D^{\alpha}_{a+} (. - a)^{\beta-1})(x) = \begin{cases} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-1} & \text{for } \alpha - \beta \notin \mathbb{N}_0 \\ 0 & \text{for } \alpha - \beta \in \mathbb{N} \end{cases}$$

- For $\alpha \notin \mathbb{N}_0$ and $f \in C^n[a,b]$

$$({}^C D^{\alpha}_{a+} f)(a) = 0$$

Liouville fractional derivative

For $\alpha \in \mathbb{C} \backslash \mathbb{N}_0$ with $\Re\alpha \geq 0$ and $n = \lfloor \Re\alpha \rfloor + 1$

$$(D_+^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{\alpha-n+1}}\, dy\,, \quad x \in \mathbb{R}$$

properties:

- For $\alpha = n \in \mathbb{N}_0$
$$(D_+^n f)(x) = f^{(n)}(x)$$
- For $\alpha, \lambda \in \mathbb{C}$ with $\Re\alpha \geq 0$ and $\Re\lambda > 0$
$$(D_+^\alpha \exp(\lambda .))(x) = \lambda^\alpha \exp(\lambda x)$$
- For $\alpha \in \mathbb{C}$ with $\Re\alpha > 0$ and $f \in \mathcal{S}(\mathbb{R})$
$$(\mathcal{F} D_+^\alpha f)(\xi) = (-i\xi)^\alpha (\mathcal{F} f)(\xi)$$

  where $(-i\xi)^\alpha = \exp(-\alpha\pi i \,\mathrm{sgn}(\xi)/2)$.

Caputo fractional derivative on $\mathbb{R}$

For $\alpha \in \mathbb{C} \backslash \mathbb{N}_0$ with $\Re\alpha > 0$ and $n = \lfloor \Re\alpha \rfloor + 1$

$$({}^C D_+^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{x} \frac{f^{(n)}(y)}{(x-y)^{\alpha-n+1}} \, dy \,, \quad x \in \mathbb{R}$$

properties:

- For $\alpha = n \in \mathbb{N}_0$

$$({}^C D_+^\alpha f)(x) = f^{(n)}(x)$$

- For $\alpha \in \mathbb{C}$ with $\Re\alpha > 0$ and $\lambda > 0$

$$({}^C D_+^\alpha \exp(\lambda.))(x) = \lambda^\alpha \exp(\lambda x)$$

- For $\alpha \in \mathbb{C}$ with $\Re\alpha > 0$ and $f \in \mathcal{S}(\mathbb{R})$

$$(\mathcal{F}\, {}^C D_+^\alpha f)(\xi) = (-i\xi)^\alpha (\mathcal{F} f)(\xi)$$

where $(-i\xi)^\alpha = \exp(-\alpha\pi i \operatorname{sgn}(\xi)/2)$.

Cauchy problem with fractional derivative

For a finite interval $[a, b] \subset \mathbb{R}$ and $\alpha \in \mathbb{C} \backslash \mathbb{N}_0$ with $\Re\alpha \geq 0$

$$(D^{\alpha}_{a+} y)(x) = f(x, y(x))$$

with $n = \lfloor \Re\alpha \rfloor + 1$ initial conditions

$$(D^{\alpha-k}_{a+} y)(a+) = b_k \,, \quad b_k \in \mathbb{C} \,, \quad k = 1, \ldots, n$$

Theorem (Kilbas et al (2006))

For an open set $G \subset \mathbb{C}$ and a function $f : (a, b] \times G \to \mathbb{C}$ with $f(x, y)$ integrable w.r.t. x for all $y \in G$ and $f(x, y)$ Lipschitz continuous in y uniformly w.r.t. $x \in (a, b]$. Then there exists a unique solution $y(x)$ for Cauchy type problem in the space $\{\, y \in L^1(a, b) \,|\, D^{\alpha}_{a+} y \in L^1(a, b) \,\}$.

Step 1: Cauchy type problem is equivalent to Volterra integral equation

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} (x - a)^{\alpha - j} + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t, y(t))}{(x - t)^{1-\alpha}} dt \,, \quad x \in [a, b]$$

Step 2: Banach fixed point theorem

Cauchy problem with Caputo fractional derivative

For a finite interval $[a, b] \subset \mathbb{R}$ and $\alpha \in \mathbb{R}\backslash\mathbb{N}_0$ with $\alpha > 0$

$$({}^C D^{\alpha}_{a+} y)(x) = f(x, y(x))$$

with $n = \lfloor \alpha \rfloor + 1$ initial conditions

$$y^{(k)}(a+) = b_k \,, \quad b_k \in \mathbb{C} \,, \quad k = 0, \ldots, n-1$$

Theorem (Kilbas et al (2006))

For an open set $G \subset \mathbb{C}$ and a function $f : (a, b] \times G \to \mathbb{C}$ with $f(x, y)$ continuous w.r.t. x for all $y \in G$ and $f(x, y)$ Lipschitz continuous in y uniformly w.r.t. $x \in (a, b]$. Then there exists a unique solution $y(x)$ for Cauchy type problem in the space $\{\, y \in C^{\lfloor \alpha \rfloor}[a, b] \,|\, {}^C D^{\alpha}_{a+} y \in C[a, b] \,\}$.

