Hyperbolic Differential-Operator Equations

with the time differentiation in boundary conditions

Yakov Yakubov
yakubov@post.tau.ac.il

Department of Applied Mathematics
School of Mathematical Sciences
Tel-Aviv University

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Outline

Problems with the time differentiation in boundary conditions arise in many problems of physics and mechanics. We consider a situation in which the time differentiation appears at the same (second) order in both the equation and boundary conditions. Physically, such a problem may represent the longitudinal displacements of an inhomogeneous rod under the action of forces at the two ends which are proportional to the acceleration. In particular, this situation is realized if there are massive loads at the ends – C. C. Lin and L. A. Segel, "Mathematics Applied to Deterministic Problems in the Natural Sciences", 1974.

First, we give an abstract interpretation of such initial boundary value problems for hyperbolic equations that a part of boundary value conditions may contain the differentiation on the time t of the same (second) order as the equation.

We prove the well-posedness of these abstract problems but we present, in the talk, some particular cases, in which we have succeeded to expand the unique solution to the series of eigenvectors of the corresponding spectral problem.

Then, we show an application of the abstract result to partial (hyperbolic) differential equations. In fact, we have obtained a generalization of the classical Fourier method of separation of variables to the case, in which boundary conditions may contain the differentiation on the time t.

Some Hilbert spaces

Let H and F be Hilbert spaces. The set $H \oplus F$ of all vectors of the form (u, v), where $u \in H$ and $v \in F$, with usual coordinatewise linear operations and the norm $\|(u, v)\|_{H\oplus F} := \left(\|u\|_H^2 + \|v\|_F^2\right)^{\frac{1}{2}}$, is a Hilbert space and called the **orthogonal sum** of Hilbert spaces H and F.

For a linear operator A closed in the Hilbert space H, the domain $D(A)$ is turned into the Hilbert space $H(A)$ with respect to the norm $\|u\|_{H(A)} := \left(\|u\|_H^2 + \|Au\|_H^2\right)^{\frac{1}{2}}$.

Some Hilbert spaces - continued

If H_1 and H are two Hilbert spaces with $H_1 \subset H$, $\overline{H}_1|_H = H$, then H_1 can be represented as the domain $D(S) = H_1$ of a suitable positive definite selfadjoint operator S in H. Then, the interpolation space, for $0 < \theta < 1$,

$$(H_1, H)_{\theta,2} = H(S^{1-\theta}).$$

Some abstract functional spaces

$W_p^\ell((0,1);H)$ with $1 \le p < \infty$ and $0 \le \ell$ is an integer, denotes the Banach space of functions $u(x)$ with values in H which have generalized derivatives up to the ℓ-th order inclusive on $(0,1)$ and the norm

$$\|u\|_{W_p^\ell((0,1);H)} := \sum_{k=0}^{\ell} \left(\int_0^1 \|u^{(k)}(x)\|_H^p dx \right)^{\frac{1}{p}}$$

is finite.

Some abstract functional spaces - continued

$C^k([0, T]; H)$, where $0 \leq k$ is an integer, denotes the Banach space of functions $u(x)$ with values from H which have continuous derivatives up to the k-th order inclusive on $[0, T]$ and the norm

$$\|u\|_{C^k([0,T];H)} := \sum_{\ell=0}^{k} \max_{x\in[0,T]} \|u^{(\ell)}(x)\|_H < \infty.$$

Denote by $C^2([0, T]; H_1, H_2, H_3)$ the space

$$C([0, T]; H_1) \cap C^1([0, T]; H_2) \cap C^2([0, T]; H_3).$$

Formulation of the problem

Let H and H^{ν}, $\nu = 1, ..., s$, be Hilbert spaces. Consider the following abstract initial boundary value problem

$$L(D_t)u := u''(t) + Au'(t) + Bu(t) = h(t), \quad (1)$$
$$L_{\nu}(D_t)u := (A_{\nu 0}u(t))'' + A_{\nu 2}u(t) = 0, \ \nu = 1, ..., s, \quad (2)$$
$$u(0) = \varphi_0, \ u'(0) = \varphi_1, \quad (3)$$

where $t \in [0, T]$; A and B are linear operators in H; $A_{\nu 0}$ and $A_{\nu 2}$ are linear operators from H into H^{ν}; $u(t)$ from $[0, T]$ into H is an unknown function. Note that operators A, B, $A_{\nu 0}$, and $A_{\nu 2}$ are, generally speaking, unbounded.

