Approximation of mean curvature motion with nonlinear Neumann conditions

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joint work with M. Falcone

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The boundary value problem

The PDE

$$\frac{\partial u}{\partial t} - \text{trace}\left(\left(I - \frac{Du \otimes Du}{|Du|^2}\right) D^2 u\right) = 0, \quad \text{in } \Omega \times (0, T),$$

where Ω is a bounded domain of $\mathbb{R}^d$ with a $W^{3,\infty}$ boundary.

Initial condition

$$u(x, \cdot) = u_0,$$

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Boundary condition: the normal vectors to the level sets make a given angle with the outward normal vector

$$\frac{\partial u}{\partial n} = \theta |Du| \quad \text{on } \partial\Omega \times (0,T),$$

where θ is a Lipschitz continuous function with $|\theta(x)| \le \overline{\theta} < 1$.

Other forms of the PDE

$$\frac{\partial u}{\partial t} - \Delta u + \frac{(D^2u\, Du, Du)}{|Du|^2} = 0$$

or

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\frac{Du}{|Du|}\right)|Du| = 0.$$

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Goal

- Propose a semi-Lagrangian scheme for the PDE (extension of Carlini-Falcone-Ferretti, JCP 2005 and Interfaces and Free Boundaries, 2011)
- Couple it with a finite difference scheme to deal with the boundary conditions

Hereafter, $d = 2$.

Some references

MCM via viscosity solution techniques

Evans-Spruck, J. Diff. Geom., 1991
Evans - Soner-Souganidis, Comm. Pure Appl. Math, 1992, ...
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Ishii-Ishii, SIAM J. Math. Anal. 2001

Numerical methods

Osher-Sethian, JCP, 1988
Merriman-Bence-Osher, AMS LN, 1993
Crandall-Lions, Numer. Math., 1996
Barles-Georgelin, SIAM J. Num. Anal., 1995
Catté-Dibos-Koepfler, SIAM J. Num. Anal., 1995

Viscosity solutions

- $F(p,X) = -\text{trace}\left((I_d - \frac{p\otimes p}{|p|^2})X\right)$ is defined for $p \neq 0$
- $\underline{F}$ and $\overline{F}$ are the LSC and USC envelopes of F
-

$$\overline{G}(x,\eta,p,X) = \begin{cases} \eta + \overline{F}(p,X) & \text{if } x \in \Omega, \\ \max(\eta + \overline{F}(p,X), p\cdot\mathbf{n} - \theta|p|) & \text{if } x \in \partial\Omega \end{cases}$$

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- $\underline{G}$ is used for subsolutions, $\overline{G}$ is used for supersolutions
- Strong comparison principle (Barles 1999)

Semi-Lagrangian schemes for MCM: (CFF 2011)

Representation formula in $\Omega = \mathbb{R}^2$: Soner-Touzi

For any regular solution u of the PDE s.t. $Du \neq 0$,

$$u(x,t) = \mathbb{E}\{u_0(y(t;x,t))\},$$

where

$$\begin{cases} dy(s;x,t) = \sqrt{2}\,\mathcal{P}\Big(Du(y(s;x,t),t-s)\Big)\,dW(s), \\ y(0;x,t) = x, \end{cases}$$

$$\mathcal{P}(q) = I - \frac{qq^T}{|q|^2} \quad \text{i.e.} \quad \mathcal{P}(Du) = \frac{1}{|Du|^2}\begin{pmatrix} u_{x_2}^2 & -u_{x_1}u_{x_2} \\ -u_{x_1}u_{x_2} & u_{x_1}^2 \end{pmatrix}.$$

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The representation formula implies that

$$u(x, t+\Delta t) = \mathbb{E}\{u(y(\Delta t; x, t+\Delta t), t)\}.$$

A one dimensional Brownian

The projector $\mathcal{P}(q)$ is of the form

$$\mathcal{P}(q) = \sigma(q)\sigma^T(q), \quad \text{with} \quad \sigma(q) = \frac{1}{|q|}\begin{pmatrix} -q_2 \\ q_1 \end{pmatrix}$$

For the real valued Brownian $\widehat{W}$ def. by $d\widehat{W}(s) \equiv \sigma^T dW(s)$,

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Remarks

- $\sigma(Du)$ is tangent to the level sets of u.
- If $d > 2$, σ is a $d \times (d-1)$-matrix and $\widehat{W}$ is a $(d-1)$-dimensional Brownian motion.

