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Introduction

The KP-II equation

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

was physically introduced to study the stability of the Korteweg de Vries soliton solutions under the influence of a weak transverse perturbation Kadomtsev Petviashili [1970]

KP hierarchy

$$L = \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + \cdots$$

where

$$\partial_x^n f = \sum_{j \geq 0} \binom{n}{j} \partial_x^j(f) \partial_x^{n-j}, \quad n \in \mathbb{Z},$$

$$u_i = u_i(\mathbf{t}), \quad \mathbf{t} = (x = t_1, t_2 = y, t_3 = t, \ldots),$$

and weights: $\mathrm{wt}(u_i) = i, \quad \mathrm{wt}(\partial_x^n) = n$

KP hierarchy: $\partial_{t_n} L = [B_n, L], \qquad B_n = (L^n)_+, \; n \geq 1.$

$$[A, B] = AB - BA$$

Zakharov-Shabat equations

Computation of $\partial_{t_m}(L^n)$ and decomposition of $L^n = B_n + (L^n)_-$ easily imply Zakharov-Shabat (Z-S) equations

$$\partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m] = 0$$

which, in turn, imply commutation of flows $\partial_{t_n}(\partial_{t_m} L) = \partial_{t_m}(\partial_{t_m} L)$. For any given pair (n, m) ($n > m$) the Z-S equations give a closed system of $n-1$ equations in $u_2, \ldots, u_n$ in $t_1 = x, t_n, t_m$.

Example:
$B_2 = (L^2)_+ = \partial_x^2 + 2u_2,$
$B_3 = (L^3)_+ = \partial_x^3 + 3u_2\partial_x + 3(u_2)_x + 3u_3$

Z-S equations with $t_2 = y$, $t_3 = t$ and $u = 2u_2$ give KP-II.

Dressing operator W

KP hierarchy is also given by compatibility of

$$\begin{cases} L\psi(\mathbf{t},k) = k\psi(\mathbf{t},k), \\ \partial_{t_n}\psi(\mathbf{t},k) = B_n\psi(\mathbf{t},k), \quad n \geq 1. \end{cases}$$

Gauge transformation:

$$L \quad \mapsto \quad \partial_x = W^{-1}LW$$

with dressing operator

$$W = 1 - w_1\partial_x^{-1} - w_2\partial_x^{-2} - \cdots$$

then:

$$\begin{aligned} &u_2 = \partial_x w_1, \\ &u_{j+1} = \partial_x w_j + F_{j+1}(w_1, w_2, \ldots, w_{j-1}), \qquad j \geq 2, \end{aligned}$$

with F_{j+1} differential polynomial of weight $j+1$ ($\mathrm{wt}(w_j) = j$).

Sato equations

The w_j may be considered as primary variables whose x-derivatives determine the KP variables u_k. The evolution of the w_j variables w.r.t. t_n are governed by Sato equations

$$\partial_{t_n} W = (W \partial_x^n W^{-1})_+ W - W \partial_x^n. \quad n \geq 1.$$

If W satisfies Sato equations, then

$L = W \partial_x W^{-1}$ satisfies Lax equations for the KP hierarchy

and $B_n = (L^n)_+ = (W \partial_x^n W^{-1})_+$ satisfy the Z-S equations.

That is the KP hierarchy is generated by the inverse gauge (dressing) transformation $L = W\partial_x W^{-1}$. The KP linear system

$$\begin{cases} L\psi = k\psi, \\ \partial_{t_n}\psi = B_n\psi, \quad n \geq 1. \end{cases}$$

is obtained by the dressing action $\psi = W\psi_0$ where the vacuum eigenfunction ψ_0 satisfies

$$\begin{cases} \partial_x\psi_0 = k\psi_0, \\ \partial_{t_n}\psi_0 = \partial_x^n\psi_0 = k^n\psi_0, \quad n \geq 1. \end{cases}$$

In the following we use the normalization

$$\psi_0(\mathbf{t}, k) = e^{\theta(\mathbf{t},k)}, \qquad \theta(\mathbf{t}, k) = \sum_{n=1}^{\infty} k^n t_n.$$

The heat hierarchy

Suppose that $f_1, \ldots, f_N$ satisfy the heat hierarchy $(t_1 = x, t_2 = y, t_3 = t)$

$$\partial_{t_n} f_i = \partial_x^n f_i, \qquad n \geq 1,$$

let

$$\tau(x, y, t) = \mathrm{Wr_x}(f_1, \ldots, f_N),$$

then

$$u(x, y, t) = 2\partial_x^2 log(\tau(x, y, t)),$$

is a solution to KP-II.