The corresponding spectral problem is

$$\lambda^2 u + \lambda A u + B u = 0, \tag{4}$$
$$\lambda^2 A_{\nu 0} u + A_{\nu 2} u = 0, \quad \nu = 1, \ldots, s. \tag{5}$$

Let there exist a Hilbert space $H_0 \subset H$ such that the operators A and B from H_0 into H act boundedly and the operators $A_{\nu 0}$ and $A_{\nu 2}$ from H_0 into H^ν act boundedly. Then, a number λ_0 is called an **eigenvalue** of the problem (4)-(5) if the problem with $\lambda = \lambda_0$ has a nontrivial solution $u_0 \in H_0$. The nontrivial solution $u_0 \in H_0$ is called an **eigenvector** corresponding to the eigenvalue λ_0 of the problem (4)-(5).

Theorem (abstract)

Let the following conditions be satisfied

1) B *is an operator in the Hilbert space* H *with dense domain* $D(B)$; A *is an operator in* H *with* $D(A) \supset (H(B), H)_{\frac{1}{2},2}$; *the embedding* $H(B) \subset H$ *is compact;*
2) *the operators* $A_{\nu 0}$, $\nu = 1, \ldots, s$, *from* $H(B)$ *into* H^{ν} *act compactly and the operators* $A_{\nu 2}$, $\nu = 1, \ldots, s$, *from* $H(B)$ *into* H^{ν} *act boundedly;*
3) *the linear manifold* $\left\{ v \mid v := (u, A_{10}u, \ldots, A_{s0}u),\ u \in D(B) \right\}$ *is dense in the Hilbert space* $H \oplus H^1 \oplus \cdots \oplus H^s$;

Theorem (abstract-continued)

4) *for* $u \in D(B),\ v \in D(B)$,

$$(Bu, v)_H + (A_{12}u, A_{10}v)_{H^1} + \dots + (A_{s2}u, A_{s0}v)_{H^s} = (u, Bv)_H + (A_{10}u, A_{12}v)_{H^1} + \dots + (A_{s0}u, A_{s2}v)_{H^s};$$

5) *for* $u \in D(B)$ *and some* $c \neq 0$

$$(Bu, u)_H + (A_{12}u, A_{10}u)_{H^1} + \dots + (A_{s2}u, A_{s0}u)_{H^s} \geq c^2(\|u\|^2_H + \|A_{10}u\|^2_{H^1} + \dots + \|A_{s0}u\|^2_{H^s});$$

Theorem (abstract-continued)

6) *all sufficiently large numbers* $\lambda > 0$ *are regular points for the operator pencil* $\mathbb{L}(\lambda)$*:*
$u \to \mathbb{L}(\lambda)u := \left((\lambda I + B)u, (\lambda A_{10} + A_{12})u, \ldots, (\lambda A_{s0} + A_{s2})u\right)$
which acts boundedly from $H(B)$ *onto* $H \oplus H^1 \oplus \cdots \oplus H^s$*,*
and, for $\lambda > 0, \ \lambda \to \infty$*,*

$$\|\mathbb{L}(\lambda)^{-1}\|_{B(H \overset{s}{\underset{\nu=1}{\oplus}} H^\nu, H)} \leq C|\lambda|^{-1},$$

$$\|A_{\nu 0}\mathbb{L}(\lambda)^{-1}\|_{B(H \overset{s}{\underset{\nu=1}{\oplus}} H^\nu, H^\nu)} \leq C|\lambda|^{-1}, \quad \nu = 1, \ldots, s;$$