Semi-discrete scheme when $Du \neq 0$ (1/2)

Euler scheme for the stochastic process: $y_k \approx y(t_k; x, t)$ with

$$y_{k+1} = y_k + \sqrt{2}\,\sigma\Big(Du(y(k\Delta t; x, t), t - k\Delta t)\Big)\,\Delta\widehat{W}_k,$$

where $\Delta\widehat{W}_k \approx$ Gaussian variable with mean value 0 and variance Δt.

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For first order accuracy, it is enough that

$$\mathbb{P}\{\Delta\widehat{W}_k = \pm\sqrt{\Delta t}\} = \frac{1}{2}.$$

For $t = t_n = n\Delta t$ and $k = 0$:

$$\mathbb{P}\left\{y_1 = x \pm \sqrt{2\Delta t}\,\sigma(Du(x, t_n), t_n)\right\} = \frac{1}{2}.$$

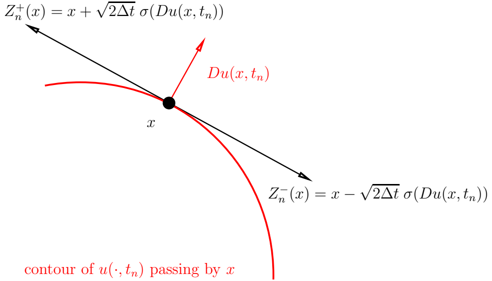
Semi-discrete scheme when $Du \neq 0$ (2/2)

This leads to the semi-discrete scheme for u:

$$u(x, t_{n+1}) = \frac{1}{2}\Big(u\left(Z_n^+(x), t_n\right) + u\left(Z_n^-(x), t_n\right)\Big),$$

where

$$Z_n^{\pm}(x) \equiv x \pm \sqrt{2\Delta t}\, \sigma(Du(x, t_n)),$$



Fully-discrete scheme for MCM when $Du \neq 0$

Consider a mesh $\mathcal{T}_h$ of Ω and call ξ a node of $\mathcal{T}_h$.

The values $u(\xi, t_n)$ are approximated by $u_h^n(\xi)$ with

$$u_h^{n+1}(\xi) = \frac{1}{2}\mathcal{I}_h[u_h^n]\left(\xi + \sqrt{\Delta t}\sigma^n(\xi)\right) + \frac{1}{2}\mathcal{I}_h[u_h^n]\left(\xi - \sqrt{\Delta t}\sigma^n(\xi)\right),$$

where $\mathcal{I}_h$ is an interpolation operator,

$$\sigma^n(\xi) = \sqrt{2}\,\sigma(D_h u_h^n(\xi)),$$

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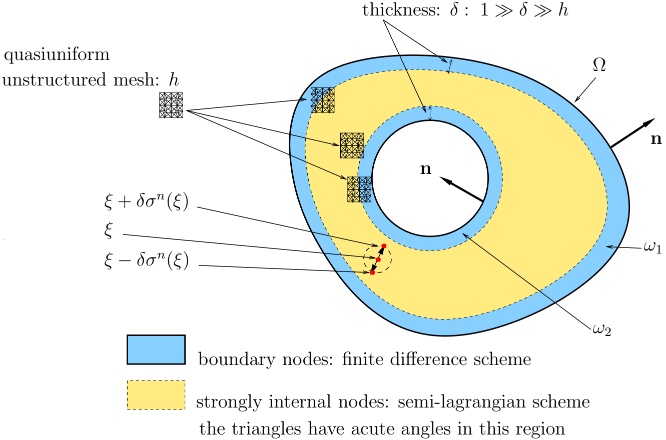
Questions

- Which scheme in the regions where $|D_h u_h^n|$ is small?
- Which scheme near the boundary?

A subdivision of Ω depending on a small parameter δ



- In the layers ω_k, one can define a system of orthogonal coordinates by projecting the points orthogonally onto $\partial\Omega$.
- Similarly, one can lift the outward unit vector $\mathbf{n}$ into ω_k.

The scheme in the layer $\overline{\omega_\ell}$

The nonlinear Neumann condition is not only imposed at the nodes on $\partial\Omega$, but also at all the boundary nodes in $\overline{\omega_\ell}$.