A broad class of solutions to KP-II may be constructed using the heat hierarchy.
In this talk we shall be interested in the class of real solutions $u(x, y, t)$ to KP-II associated to

$$f_i(\mathbf{t}) = \sum_{j=1}^{M} a_{ij} E_j(\mathbf{t}), \qquad a_{ij} \in \mathbb{R}, \; i = 1, \ldots, N$$

where

$$E_j(\mathbf{t}) \equiv \psi_0(\mathbf{t}, k_j) = \exp(\theta(\mathbf{t}, k_j)), \qquad \theta(\mathbf{t}, k) = \sum_{n \geq 1} k^n t_n,$$

and

$$k_1 < k_2 < \cdots < k_M.$$

$$f_i(\mathbf{t}) = \sum_{j=1}^{M} a_{ij} \exp(k_j x + \cdots), \qquad i = 1, \ldots, N.$$

We require that the $N \times M$ matrix of the coefficients $A = (a_{ij})$ has all $N \times N$ non-negative minors.

That ensures that $\tau(x, y, t) = \mathrm{Wr}_x(f_1, \ldots, f_N)$ has no zeros in the (x, y)-plane for all t.

This of course guarantees that the corresponding KP–II solution $u(x, y, t) = 2\partial_x^2 \log(\tau)$ is non-singular.

Aymptotically, as $y \to \pm\infty$, there exist certain (non-decaying) directions invariant in t and along which the solution has the form of a plane wave similar to the one-soliton solution to KP

$$u(x,y,t) \approx \frac{1}{2}(k_r - k_s)^2 \mathrm{sech}^2(\frac{1}{2}(\theta(k_r,\mathbf{t}) - \theta(k_s,\mathbf{t}))).$$

N = the number of the asymptotic line solitons as $(y \to +\infty)$,

$M - N$ = the number of the asymptotic line solitons as $(y \to -\infty)$.

For the above reasons, this type of KP-II solution $u(x,y,t)$ is called $(M-N, N)$-line soliton solution.

Let the dressing operator satisfy

$$W = 1 - \sum_{j=1}^{N} w_j \partial_x^{-j},$$

and define the differential operator

$$W_N = W\partial_x^N = \partial_x^N - w_1\partial_x^{N-1} - \cdots - w_N.$$

Sato equation $\Longrightarrow \quad \partial_{t_n} W_N = B_n W_N - W_N \partial_x^n.$

Suppose that

$$\begin{cases} W_N f = 0, \\ \partial_{t_n} f = \partial_x^n f, \end{cases}$$

then $\partial_{t_n}(W_N f) = 0$, that is the above linear system is compatible.

The above proposition allows to construct the dressing operator associated to a given $\tau = \mathrm{Wr}_x(f_1, \ldots, f_N)$, since

$$0 = W_N f = \partial_x^N f - w_1 \partial_x^{N-1} f - \cdots w_N f.$$

$Gr(N, M)$

$Gr(N, M)$ is the set of N-dimensional subspaces of $\mathbb{R}^M$.

Example 1: $Gr(1, 2)$ is the set of all lines passing through the origin. Then clearly
$Gr(1, 2) = \{(1 : a), : a \in \mathbb{R}\} \cup \{(0 : 1)\} \cong \mathbb{R} \cup \{\infty\}$, which is the projective line $\mathbb{R}P^1$ and can be identified with the circle S^1.