Theorem (abstract-continued)

7) *A is a skew-symmetric operator in H, i.e., $A^*u = -Au$, $u \in D(A)$ and A from $(H(B), H)_{\frac{1}{2},2}$ into H is bounded;*

8) *$h \in W_p^1((0, T); H)$ for some $p > 1$;*

9) *$\varphi_0 \in H(B)$, $\varphi_1 \in H(B)$.*

Then, there exists a unique solution $u(t)$ of problem (1)-(3) such that the function $t \to (u(t), A_{10}u(t), \ldots, A_{s0}u(t))$ from $[0, T]$ into $H \oplus H^1 \oplus \cdots \oplus H^s$ is twice continuously differentiable and from $[0, T]$ into $H(B) \oplus H^1 \oplus \cdots \oplus H^s$ is continuous and the solution can be expanded to the series

Theorem (abstract-continued)

$$u(t) = \sum_{k=1}^{\infty} \frac{C_k(t)}{D_k} e^{\lambda_k t} u_k,$$

where λ_k *are purely imaginary eigenvalues and* u_k *are the corresponding eigenvectors of the spectral problem (4)-(5), and the series converges in the sense of the space* $C^2([0,T]; H(B), (H(B), H)_{\frac{1}{2},2}, H)$

Theorem (abstract-finished)

with

$$D_k = (Bu_k, u_k)_H + \sum_{\nu=1}^{s} (A_{\nu 2} u_k, A_{\nu 0} u_k)_{H^\nu} + |\lambda_k|^2 (\|u_k\|_H^2 + \sum_{\nu=1}^{s} \|A_{\nu 0} u_k\|_{H^\nu}^2),$$

$$C_k(t) = (B\varphi_0 - \lambda_k \varphi_1, u_k)_H + \sum_{\nu=1}^{s} (A_{\nu 2}\varphi_0 - \lambda_k A_{\nu 0}\varphi_1, A_{\nu 0} u_k)_{H^\nu} - \lambda_k \int_0^t e^{-\lambda_k \tau} (h(\tau), u_k)_H d\tau.$$

Formulation of a possible application problem

Consider, in the domain $[0, T] \times [0, 1]$, an initial boundary value problem for the hyperbolic equation

$$L(D_t)u := D_{tt}^2 u(t,x) + ia(x)D_t u(t,x) - D_x(b(x)D_x u(t,x)) \\ + c(x)u(t,x) = h(t,x), \quad (t,x) \in [0, T] \times [0, 1], \tag{6}$$

$$\begin{aligned} L_1(D_t)u &:= \alpha D_{tt}^2 \left[u(t,0)\right] + D_x u(t,0) = 0, \quad t \in [0, T], \\ L_2(D_t)u &:= \beta D_{tt}^2 \left[u(t,1)\right] + D_x u(t,1) = 0, \quad t \in [0, T], \end{aligned} \tag{7}$$

$$u(0,x) = \varphi_0(x), \quad D_t u(0,x) = \varphi_1(x), \quad x \in [0, 1], \tag{8}$$

where α, β are real numbers, $i = \sqrt{-1}$, $D_t := \frac{\partial}{\partial t}$, $D_x := \frac{\partial}{\partial x}$.

The corresponding spectral problem is

$$
\begin{aligned}
&\lambda^2 u(x) + ia(x)\lambda u(x) - (b(x)u'(x))' + c(x)u(x) = 0,\\
&\alpha\lambda^2 u(0) + u'(0) = 0,\\
&\beta\lambda^2 u(1) + u'(1) = 0.
\end{aligned}
\tag{9}
$$

A number λ_0 is called an **eigenvalue** of problem (9) if the problem with $\lambda = \lambda_0$ has a nontrivial solution $u_0(x) \in W_2^2(0,1)$. The nontrivial solution $u_0(x) \in W_2^2(0,1)$ is called an **eigenfunction** corresponding to the eigenvalue λ_0 of problem (9).