We use the lifting of $\mathbf{n}$ and a monotone scheme:

$$\mathcal{B}^\ell(\xi_i, u_i^{n+1}, [u^{n+1}]_\ell, [[u^n]]) = 0, \quad \text{for all } i \text{ s.t. } \xi_i \in \overline{\omega_\ell}$$

where

$$[u]_\ell = \{u_j, 1 \le j \le N_h,\ j \ne i,\ \xi_j \in \overline{\omega_\ell}\},$$
$$[[u]] = \{u_j, 1 \le j \le N_h,\ \xi_j \text{ is strongly internal}\}.$$

For example, a first order Godunov like scheme can be used.

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Given the values at the strongly internal nodes, this is a system of nonlinear equations which can be solved by combining Gauss-Seidel sweeps with different orderings.

The scheme at the strongly internal nodes (1/2)

Ingredients

- $\mathcal{I}_h$: Lagrange interpolation operator associated with $\mathcal{P}^1$ finite elements

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- D_h : discrete gradient reconstructed at the mesh nodes. For example,

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- Two internal regions: given two positive numbers C and s, the two sets of indices $\mathcal{J}_1^n$ and $\mathcal{J}_2^n$ are defined as follows:

$$\mathcal{J}_1^n = \{i : \ \xi_i \text{ is strongly internal and } \ |D_i^n| \geq C h^s\},$$
$$\mathcal{J}_2^n = \{i : \ \xi_i \text{ is strongly internal and } \ |D_i^n| < C h^s\}.$$

The scheme at the strongly internal nodes (2/2)

The modified semi-Lagrangian scheme with threshold Ch^s

- If $i \in \mathcal{J}_1^n$,

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\delta^2}\Big(\mathcal{I}_h[u^n](\xi_i + \delta\sigma_i^n) + \mathcal{I}_h[u^n](\xi_i - \delta\sigma_i^n) - 2u_i^n\Big)$$

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- If $i \in \mathcal{J}_2^n$,

$$u_i^{n+1} = -A_{ii}^{-1} \sum_{j \neq i} A_{ij} u_j^n$$

where

- $\sigma_i^n = \sigma(D_i^n)$
- A is the matrix arising from the $\mathcal{P}^1$ finite element discretization of $-\Delta$, when the functions are expanded in the nodal basis.

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Remark The scheme in $\mathcal{J}_2^n$ is a discrete version of

$$\frac{\partial w}{\partial t} - \epsilon(x)\Delta w = 0, \qquad \text{with} \qquad \epsilon \sim h^2/\Delta t.$$

Analysis of the scheme

Invariance w.r.t. addition of constants

We write the scheme in the form $u^{n+1} = \mathcal{S}^{\Delta t}(u^n)$.
We have

$$\mathcal{S}^{\Delta t}(u^n + k) = \mathcal{S}^{\Delta t}(u^n) + k.$$

Monotonicity

The scheme is not monotone

Consistency (the scheme is written $\mathcal{G}^{\Delta t}(i,n,u^{n+1},u^n)=0$)

Definition

For a smooth function Φ, for any sequence $(h_m,\Delta t_m,\delta_m)$ tending to 0 with $h_m=o(\delta_m)$, for $\xi_{i,m}\to x$, and for $t_{n_m}\to t$,

$$\begin{aligned}\underline{G}(x,\frac{\partial\Phi}{\partial t}(x,t),D\Phi(x,t),D^2\Phi(x,t)) &\leq \liminf_{m\to\infty}\mathcal{G}^{\Delta t}(i_m,n_m,\Phi^{n_m+1},\Phi^{n_m})\\ &\leq \limsup_{m\to\infty}\mathcal{G}^{\Delta t}(i_m,n_m,\Phi^{n_m+1},\Phi^{n_m})\\ &\leq \overline{G}(x,\frac{\partial\Phi}{\partial t}(x,t),D\Phi(x,t),D^2\Phi(x,t))\end{aligned}$$

with $\Phi^n=(\Phi(\xi_j,n\Delta t))_{j=1,\dots,N_h}$.

Proposition

Assume that D_h is a first order approximation of D. Assume that $h^2/\Delta t=o(1)$, $h/\delta=o(1)$ and $h^{1-s}/\delta=o(1)$. Then the scheme is consistent.