Example 2: $Gr(1, M) = \mathbb{R}P^{M-1} = \bigcup_{j=1}^{M} F_j$, (Schubert decomposition) where
cell $F_j = \{(0 : \cdots : 0 : 1 : a_1 : \cdots : a_{M-j}) \; ; \; a_k \in \mathbb{R}\}$

Let $A = (a_{ij})$ be a $N \times M$ matrix of maximal rank N. It may be put in canonical RREF with a distinguished set of pivot columns $\mathcal{I} = \{1 \leq i_1 < i_2 < \cdots < i_N \leq M\}$.

For any given A its uniquely defined RREF provides a coordinate for a point of $Gr(N, M)$.

The set of matrices A whose RREF has the same pivot set $\mathcal{I}$ forms a cell $W_{\mathcal{I}}$ which provides the Schubert decomposition of the Grassmannian:

$$Gr(N, M) = \bigcup_{\mathcal{I}} W_{\mathcal{I}}.$$

$$\dim(Gr(N, M)) = N(M - N)$$

$$\dim(W_{\mathcal{I}}) = N(M - N) + \frac{1}{2}N(N - 1) - (i_2 + \cdots + i_N).$$

$\{f_1, \ldots, f_N\}$ solutions of the linear heat hierarchy of the form

$$f_i = \sum_{j=1}^{M} a_{ij} E_j, \qquad i = 1, \ldots, N$$

with $E_j = \exp(\sum_{n \geq 1} k_j^n t_n)$, $j = 1, \ldots, M$.
$\{f_1, \ldots, f_N\}$ linearly independent $\implies A = (a_{ij})$ has rank N.

$$\tau = \mathrm{Wr}_x(f_1, \ldots, f_N) = \sum_{1 \leq i_1 < \cdots < i_N \leq M} \xi(i_1, \ldots, i_N) E(i_1, \ldots, i_N),$$

where $E(i_1, \ldots, i_N) = \mathrm{Wr}_x(E_{i_1}, \ldots, E_{i_N})$, and the minors $\xi(i_1, \ldots, i_N)$ are the Plücker coordinates of the corresponding point on $Gr(N, M)$.

Then for the case under consideration, the solutions $u(x, y, t)$ correspond to τ-functions with non-negative Plücker coordinates (i.e. non-negative maximal minors).

The corresponding $(N \times M)$ matrices A are totally non-negative and parametrize the totally non-negative Grassmannian $Gr^{+}(N, M) \subset Gr^{(}N, M)$ classified in combinatorial way in term of networks by Postnikov.

Kodama: take A totally non-negative and irreducible (i.e. once A is in RREF in each column there is a non zero element and in each row there is a non zero element other then the pivot). Then associate to the corresponding cell $W_{\mathcal{I}}^{+,0}$ in $Gr(N, M)_0^{+}$ a derangement of the indexes $\{1, \ldots, M\}$ which allows to identify the asymptotic directions of the line solitons in the (x, y) - plane.

Description of points in $Gr(1,M)_0^+$ in terms of divisors

Aim: to associate to each $(N, M-N)$-soliton solution $u(\mathbf{t})$ the corresponding point in $Gr(N,M)_0^+$ (totally non negative and irreducible part of $Gr(N,M)$) via a set of divisors.

Model: $Gr(1,M)_0^+ = F_1^+ = \{(1 : a_1 : \cdots : a_{M-1}),\ a_j \in \mathbb{R}^+\}$.

$$f = E_1 + a_1 E_2 + \cdots + a_{M-1} E_M, \qquad E_j = \psi_0(\mathbf{t}, k_j)$$

$$\psi = \left(1 - \frac{w_1}{k}\right)\psi_0, \qquad w_1 = f_x/f.$$

Idea: $0 = W_1 f \equiv (\partial_x - w_1) f \quad \Longleftrightarrow \quad 0 = \sum_{j=1}^M \mathrm{Res}_{k=k_j} \Phi(k, \mathbf{t})$.