Step 1: Cauchy type problem is equivalent to Volterra integral equation

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t, y(t))}{(x-t)^{1-\alpha}} dt \,, \quad x \in [a, b]$$

Step 2: Banach fixed point theorem

Lévy strictly α-stable distributions on $\mathbb{R}$

A random variable is said to be strictly stable (or to have a strictly stable distribution), if it has the property that linear combinations of two independent copies of the variable have the same distribution, up to a scaling.

random variable X with Lévy strictly stable distribution

characteristic function $E[\exp(i\xi X)] = \exp(\psi(\xi))$

characteristic exponent $\psi(\xi) := -c_0|\xi|^\alpha \exp\big(-i\,\mathrm{sgn}(\xi)\theta\alpha\frac{\pi}{2}\big)$

parameters

- α index of stability $0 < \alpha \leq 2$
- θ skewness parameter $|\theta| \leq \min\left(\frac{2-\alpha}{\alpha}, 1\right)$
- c_0 scaling parameter $c_0 > 0$

Lévy α-stable distributions on $\mathbb{R}$

A random variable is said to be stable (or to have a stable distribution), if it has the property that linear combinations of two independent copies of the variable have the same distribution, up to location and scale parameters.

random variable X with Lévy stable distribution

characteristic function $E[\exp(i\xi X)] = \exp(\psi(\xi))$

characteristic exponent

$$\psi(\xi) = \begin{cases} -c|\xi|^{\alpha}\left(1 - i\beta(\operatorname{sgn}\xi)\tan\frac{\alpha\pi}{2}\right) + i\tau\xi & \text{for } \alpha \neq 1\,, \\ -c|\xi|\left(1 - i\beta\frac{2}{\pi}(\operatorname{sgn}\xi)\log|\xi|\right) + i\tau\xi & \text{for } \alpha = 1\,. \end{cases}$$

parameters

- α index of stability $0 < \alpha \leq 2$
- β skewness parameter $-1 \leq \beta \leq 1$
- c scaling parameter $0 < c$
- τ location parameter $\tau \in \mathbb{R}$

Lévy operator

Theorem (Sato (1999) Theorem 31.5)

Suppose $\{X_t\}$ *is a Lévy process on* $\mathbb{R}^d$ *with generating triplet* (A, ν, γ), *whereat* $A = (A_{jk}) \in \mathbb{R}^{d \times d}$ *and* $\gamma = (\gamma_j) \in \mathbb{R}^d$. *The associated family of operators* $\{P_t \mid t \geq 0\}$ *is a strongly continuous semigroup on* $C_0(\mathbb{R}^d)$ *with norm* $\|P_t\| = 1$. *Let* L *be its infinitesimal generator. Then* $C_c^\infty(\mathbb{R}^d)$ *is a core of* L, $C_0^2(\mathbb{R}^d) \subset \mathcal{D}(L)$, *and*

$$Lf(x) = \frac{1}{2} \sum_{j,k=1}^{d} A_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(x) + \sum_{j=1}^{d} \gamma_j \frac{\partial f}{\partial x_j}(x) + \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \sum_{j=1}^{d} y_j \frac{\partial f}{\partial x_j}(x) 1_D(y) \right) \nu(\mathrm{d}y) \quad (9)$$

for $f \in C_0^2(\mathbb{R}^d)$ *and* $D = \{x \in \mathbb{R}^d \mid |x| \leq 1\}$.

Lévy operator: examples for $d = 1$

1. Any non-trivial α-stable distribution with $0 < \alpha < 2$ has absolutely continuous Lévy measure

$$\nu(dx) = \begin{cases} c_1 x^{-1-\alpha} & \text{on } (0,\infty)\,, \\ c_2 |x|^{-1-\alpha} & \text{on } (-\infty, 0)\,, \end{cases}$$

   with $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 > 0$.

2. If the Lévy measure ν is non-singular, then

$$\int_{\mathbb{R}} \left(f(x+y) - f(x) - y \frac{\partial f}{\partial x}(x) 1_D(y) \right) \nu(\,\mathrm{d}y) = (K * f - \mu f)(x)$$

   for some $K \in L^1(\mathbb{R})$ and $\mu \in \mathbb{R}$.