Theorem (application)

Let the following conditions be satisfied

1) $a \in C[0,1]$ *and is real-valued;* $b \in C^1[0,1]$, $b(x) > 0$ *for* $x \in [0,1]$; $c \in C[0,1]$, $c(x) > 0$ *for* $x \in [0,1]$;
2) $\alpha < 0$, $\beta > 0$;
3) $h \in W_p^1((0,T); L_2(0,1))$ *for some* $p > 1$;
4) $\varphi_0 \in W_2^2(0,1)$, $\varphi_1 \in W_2^2(0,1)$.

Theorem (application-continued)

Then, there exists a unique solution $u(t,x)$ of the problem (6)-(8) such that the function $t \to (u(t,x), \alpha u(t,0), \beta u(t,1))$ from $[0, T]$ into $L_2(0,1) \oplus \mathbb{C} \oplus \mathbb{C}$ is twice continuously differentiable and from $[0, T]$ into $W_2^2(0,1) \oplus \mathbb{C} \oplus \mathbb{C}$ is continuous and the solution can be expanded to the series

$$u(t,x) = \sum_{k=1}^{\infty} \frac{C_k(t)}{D_k} e^{\lambda_k t} u_k(x),$$

Theorem (application-continued)

with

$$D_k = \int_0^1 b(x)|u_k'(x)|^2 dx + \int_0^1 c(x)|u_k(x)|^2 dx + |\lambda_k|^2 \Big(\int_0^1 |u_k(x)|^2 dx - b(0)\alpha |u_k(0)|^2 + b(1)\beta |u_k(1)|^2 \Big),$$

$$C_k(t) = \int_0^1 \Big(-(b(x)\varphi_0'(x))' + c(x)\varphi_0(x) - \lambda_k \varphi_1(x) \Big) u_k(x) dx - b(0)(\varphi_0'(0) - \lambda_k \alpha \varphi_1(0)) u_k(0) + b(1)(\varphi_0'(1) - \lambda_k \beta \varphi_1(1)) u_k(1) - \lambda_k \int_0^t e^{-\lambda_k \tau} \int_0^1 h(\tau, x) u_k(x) dx d\tau,$$

Theorem (application-finished)

and λ_k are purely imaginary eigenvalues and $u_k(x)$ are the corresponding eigenfunctions of the spectral problem (9), and the series converges in the sense of the space $C^2([0, T]; W_2^2(0,1), W_2^1(0,1), L_2(0,1))$.[a]

[a] Without loss of generality we can assume that $u_k(x)$ are real-valued functions since $\overline{u_k(x)}$ are also solutions of (9) with purely imaginary $\lambda = \lambda_k$ and, therefore, $u_k(x) \pm \overline{u_k(x)}$ are also solutions of (9).

Generalization of the classical Fourier method

If we formally take $\alpha = \beta = 0$, $a(x) \equiv h(t,x) \equiv 0$, $c(x) \equiv c^2 > 0$, $b(x) \equiv b^2 > 0$ then problem (6)-(8) turns out to be a classical problem and the expansion formula of the application theorem coincides with the known one

$$u(t,x) = \int_0^1 \varphi_0(x)dx \cos(ct) + \frac{1}{c}\int_0^1 \varphi_1(x)dx \sin(ct)$$
$$+ 2\sum_{k=1}^{\infty}\Big(\int_0^1 \varphi_0(x)\cos(k\pi x)dx \cos(t\sqrt{k^2\pi^2 b^2 + c^2})$$
$$+ \frac{1}{\sqrt{k^2\pi^2 b^2 + c^2}}\int_0^1 \varphi_1(x)\cos(k\pi x)dx \sin(t\sqrt{k^2\pi^2 b^2 + c^2})\Big)$$
$$\times \cos(k\pi x),$$

which is obtained by the classical Fourier method of separation of variables.