We introduce

$$\psi_D(k,\mathbf{t}) = \left(1 + \frac{\chi_1(\mathbf{t})}{k-\gamma_1}\right)\psi_0$$

s.t.$(k-\gamma_1)\psi_D = W_1\psi_0 = k\psi$ and we use the following natural normalization $\chi_1(\mathbf{0}) = 0$.
That implies that $\gamma_1 = w_1(\mathbf{0}) \in \cup]k_j, k_{j+1}[$.
We take

$$\psi_U(k,\mathbf{t}) = c_0(\mathbf{t}) + \frac{c_1(\mathbf{t})}{k-\delta_1} + \cdots + \frac{c_{M-2}(\mathbf{t})}{k-\delta_{M-2}},$$

with the compatible normalization $c_0(\mathbf{0}) = 1$, $c_l(\mathbf{0}) = 0$, $l = 1, \ldots, M-2$.
Let us consider $\psi_D(k_j,\mathbf{t}) = \psi_U(k_j,\mathbf{t})$, $\quad j = 1, \ldots, M$.

$$\psi_D(k_j, \mathbf{t}) = \psi_U(k_j, \mathbf{t}), \quad j = 1, \dots, M. \tag{1}$$

Theorem: a) If we assign a divisor $D = (\gamma_1, \delta_1, \dots, \delta_{M-2})$ s.t. $D \cap]k_j, k_{j+1}[\neq \emptyset$, then system (1) possesses a unique solution $(\chi_1, c_0, \dots, c_{M-2})$. Moreover $w_1 = \gamma_1 - \chi_1$ is associated a real and regular $(1, M-1)$-soliton solution to KP-II and

$$f_D(t) = \sum_{j=1}^{M} \beta_j E_j, \qquad \beta_j = \frac{\prod_l (k_j - \delta_l)}{(k_j - \gamma_1) \prod_s' (k_j - k_s)}.$$

b) Viceversa, let $w_1 = f_x / f$ be given and let $\gamma_1 = w_1(\mathbf{0})$. Then (1) gives a unique divisor $D^{(1)} = (\delta_1, \dots, \delta_{M-2})$ and uniquely defined functions c_l, $l = 0, \dots, M-2$. Moreover $D = \gamma_1 \cup D^{(1)}$ satisfies the compatibility condition $D \cap]k_j, k_{j+1}[\neq \emptyset$ and $Wr(f, f_D) = 0$.

Description of points in $Gr(N, M)_0^+$ in terms of divisors

In this case $Gr(N, M)_0^+ = \bigcup_{\mathcal{I}} W_{\mathcal{I},0}^+$.

$$f_i = \sum_{j=1}^{M} a_{ij} E_j, \quad i = 1, \ldots, N, \qquad E_j = \psi_0(\mathbf{t}, k_j)$$

$$\psi = \left(1 + \frac{w_1}{k} + \cdots + \frac{w_N}{k^N}\right)\psi_0, \qquad w_1 = \tau_x/\tau, \quad \tau = \mathrm{Wr}(f_1, \ldots, f_N).$$

Idea:

$$0 = (\partial_x^N - w_1 \partial_x^{N-1} \ldots) f_i^{(D)} \quad \Longleftrightarrow \quad 0 = \sum_{j=1}^{M} \mathrm{Res}_{k=k_j} \Phi^{(i)}(k, \mathbf{t}).$$

Description of points in $Gr(N, M)_0^+$ in terms of divisors

In this case we need a system of divisors!

Let $D = (\gamma_1, \dots, \gamma_N)$ and $D^{(i)} = (\delta_1^{(i)}, \dots, \delta_{M-2}^{(i)})$, $i = 1, \dots, N$.

$$\begin{cases} \gamma_1 + \cdots + \gamma_N = w_1(\mathbf{0}), \\ \vdots \\ \gamma_1 \cdots \gamma_N = (-1)^N w_N(\mathbf{0}), \end{cases}$$

Example: Top cell (maximal dimension):

$$D^{(i)} = (\delta_1^{(i)}, \dots, \delta_{M-N-1}^{(i)}, k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_N)$$

$$D_0^{(i)} = (\delta_1^{(i)}, \dots, \delta_{M-N-1}^{(i)}, \gamma_1, \dots, \gamma_N), \quad i = 1, \dots, N$$

s.t.

$$D_0^{(i)} \cap]k_j, k_{j+1}[\neq \emptyset, \quad \forall i = 1, \dots, N, \quad j = 1, \dots, M-1.$$