Monotonicity

Hereafter, $(h_m, \Delta t_m, \delta_m)$ and (ξ_{j_m}, t_{n_m}) are generic sequences s.t.

$$(h_m, \Delta t_m, \delta_m) \to 0 \quad \text{and} \quad (\xi_{j_m}, t_{n_m}) \to (\xi, t).$$

Relaxed monotonicity (CFF 2011)

The scheme $S^{\Delta t}$ is said to be monotone in the generalized sense if for any smooth function ϕ and grid functions v^m:

- If $v^m \le \phi^{n_m-1}$, then
$$S^{\Delta t_m}(v^m; j_m) \le \widetilde{S}^{\Delta t_m}(\phi^{n_m-1}; j_m) + o(\Delta t_m),$$
- If $\phi^{n_m-1} \le v^m$, then
$$\widetilde{S}^{\Delta t_m}(\phi^{n_m-1}; j_m) \le S^{\Delta t_m}(v^m; j_m) + o(\Delta t_m)$$

where $\widetilde{S}^{\Delta t}$ is a (possibly different) scheme consistent in the sense defined above.

A regularized scheme with a vanishing viscosity

The regularized scheme

- If $i \in \mathcal{J}_1^n$, then $\widehat{\mathcal{S}}(u^n; i) = \mathcal{S}(u^n; i) - W \dfrac{\Delta t}{\delta h^{s+1}} \sum_{j=1}^{N_h} A_{ij} u_j^n$,

  where W is a suitable positive constant
- If $i \in \mathcal{J}_2^n$, then $\widehat{\mathcal{S}}(u^n; i) = -(A_{ii})^{-1} \sum_{j \neq i} A_{ij} u_j^n$

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Theorem

Assume that D_h is a first order approximation of D. Take

$$0 < s < 1, \quad h = \delta^\gamma, \quad \Delta t = \beta \delta^{1+\gamma(1+s)}, \quad \textit{with } \gamma(1-s) > 1,$$

then for β small enough and suitable W, the regularized scheme is monotone in the generalized sense and consistent.

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Corollary (CFF 2011)

The regularized scheme is convergent.

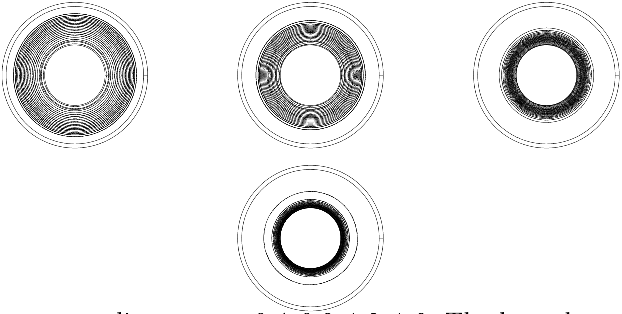
An example proposed by G. Barles

$$\Omega = \{r < |x| < R\}, \qquad u_0(x) = \phi(|x|^2).$$

$\forall \theta \in (-1, 1)$, the viscosity solution is

$$u(x,t) = \phi(\min(|x|^2 + 2t, R^2)).$$

The PDE holds up to the boundary $|x| = r$, and the boundary condition is lost there. Near $|x| = R$, $u(\cdot, t)$ is constant.



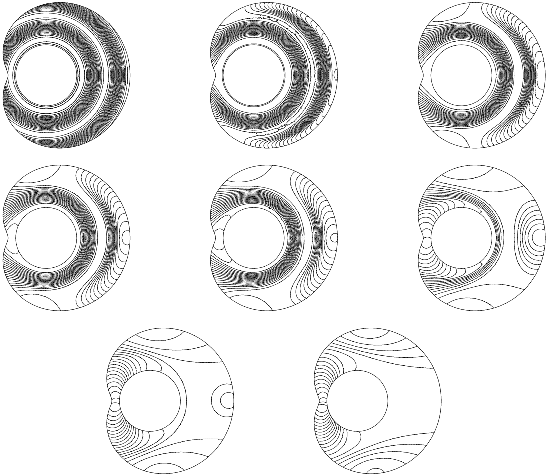
$h = 0.01$: contour lines at $t = 0.4, 0.8, 1.2, 1.6$. The boundary zones are also displayed.

Results (no artificial viscosity)

Table: $\|\text{ Error}\|_\infty$ for $s = 0.5$, $h = 1/N$, $\Delta t = h/10$, $\delta = \sqrt{2\Delta t}$

N	50	100	200	400
Error	0.116	0.082	0.055	0.041
Rel. Error	4.53%	3.2%	2.14%	1.6%

Another example: $s = 0.5$, $\delta = 0.1$, $h \sim 0.01$, $\Delta t = 0.001$



$\theta = -0.5$: contour lines at $t = 0, 0.08, 0.16, 0.24, 0.32, 0.8, 1.2, 2$